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Research Article

Global Existence and Blow-Up Solutions and Blow-Up Estimates for Some Evolution Systems with p-Laplacian with Nonlocal Sources

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This paper deals with p-Laplacian systems $u_t - \text{div}(|\nabla u|^{p-2}\nabla u) = \int_\Omega v^\alpha(x,t)dx$, $x \in \Omega$, $t > 0$, $v_t - \text{div}(|\nabla v|^{q-2}\nabla v) = \int_\Omega u^\beta(x,t)dx$, $x \in \Omega$, $t > 0$, with null Dirichlet boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, where $p, q \geq 2$, $\alpha, \beta \geq 1$. We first get the nonexistence result for related elliptic systems of nonincreasing positive solutions. Secondly by using this nonexistence result, blow up estimates for above p-Laplacian systems with the homogeneous Dirichlet boundary value conditions are obtained under $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$ $(R > 0)$. Then under appropriate hypotheses, we establish local theory of the solutions and obtain that the solutions either exist globally or blow up in finite time.

1. Introduction

In this paper, we study the following nonlocal p-Laplacian systems in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$:

$$
\begin{gathered}
u_t - \text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \int_\Omega v^\alpha(x,t)dx, \quad x \in \Omega,\ t > 0, \\
v_t - \text{div}\left(|\nabla v|^{q-2}\nabla v\right) = \int_\Omega u^\beta(x,t)dx, \quad x \in \Omega,\ t > 0, \\
u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega,
\end{gathered}
\tag{1.1}
$$

where $p,q \geq 2$, $\alpha,\beta \geq 1$. $u_0(x) \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, $v_0(x) \in L^\infty(\Omega) \cap W_0^{1,q}(\Omega)$ and $\partial u_0(x)/\partial\eta, \partial v_0(x)/\partial\eta < 0$ on $\partial\Omega$, η denotes the unit outer normal vector on the boundary.

As well as the nonexistence of positive solutions of the related elliptic systems,

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= \int_\Omega v^\alpha(x)dx, \quad x \in \Omega, \\ -\operatorname{div}\left(|\nabla v|^{q-2}\nabla v\right) &= \int_\Omega u^\beta(x)dx, \quad x \in \Omega. \end{aligned} \tag{1.2}$$

Equations (1.1) are the classical reaction-diffusion system of Fujita-type for $p = q = 2$. If $p \neq 2$, $q \neq 2$, (1.1) appears in the theory of non-Newtonian fluids [1, 2] and in nonlinear filtration theory [3]. In the non-Newtonian fluids theory, the pair (p,q) is a characteristic quantity of the medium. Media with $(p,q) > (2,2)$ are called dilatant fluids and those with $(p,q) < (2,2)$ are called pseudoplastics. If $(p,q) = (2,2)$, they are Newtonian fluids.

In the past two decades, many physical phenomena were formulated into nonlocal mathematical models (see [4–9] and the references therein) and studied by many authors. Degenerate parabolic equations involving a nonlocal source, which arise in a population model that communicates through chemical means, were studied in [10, 11].

As a matter of course, (1.1) with $p = q = 2$ give semilinear parabolic equations and have been studied by many authors. Over the last few years, much effort has been devoted to the study of blow-up properties for nonlocal semilinear parabolic equations of the type $v_t = \triangle v + g(t)$ (see [12–14]). Conditions on blowing up, blow-up set, blow-up rate, and asymptotic behavior of solutions are obtained, see [4, 5]. The problem concerning (1.1) includes the existence and multiplicity of global solutions, blowing-up, blow-up rates and blow-up sets, uniqueness and nonuniqueness, and so forth. For (1.2), there are problems such as existence and nonexistence, uniqueness and nonuniqueness, and so on. On the contrary, it seems that little is known about the result for quasilinear reaction-diffusion system (non-Newtonian filtration systems) and quasilinear elliptic system (e.g., [15–18]). For the scalar problem, a few authors (see [8, 19]) investigated the following equation:

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = u^q, \tag{1.3}$$

with initial and boundary conditions. Roughly speaking, their results are

(1) the solution u exists globally if $q < p - 1$, and

(2) u blows up in finite time if $q > p - 1$ and $u_0(x)$ is sufficiently large.

The authors in [7] studied the following equation:

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \int_\Omega u^q(x,t)dx, \tag{1.4}$$

with null Dirichlet conditions and obtained that the solution either exists globally or blows up in finite time. Under appropriate hypotheses, they have local theory of the solution and obtain that the solution either exists globally or blows up in finite time.

The authors in [9] deal with the following reaction-diffusion system:

$$
\begin{aligned}
u_t - \triangle u &= \int_{\Omega} f(v(y,t))\,dy, \quad x \in \Omega,\ t > 0,\\
v_t - \triangle v &= \int_{\Omega} g(u(y,t))\,dy, \quad x \in \Omega,\ t > 0,
\end{aligned}
\tag{1.5}
$$

with initial and boundary conditions. They proved that there exists a unique classical solution and the solution either exists globally or blows up in finite time. Furthermore, they obtain the blow-up set and asymptotic behavior provided that the solution blows up in finite time.

For p-Laplacian systems, Yang and Lu in [15] studied the following equations:

$$
\begin{aligned}
u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) &= v^{\alpha},\\
v_t - \operatorname{div}\left(|\nabla v|^{q-2}\nabla v\right) &= \omega^{\beta},\\
\omega_t - \operatorname{div}\left(|\nabla \omega|^{m-2}\nabla \omega\right) &= u^{\gamma}, \quad x \in \Omega,\ t > 0.
\end{aligned}
\tag{1.6}
$$

They derive some estimates near the blow-up point for positive solutions and nonexistence of positive solutions of the relate elliptic systems.

The main purpose of this paper is to derive some estimates near the blow-up point and investigate the global existence and blow-up of solutions for problem (1.1).

The outline of the paper is as follows. In the next section, we investigate the global nonexistence for elliptic system (1.2). Section 3 is devoted to blow-up estimate for system (1.1). In Section 4, we give the local existence and uniqueness of system (1.1). In Section 5, we give the blow-up property of solutions to (1.1).

After finishing this paper, we learn from a recent paper by Li [20] that he obtained the results of global existence and blow-up of solutions for (1.1). As we will show in Sections 4 and 5, our proof for the results of global existence and blow-up of solutions given here is simpler than [20].

2. Nonexistence for elliptic system (1.2)

Motivated by [12, 13, 15, 16, 18], we consider radially symmetric solutions of the elliptic system (1.2), that is, suppose that $u(x) = u(r)$, $v(x) = v(r)$ with $r = |x|$.

Let

$$
\begin{aligned}
z_1 &= \frac{(p+1)(q-1)+\alpha(q+1)}{\alpha\beta-(p-1)(q-1)} - \frac{N-p}{p-1},\\
z_2 &= \frac{(q+1)(p-1)+\beta(p+1)}{\alpha\beta-(p-1)(q-1)} - \frac{N-q}{q-1}.
\end{aligned}
\tag{2.1}
$$

We give the following theorem

Theorem 2.1. *Assume that*

(i) $N > \max\{p,q\}$, $\alpha\beta > (p-1)(q-1)$ *with* $p,q > 1$;

(ii) $z_1 \geq 0$ *or* $z_2 \geq 0$.

Then system (1.2) has no positive radially symmetric solution.

To prove Theorem 2.1, system (1.2) can be written in radial coordinates as

$$(\Phi_p(u'))' + \frac{N-1}{r}\Phi_p(u') + \int_0^r v^\alpha = 0, \tag{2.2}$$

$$(\Phi_q(v'))' + \frac{N-1}{r}\Phi_q(v') + \int_0^r u^\beta = 0, \tag{2.3}$$

$$u(0) > 0, \quad v(0) > 0, \quad u'(0) = v'(0) = 0, \tag{2.4}$$

in $\mathbb{R}^N$ with $N \geq \max\{p,q\}$, where $\Phi_p(u) = |u|^{p-2}u$, $\Phi_q(v) = |v|^{q-2}v$, $p,q > 1$.

By the similar argument of [15, Lemma 2], we can prove the following lemmas.

Lemma 2.2. *Let (u,v) be a positive and radially symmetric solution of (2.2)–(2.4). Then for $r > 0$,*

$$\begin{aligned} \left(\frac{r^{p+1}}{N}\right)^{1/(p-1)} v^{\alpha/(p-1)} &\leq -ru' \leq \frac{N-p}{p-1}u(r), \\ \left(\frac{r^{q+1}}{N}\right)^{1/(q-1)} u^{\beta/(q-1)} &\leq -rv' \leq \frac{N-q}{q-1}v(r). \end{aligned} \tag{2.5}$$

From (2.5), we have the following lemma.

Lemma 2.3. *Suppose that the conditions in Theorem 2.1 are satisfied. Let (u,v) be a positive and radially symmetric solution of (2.2)–(2.4). Then*

$$\begin{aligned} u(r) &\leq Cr^{-((p+1)(q-1)+\alpha(q+1))/(\alpha\beta-(p-1)(q-1))}, \\ v(r) &\leq Cr^{-((q+1)(p-1)+\beta(p+1))/(\alpha\beta-(p-1)(q-1))}, \end{aligned} \tag{2.6}$$

in which $C = C(N,\alpha,\beta,p,q)$.

Proof of Theorem 2.1. Let (u,v) be a nontrivial positive and radially symmetric solution of (2.2)–(2.4). We consider first the case $z_1 > 0$ or $z_2 > 0$.

By Lemma 2.2,

$$(r^{N-p}u^{p-1}(r))' = r^{N-p-1}u^{p-2}[(p-1)ru'(r) + (N-p)u(r)] \geq 0, \tag{2.7}$$

we have $u(r) \geq cr^{-(N-p)/(p-1)}$ and $(u(r)r^{(N-p)/(p-1)})$, $(v(r)r^{(N-q)/(q-1)})$ are nondecreasing on $(0,+\infty)$. From Lemma 2.3 and for $r > r_0 > 0$, we obtain that $r^{z_1} \leq C$ or $r^{z_2} \leq C$. Since $z_1 > 0$ or $z_2 > 0$, this leads to a contradiction for r sufficiently large.

Suppose next that $z_1 = 0$ (the case $z_2 = 0$ being similar). From (2.2), it follows that for $r \geq r_0 \geq 0$,

$$r^{N-1}|u'(r)|^{p-1} - r_0^{N-1}|u'(r_0)|^{p-1} = \int_{r_0}^{r} s^{N-1}\left(\int_0^s v^{\alpha}(t)dt\right)ds. \tag{2.8}$$

By Lemma 2.2, we have $v^{\alpha}(t) \geq Ct^{\alpha(q+1)/(q-1)}u^{\alpha\beta/(q-1)}$ and hence

$$r^{N-1}|u'(r)|^{p-1} \geq C\int_{r_0}^{r} s^{N-1}\left(\int_0^s t^{\alpha(q+1)/(q-1)}u^{\alpha\beta/(q-1)}dt\right)ds. \tag{2.9}$$

Now taking into account that $u(t) \geq Ct^{(p-N)/(p-1)}$, we obtain

$$\begin{aligned} r^{N-1}|u'(r)|^{p-1} &\geq C\int_{r_0}^{r} s^{N-1}\left(\int_0^s t^{\alpha(q+1)/(q-1)}t^{\alpha\beta(p-N)/((p-1)(q-1))}dt\right)ds \\ &= C\int_{r_0}^{r} s^{-1}ds = C\ln\left(\frac{r}{r_0}\right), \end{aligned} \tag{2.10}$$

where we have used the assumption $z_1 = 0$.

On the other hand, from

$$ru' + \frac{N-p}{p-1}u(r) \geq 0, \quad \text{for } r > 0, \tag{2.11}$$

we find that

$$\left(\frac{N-p}{p-1}\right)^{p-1}u^{p-1}(r) \geq |u'(r)|^{p-1}r^{p-1}. \tag{2.12}$$

Together with (2.10), this implies that

$$r^{(N-p)/(p-1)}u(r) \geq C\left(\ln\left(\frac{r}{r_0}\right)\right)^{1/(p-1)}. \tag{2.13}$$

This is impossible, however, since from Lemma 2.3, estimate implies that

$$r^{(N-p)/(p-1)}u(r) \leq Cr^{z_1} = C. \tag{2.14}$$

This contradiction concludes the proof of the theorem. □

3. Blow-up estimate of system (1.1)

Motivated by Weissler [12], Caristi and Mitidieri [13], and Yang and Lu [15], we use the nonexistence result of the elliptic system (1.2) obtained in Section 2 to establish the blow-up estimates for the quasilinear reaction-diffusion system (1.1). In this section, we impose the condition $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$ $(R > 0)$ to system (1.1).

Theorem 3.1. *Let (u,v) be a solution of (1.1). Assume that*

(i) *$u(\cdot,t)$, $v(\cdot,t)$ are nonnegative, radially symmetric, and radially decreasing functions of $r=|x|$;*
(ii) *$u_t(x,t)$, $v_t(x,t)$ attain the maxima at $x=0$ for every $t\in(0,T)$;*
(iii) *$u_t(x,t)\geq 0$, $v_t(x,t)\geq 0$ for $(x,t)\in Q_T=B_R\times(0,T)$;*
(iv) *u,v have a blow-up time $T<+\infty$;*
(v) *integer $N>\max\{p,q\}$, $\alpha\beta>(p-1)(q-1)$ with $p,q\geq 2$ with $z_1\geq 0$ or $z_2\geq 0$;*
(vi) *there are positive constants k_1 and k_2 and $\eta<T$ such that*

$$k_2(u(0,t))^{\delta_2/\delta_1}\leq v(0,t)\leq k_1(u(0,t))^{\delta_2/\delta_1}\quad \text{for } t\in(\eta,T). \tag{3.1}$$

Then there are positive constants c_1, c_2 and $t_1\in(0,T)$ such that

$$u(x,t)\leq u(0,t)\leq c_1(T-t)^{-\delta_1},\qquad v(x,t)\leq v(0,t)\leq c_2(T-t)^{-\delta_2} \tag{3.2}$$

for $(x,t)\in Q_T\times Q_{t_1}$, where

$$\delta_1=\frac{\alpha q+(q-1)p}{\alpha(p\beta+q(p-2))-p(q-1)},\qquad \delta_2=\frac{\beta p+(p-1)q}{\beta(q\alpha+p(q-2))-q(p-1)}. \tag{3.3}$$

Proof. Define $m(t)=u(0,t)^{1/\tau_1}$, $n(t)=v(0,t)^{1/\tau_2}$ for $t\in(0,T)$, where

$$\tau_1=\frac{\alpha q+(q-1)p}{\alpha\beta-(p-1)(q-1)},\qquad \tau_2=\frac{\beta p+(p-1)q}{\alpha\beta-(p-1)(q-1)}. \tag{3.4}$$

By putting $\gamma(t)=m(t)+n(t)$, $\omega_1(t)=(u(r/\gamma(t),t))/\gamma(t)^{\tau_1}, \omega_2(t)=(v(r/\gamma(t),t))/\gamma(t)^{\tau_2}$, $r=|x|$, using the symmetry and Assumptions (ii)–(iii) in Theorem 3.1, it follows that

$$0\leq(\Phi_p(\omega_1'))'+\frac{N-1}{r}\Phi_p(\omega_1')+\int_0^r\omega_2^{\alpha}\leq\frac{u_t(0,t)}{\gamma(t)^{p+(p-1)\tau_1}}+\frac{v_t(0,t)}{\gamma(t)^{q+(q-1)\tau_2}}, \tag{3.5}$$

$$0\leq(\Phi_q(\omega_2'))'+\frac{N-1}{r}\Phi_q(\omega_2')+\int_0^r\omega_1^{\beta}\leq\frac{u_t(0,t)}{\gamma(t)^{p+(p-1)\tau_1}}+\frac{v_t(0,t)}{\gamma(t)^{q+(q-1)\tau_2}} \tag{3.6}$$

for any $t\in(0,T)$ and $r\in[0,R\gamma(t))$.

Since $u(x,t)$, $v(x,t)$ achieve their maxima at $x=0$, we easily see that ω_1 and ω_2 are bounded. Indeed,

$$0\leq\omega_1(r,t)\leq\frac{u(0,t)}{\gamma(t)^{\tau_1}}\leq 1,\qquad 0\leq\omega_2(r,t)\leq\frac{v(0,t)}{\gamma(t)^{\tau_2}}\leq 1. \tag{3.7}$$

Multiplying (3.5) by $w_{1,r}$ (where $w_{1,r}$ express partial derivation of ω_1 for r), and then integrating with respect to r on $(0,r)$, we have

$$\frac{(p-1)}{p}|\omega_{1,r}|^p+\omega_1\int_0^r \omega_2^\alpha(s)ds-\int_0^r \omega_{1,r}\omega_2^\alpha ds\le 0. \tag{3.8}$$

From (3.8) and $\omega_{1,r}\le 0$, it follows that

$$|\omega_1|\le\left(\frac{K_1 p}{p-1}\right)^{1/p} \tag{3.9}$$

for $t\in(0,T)$ and $r\in[0,R\gamma(t))$. Similarly, we get

$$|\omega_2|\le\left(\frac{K_2 q}{q-1}\right)^{1/q} \tag{3.10}$$

for $t\in(0,T)$ and $r\in[0,R\gamma(t))$, where K_1, K_2 are positive constants.

Now we proceed by contradiction to claim that

$$\liminf_{t\to T}\frac{u_t(0,t)}{\gamma(t)^{p+(p-1)\tau_1}}+\frac{v_t(0,t)}{\gamma(t)^{q+(q-1)\tau_2}}=C>0. \tag{3.11}$$

Otherwise, suppose that there exists a sequences $\{t_n\}\subseteq(0,T)$ with $t_n\to T$ such that

$$\liminf_{t_n\to T}\frac{u_t(0,t_n)}{\gamma(t)^{p+(p-1)\tau_1}}+\frac{v_t(0,t_n)}{\gamma(t)^{q+(q-1)\tau_2}}=0. \tag{3.12}$$

By using Ascoli-Arzelá theorem, there exists a sequence (still denoted by $\{t_n\}$) such that

$$\omega_1(\cdot,t_n)\longrightarrow\overline{\omega}_1(\cdot),\quad \omega_2(\cdot,t_n)\longrightarrow\overline{\omega}_2(\cdot),\quad \text{as } n\longrightarrow+\infty, \tag{3.13}$$

hold uniformly on a compact subset of $[0,+\infty)$. Now in the sense of distributions,

$$\begin{aligned}(\Phi_p(\overline{\omega}_1'))'+\frac{N-1}{r}\Phi_p(\overline{\omega}_1')+\int_0^r\overline{\omega}_2^\alpha=0,\\ (\Phi_q(\overline{\omega}_2'))'+\frac{N-1}{r}\Phi_q(\overline{\omega}_2')+\int_0^r\overline{\omega}_1^\beta=0.\end{aligned} \tag{3.14}$$

The absolute continuity of ω_1, ω_2 implies that $\overline{\omega}_1$, $\overline{\omega}_2$ are $C^1(0,+\infty)$. By the local existence and uniqueness of initial value problem for (3.14) and using the argument in [4, 5], we conclude that $\overline{\omega}_1,\overline{\omega}_2>0$ on $(0,+\infty)$ with $\overline{\omega}_1'(0)=\overline{\omega}_2'(0)=0$.

If $N=2$, $p>2$, we proceed as follow. From (3.14), we infer that $r\Phi_p(\overline{\omega}_1')$, $r\Phi_q(\overline{\omega}_2')$ are decreasing and that there exist $M>0$ and $r_0>0$ such that

$$r\Phi_p(\overline{\omega}_1')\le M\quad \text{for } r\in(r_0,+\infty). \tag{3.15}$$

The last inequality implies that

$$\begin{aligned}\overline{\omega}_1(s) &\geq \overline{\omega}_1(s) - \overline{\omega}_1(t) = (-M)^{1/(p-1)} \int_s^t r^{-1/(p-1)} dr \\ &= (-M)^{1/(p-1)} \left(t^{(p-2)/(p-1)} - s^{(p-2)/(p-1)}\right)\end{aligned} \tag{3.16}$$

for $r_0 \leq s \leq t$. Letting $t \to +\infty$ in (3.16), we obtain a contraction.

If $N = 2$, $p = 2$, proceeding similarly as above implies that

$$\overline{\omega}_1(s) > \overline{\omega}_1(s) - \overline{\omega}_1(t) > (-M)[\ln(t) - \ln(s)] \tag{3.17}$$

for $r_0 \leq s \leq t$. Letting $t \to +\infty$ in the inequality, we obtain a contraction.

Finally, if $N > \max\{p,q\} \geq 2$ holds, we know from Theorem 2.1 that system (3.14) has no positive solutions. We conclude that (3.11) is true. It follows from (3.11) that there exists $t_1 \in (0,T)$ such that for any $t \in (t_1,T)$, we have

$$0 \leq \frac{u_t(0,t)}{\gamma(t)^{p+(p-1)\tau_1}} + \frac{v_t(0,t)}{\gamma(t)^{q+(q-1)\tau_2}} \leq \frac{u_t(0,t)}{u(0,t)^{(1+\delta_1)/\delta_1}} + \frac{v_t(0,t)}{v(0,t)^{(1+\delta_2)/\delta_2}}. \tag{3.18}$$

Integrating (3.18) on $(t,s) \subseteq (t_1,T)$ and then letting $s \to T$, we obtain

$$c(T-t) \leq \delta_1 u(0,t)^{-1/\delta_1} + \delta_2 v(0,t)^{-1/\delta_2}. \tag{3.19}$$

By using condition (vi) in (3.19), we have

$$u(x,t) \leq u(0,t) \leq c_1(T-t)^{-\delta_1} \quad \text{for any } (x,t) \in Q_T \backslash Q_{t_1}. \tag{3.20}$$

In the same way, we have the blow-up estimate for v. The proof is complete. □

Remark 3.2. From the condition in Theorem 3.1, we fell that the condition (vii) is rather strong. We guess that the condition (vii) may be removed and a better result can be obtained:

$$u(0,t) = O\left((T-t)^{-\delta_1}\right), \quad v(0,t) = O\left((T-t)^{-\delta_2}\right), \quad \text{as } t \longrightarrow T. \tag{3.21}$$

Further discussion on this problem will be made.

4. Local existence and uniqueness

In this section, we study the global existence of (1.1) under appropriate hypotheses. From the point of physics, we need only to consider the nonnegative solutions. Moreover, if we assume $u_0(x), v_0(x) \geq 0$, by Lemma 4.5 (proved later), we can show that $(u(x,t),v(x,t)) \geq 0$ a.e. in $\Omega \times (0,T)$. Since (1.1) are the degenerate parabolic equations for $|\nabla u| = 0$, $|\nabla v| = 0$, one cannot expect the existence of classical solution of (1.1). As it is now well known that degenerate equations need not posses classical solutions, most of studies of p-Laplacian equations concerned with weak solutions (see [7, 9]). We begin by giving a precise denition of a weak solution for problem (1.1). Let $Q_T = \Omega \times (0,T)$, $T > 0$,

$$\Psi \equiv \{\psi(x,t) \in C^{1,1}(Q_T);\ \psi(x,T) = 0,\ \psi(x,t)|_{\partial\Omega} = 0\}. \tag{4.1}$$

Definition 4.1. A pair of function $(u(x,t),v(x,t))$ is called a sub-(or super-) solutions of (1.1) on Q_T if and only if $(u,v)\in C(0,T;L^\infty(\Omega))\cap L^p(0,T;W_0^{1,p}(\Omega))$, $(u_t,v_t)\in L^2(0,T;L^2(\Omega))$, $(u(x;t);v(x;t))\geq(\leq)0$, $(u(x,t),v(x,t))|_{t=0}\geq(\leq)(u_0(x),v_0(x))$, and

$$\begin{aligned}
&\int_\Omega u(x,t_2)\psi_1(x,t_2)\,dx-\int_\Omega u(x,t_1)\psi_1(x,t_1)\,dx\\
&\quad\geq(\leq)\int_{t_1}^{t_2}\int_\Omega u\psi_{1\,t}\,dx\,dt-\int_{t_1}^{t_2}\int_\Omega|\nabla u|^{p-2}\nabla u\nabla\psi_1\,dx\,dt+\int_{t_1}^{t_2}\int_\Omega\psi_1(x,t)\int_\Omega v^\alpha(x,t)\,dx\,dt,\\
&\int_\Omega v(x,t_2)\psi_2(x,t_2)\,dx-\int_\Omega v(x,t_1)\psi_2(x,t_1)\,dx\\
&\quad\geq(\leq)\int_{t_1}^{t_2}\int_\Omega v\psi_{2\,t}\,dx\,dt-\int_{t_1}^{t_2}\int_\Omega|\nabla v|^{q-2}\nabla v\nabla\psi_2\,dx\,dt+\int_{t_1}^{t_2}\int_\Omega\psi_2(x,t)\int_\Omega u^\beta(x,t)\,dx\,dt
\end{aligned}\tag{4.2}$$

hold for all $0<t_1<t_2<T$, where $\psi_i(x,t)\in\Psi$ $(i=1,2)$. A weak solution of (1.1) is a vector function which is both a subsolution and a supersolution of (1.1). For every $T<\infty$, if (u,v) is a solution of (1.1), we say (u,v) is global.

Remark 4.2. Clearly, every nonnegative classical (sub-, super-) solution of (1.1) is a weak (sub-, super-) solution of (1.1) in the sense of Definition 4.1.

By a modification of the method given in [7], we obtain the following results.

Theorem 4.3 (local existence). *There exists a T_0 such that (1.1) admit a solution $(u,v)\in C(0,T_0;L^\infty(\Omega))\cap L^p(0,T_0;W_0^{1,p}(\Omega))$.*

Theorem 4.4 (uniqueness). *The solution (u,v) of (1.1) is uniqueness determined by the initial data $(u_0,v_0)\in L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$.*

In order to prove Theorem 4.3-Theorem 4.4, as in [7], we establish a comparison lemma, which will be used in later proofs and may show an independent interest.

Lemma 4.5. *Suppose $(\overline{u}(x,t),\overline{v}(x,t))$ and $(\underline{u}(x,t),\underline{v}(x,t))$ are super and lower solutions of (1.1), respectively, then $(\underline{u}(x,t),\underline{v}(x,t))\leq(\overline{u}(x,t),\overline{v}(x,t))$ a.e. in Q_T.*

Proof of this lemma is similar as in [7] only need a little modification, we omit it here.

Proof of Theorem 4.3. Consider the following approximate problems for (1.1):

$$\begin{gathered}
u_{nt}-\operatorname{div}\left(\left(|\nabla u_n|^2+\varepsilon_{1n}\right)^{(p-2)/2}\nabla u_n\right)=\int_\Omega v_n^\alpha(x,t)\,dx,\quad(x,t)\in\Omega\times(0,T),\\
v_{nt}-\operatorname{div}\left(\left(|\nabla v_n|^2+\varepsilon_{2n}\right)^{(q-2)/2}\nabla v_n\right)=\int_\Omega u_n^\beta(x,t)\,dx,\quad(x,t)\in\Omega\times(0,T),\\
u_n(x,t)=v_n(x,t)=0,\quad(x,t)\in\partial\Omega\times(0,T],\\
u_n(x,0)=u_0^{\varepsilon_{1n}}(x),\quad v_n(x,0)=v_0^{\varepsilon_{2n}}(x),\quad x\in\Omega.
\end{gathered}\tag{4.3}$$

Here ε_{1n}, ε_{2n} are strictly decreasing sequence, $0<\varepsilon_{1n}$, $\varepsilon_{2n}<1$, and $\varepsilon_{1n},\varepsilon_{2n}\to 0$, as $n\to\infty$. $(u_0^{\varepsilon_{1n}},v_0^{\varepsilon_{2n}})\in C_0^\infty(\Omega)$ are approximation functions for the initial data $(u_0(x),v_0(x))$ such that $|u_0^{\varepsilon_{1n}}|_{L^\infty(\Omega)}\le|u_0|_{L^\infty(\Omega)}$, $|v_0^{\varepsilon_{2n}}|_{L^\infty(\Omega)}\le|v_0|_{L^\infty(\Omega)}$, $|\nabla u_0^{\varepsilon_{1n}}|_{L^\infty(\Omega)}\le|\nabla u_0|_{L^\infty(\Omega)}$, $|\nabla v_0^{\varepsilon_{2n}}|_{L^\infty(\Omega)}$ $\le|\nabla v_0|_{L^\infty(\Omega)}$ for all ε_{in} $(i=1,2)$, and $(u_0^{\varepsilon_{1n}},v_0^{\varepsilon_{2n}})\to(u_0,v_0)$ strongly in $W_0^{1,p}(\Omega)$.

Equations (4.3) are a nondegenerate problem for each fixed ε_{in} $(i=1,2)$. It is easy to prove that it admits a unique classic solution (u_n,v_n) by using Schauder's fixed-point theorem.

To find the limit function $(u(x,t),v(x,t))$ of the sequence $(u_n(x,t),v_n(x,t))$, we divide our proof into four steps.

Step 1. There exist a small $T_0>0$ and a constant $M>0$, independent of n, such that

$$|u_n|_{L^\infty(Q_{T_0})}\le M,\qquad |v_n|_{L^\infty(Q_{T_0})}\le M. \tag{4.4}$$

To this end, we consider the ordinary differential equation:

$$\begin{gathered} K'(t)=|\Omega|(K(t)+1)^{\hat{p}},\\ K(0)=\max\left\{\max_{x\in\overline{\Omega}}u_0(x),\max_{x\in\overline{\Omega}}v_0(x)\right\}, \end{gathered} \tag{4.5}$$

where $\hat{p}=\max\{\alpha,\beta\}$. It is obvious that there exists $T_0>0$, such that (4.5) has a bounded solution $K(t)>0$ on $[0,T_0]$. By Lemma 4.5, we get $u(x,t)\le K(t)\le M$, $v(x,t)\le K(t)\le M$, where $M=\max\{K(t)\mid t\in[0,T_0]\}$. We draw the conclusion.

Step 2. There exist constants $M_1,M_2>0$, independent of n, such that

$$|\nabla u_n|_{L^p(Q_{T_0})}\le M_1,\qquad |\nabla v_n|_{L^q(Q_{T_0})}\le M_2. \tag{4.6}$$

In fact, multiplying (4.3) by u_n, v_n and integrating over Q_{T_0}, we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega u_n^2(x,T_0)\,dx+\int_0^{T_0}\int_\Omega\left(|\nabla u_n|^2+\varepsilon_{1n}\right)^{(p-2)/2}|\nabla v_n|^2dx\,dt\\ &\quad=\frac{1}{2}\int_\Omega\left(u_0^{\varepsilon_{1n}}(x)\right)^2dx+\int_0^{T_0}\left(\int_\Omega u_n(x,t)dx\right)\left(\int_\Omega v_n^\alpha(x,t)dx\right)dt,\\ &\frac{1}{2}\int_\Omega v_n^2(x,T_0)\,dx+\int_0^{T_0}\int_\Omega\left(|\nabla v_n|^2+\varepsilon_{2n}\right)^{(q-2)/2}|\nabla v_n|^2dx\,dt\\ &\quad=\frac{1}{2}\int_\Omega\left(v_0^{\varepsilon_{2n}}(x)\right)^2dx+\int_0^{T_0}\left(\int_\Omega v_n(x,t)dx\right)\left(\int_\Omega u_n^\beta(x,t)dx\right)dt. \end{aligned} \tag{4.7}$$

By $|u_0^{\varepsilon_{1n}}|_{L^\infty(\Omega)}\le|u_0|_{L^\infty(\Omega)}$, $|v_0^{\varepsilon_{2n}}|_{L^\infty(\Omega)}\le|v_0|_{L^\infty(\Omega)}$ and (4.4), we get

$$\begin{aligned} &\int_0^{T_0}\int_\Omega|\nabla u_n|^p dx\,dt\le\frac{1}{2}|u_0|^2_{L^\infty(\Omega)}+T_0|\Omega|^2M^{\alpha+1},\\ &\int_0^{T_0}\int_\Omega|\nabla v_n|^q dx\,dt\le\frac{1}{2}|v_0|^2_{L^\infty(\Omega)}+T_0|\Omega|^2M^{\beta+1}. \end{aligned} \tag{4.8}$$

Step 3. There exist constants $M_3, M_4 > 0$, independent of n, such that

$$|u_{nt}|_{L^2(Q_{T_0})} \le M_3, \qquad |v_{nt}|_{L^2(Q_{T_0})} \le M_4. \tag{4.9}$$

To do so, multiplying (4.3) by u_{nt}, v_{nt} and integrating over Q_{T_0}, we have

$$\begin{aligned}\int_0^{T_0}\int_\Omega u_{nt}^2(x,t)\,dx\,dt &= -\int_0^{T_0}\int_\Omega (|\nabla u_n|^2+\varepsilon_{1n})^{(p-2)/2}\nabla u_n\nabla u_{nt}\,dx\,dt \\ &\quad + \int_0^{T_0}\left(\int_\Omega u_n(x,t)\,dx\right)\left(\int_\Omega v_n^\alpha(x,t)\,dx\right)dt, \\ \int_0^{T_0}\int_\Omega v_{nt}^2(x,t)\,dx\,dt &= -\int_0^{T_0}\int_\Omega (|\nabla v_n|^2+\varepsilon_{2n})^{(q-2)/2}\nabla v_n\nabla v_{nt}\,dx\,dt \\ &\quad + \int_0^{T_0}\left(\int_\Omega v_n(x,t)\,dx\right)\left(\int_\Omega u_n^\beta(x,t)\,dx\right)dt.\end{aligned} \tag{4.10}$$

By Hölder inequality, $|u_0^{\varepsilon_{1n}}|_{L^\infty(\Omega)} \le |u_0|_{L^\infty(\Omega)}$, $|v_0^{\varepsilon_{2n}}|_{L^\infty(\Omega)} \le |v_0|_{L^\infty(\Omega)}$, and (4.6), we yield

$$\begin{aligned}\int_0^{T_0}\int_\Omega u_{nt}^2(x,t)\,dx\,dt &\le -\frac{1}{2}\int_\Omega (|\nabla u_n|^2+\varepsilon_{1n})^{p/2}dx + \frac{1}{2}\int_\Omega (|\nabla u_0^{\varepsilon_{1n}}|^2+\varepsilon_{1n})^{p/2}dx \\ &\quad + |\Omega|^{(\alpha-1)/\alpha}\int_0^{T_0}\left(\int_\Omega v_n^\alpha dx\right)^{(\alpha+1)/\alpha}dt \le M_3', \\ \int_0^{T_0}\int_\Omega v_{nt}^2(x,t)\,dx\,dt &\le -\frac{1}{2}\int_\Omega (|\nabla v_n|^2+\varepsilon_{vn})^{q/2}dx + \frac{1}{2}\int_\Omega (|\nabla v_0^{\varepsilon_{2n}}|^2+\varepsilon_{2n})^{q/2}dx \\ &\quad + |\Omega|^{(\beta-1)/\beta}\int_0^{T_0}\left(\int_\Omega u_n^\beta dx\right)^{(\beta+1)/\beta}dt \le M_4'.\end{aligned} \tag{4.11}$$

Therefore, by virtue of (4.4)–(4.9) and the Ascoli-Arzelá theorem, we can choose subsequences, still denoted by $\{u_n\}$, $\{v_n\}$ for convenience, such that

$$u_n \longrightarrow u, \quad v_n \longrightarrow v, \quad \text{a.e. for } (x,t)\in\Omega\times(0,T_0), \tag{4.12}$$

$$\nabla u_n \longrightarrow \nabla u, \quad \nabla v_n \longrightarrow \nabla v, \quad \text{weakly in } L^p(0,T_0;L^p(\Omega)), \tag{4.13}$$

$$u_{nt} \longrightarrow u_t, \quad v_{nt} \longrightarrow v_t, \quad \text{weakly in } L^2(0,T_0;L^2(\Omega)), \tag{4.14}$$

$$\begin{aligned}|\nabla u_n|^{p-2}(u_n)_{x_i} &\longrightarrow \omega_{1i}, \\ |\nabla v_n|^{q-2}(v_n)_{x_i} &\longrightarrow \omega_{2i},\end{aligned} \quad \text{weakly in } L^{p/(p-1)}(0,T_0;L^{p/(p-1)}(\Omega)). \tag{4.15}$$

Step 4. We show that $\omega_{1i} = |\nabla u_n|^{p-2}u_{x_i}$, $\omega_{2i} = |\nabla v_n|^{q-2}v_{x_i}$. Multiplying (4.3) by $\psi(u_n - u)$, $\psi(v_n - u)$ and integrating over Q_{T_0}, we have

$$\begin{aligned}
&\int_0^{T_0}\int_\Omega \psi(u_n-u)u_{nt}\,dx\,dt + \int_0^{T_0}\int_\Omega \psi\left(|\nabla u_n|^2+\varepsilon_{1n}\right)^{(p-2)/2}\nabla u_n\nabla(u_n-u)\,dx\,dt \\
&\qquad + \int_0^{T_0}\int_\Omega (u_n-u)\left(|\nabla u_n|^2+\varepsilon_{1n}\right)^{(p-2)/2}\nabla u_n\nabla\psi\,dx\,dt \\
&\qquad = \int_0^{T_0}\int_\Omega \psi(u_n-u)\left(\int_\Omega v_n^\alpha(x,t)dx\right)dx\,dt, \\
&\int_0^{T_0}\int_\Omega \psi(v_n-v)v_{nt}\,dx\,dt + \int_0^{T_0}\int_\Omega \psi\left(|\nabla v_n|^2+\varepsilon_{2n}\right)^{(q-2)/2}\nabla v_n\nabla(v_n-v)\,dx\,dt \\
&\qquad + \int_0^{T_0}\int_\Omega (v_n-v)\left(|\nabla v_n|^2+\varepsilon_{2n}\right)^{(q-2)/2}\nabla v_n\nabla\psi\,dx\,dt \\
&\qquad = \int_0^{T_0}\int_\Omega \psi(v_n-v)\left(\int_\Omega u_n^\beta(x,t)dx\right)dx\,dt.
\end{aligned} \tag{4.16}$$

Using (4.4), (4.12), and (4.14), we can get

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^{T_0}\int_\Omega \psi|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)\,dx\,dt = 0, \\
&\lim_{n\to\infty}\int_0^{T_0}\int_\Omega \psi|\nabla v_n|^{q-2}\nabla v_n\nabla(v_n-v)\,dx\,dt = 0,
\end{aligned} \tag{4.17}$$

where $\psi \in C_0^{1,1}(Q_{T_0}), \psi \geq 0$. The left is the same as [8, Theorem 2.1]. Therefore, we complete our proof by a standard limiting process. □

Proof of Theorem 4.4. Assume that (u_1, v_1) and (u_2, v_2) are solutions of (1.1), using Lemma 4.5 repeatedly, we can get $(u_1, v_1) = (u_2, v_2)$ a.e. in $\Omega \times [0, T_0]$. □

5. Global existence and blow-up

In this section, we will discuss the global existence and blow-up in finite time of the solution for system (1.1). Our approach in a combination principle and super- and sub-techniques which are similar as in [7]. Firstly, we suppose $p, q > 2$.

Theorem 5.1 (global existence). *Assume that one of the following conditions hold:*

(1) $\alpha < p - 1$ *and* $\beta < q - 1$*;*
(2) $\alpha = p - 1$, $\beta = q - 1$, *and* $|\Omega|$ *is sufficiently small;*
(3) $\alpha > p - 1$, $\beta > q - 1$, *and* $u_0(x)$, $v_0(x)$ *are sufficiently small.*

Then the solution of system (1.1) exists globally.

THEOREM 5.2 (blow-up in finite time). *Assume that*

(i) $\alpha = p-1$, $\beta = q-1$, *and* $|\Omega|$ *is sufficiently large or*

(ii) $\alpha > p-1$, $\beta > q-1$, *and* $u_0(x)$, $v_0(x)$ *are sufficiently large.*

Then the solution of system (1.1) blows up in finite time.

Proof of Theorem 5.1. Let $\phi(x)$ be the solution of the elliptic problem

$$-\operatorname{div}(|\nabla\phi|^{p-2}\nabla\phi) = 1, \quad x \in \Omega, \quad \phi(x) = 0, \quad x \in \partial\Omega. \tag{5.1}$$

Then we have $\phi(x) \ge 0$ on $\overline{\Omega}$, $\partial\phi(x)/\partial\eta < 0$ on the boundary $\partial\Omega$, and there exists $M > 0$ such that $\max_{x\in\overline{\Omega}} \phi(x) = M$ (see [21, 22]).

Let $(\overline{u},\overline{v}) = (a\phi(x), a\psi(x))$, where $a > 0$ will be determined later.

(1) In the case $\alpha < p-1$ and $\beta < q-1$, we can choose $a > \max\{(|\Omega|M^\alpha)^{1/(p-\alpha-1)}, (|\Omega|M^\beta)^{1/(q-\beta-1)}, \sup_{x\in\Omega} u_0(x)/\phi(x), \sup_{x\in\Omega} v_0(x)/\psi(x)\}$, since $\partial\phi(x)/\partial\eta, \partial\psi(x)/\partial\eta < 0$ on $\partial\Omega$. Thus we have

$$\begin{aligned} \overline{u}_t - \operatorname{div}(|\nabla\overline{u}|^{p-2}\nabla\overline{u}) &= a^{p-1} \ge a^\alpha M^\alpha|\Omega| \ge a^\alpha \int_\Omega \psi^\alpha dx, \\ \overline{v}_t - \operatorname{div}(|\nabla\overline{v}|^{q-2}\nabla\overline{v}) &= a^{q-1} \ge a^\beta M^\beta|\Omega| \ge a^\beta \int_\Omega \phi^\beta dx. \end{aligned} \tag{5.2}$$

Noticing $\overline{u}(x,t) = 0$, $\overline{v}(x,t) = 0$ on $\partial\Omega \times (0,+\infty)$ and $\overline{u}(x,0) \ge u_0(x)$, $\overline{v}(x,0) \ge v_0(x)$ in Ω, we get $u(x,t) \le \overline{u}(x,t)$, $v(x,t) \le \overline{v}(x,t)$ in $\Omega \times (0,+\infty)$ by Lemma 4.5. Hence, $u(x,t)$, $v(x,t)$ exist globally.

(2) In this case, we can choose $a > \{\sup_{x\in\Omega} u_0(x)/\phi(x), \sup_{x\in\Omega} v_0(x)/\phi(x)\}$, then (5.2) can be proved that $|\Omega| \le \min\{1/M^\alpha, 1/M^\beta\}$. The left is the same as in (1).

(3) In this case, to insure inequality (5.2) holds, we need only that choose $a < \min\{(|\Omega|M^\alpha)^{1/(p-\alpha-1)}, (|\Omega|M^\beta)^{1/(q-\beta-1)}\}$, thus for the fixed a and sufficiently small $u_0(x)$, $v_0(x)$, we choose $a > \max\{\sup_{x\in\Omega} u_0(x)/\phi(x), \sup_{x\in\Omega} v_0(x)/\phi(x)\}$. The left is the same as in (1). □

Proof of Theorem 5.2. (i) Without lose of generality, we can suppose that $0 \in \Omega$. We get our conclusion by a small modification of the results of [8, Section 4].

(ii) To prove $u(x,t)$ and $v(x,t)$ blow-up in finite time, according to sub- and supersolution, we need only to find blowing up subsolutions. The proof is similar, as here we use an argument as done in [5, 7].

Let $\phi \in C^1(\overline{\Omega})$, $\phi(x) \ge 0$, $\phi(x) \not\equiv 0$, and $\phi(x)|_{\partial\Omega} = 0$. By translation, we may assume without loss of generality that $0 \in \Omega$ and $\phi(0) > 0$. Set

$$z_1(x,t) = \frac{1}{(T-t)^{\gamma_1}} V\left(\frac{|x|}{(T-t)^{\sigma_1}}\right), \qquad z_2(x,t) = \frac{1}{(T-t)^{\gamma_2}} V\left(\frac{|x|}{(T-t)^{\sigma_2}}\right) \tag{5.3}$$

with

$$V(y) = \left(1 + \frac{A}{2} - \frac{y^2}{2A}\right)_+, \quad y \ge 0, \tag{5.4}$$

where $\gamma_1, \gamma_2, \sigma_1, \sigma_2 > 0$, $A > 1$, and $0 < T < 1$ are to be determined later. Note that

$$\begin{aligned} \operatorname{supp} z_1(\cdot, t) &= \overline{B(0, R(T-t)^{\sigma_1})} \subset \overline{B(0, RT^{\sigma_1})} \subset \Omega, \\ \operatorname{supp} z_2(\cdot, t) &= \overline{B(0, R(T-t)^{\sigma_2})} \subset \overline{B(0, RT^{\sigma_2})} \subset \Omega \end{aligned} \tag{5.5}$$

for sufficiently small $T > 0$ with $R = (A(2+A))^{1/2}$.

Denote $y_1 = |x|/(T-t)^{\sigma_1}$, $y_2 = |x|/(T-t)^{\sigma_2}$, a series of computation shows

$$z_{i,t}(x,t) = \frac{\gamma_i(V(y_i) + \sigma_i y_i V'(y_i))}{(T-t)^{\gamma_i+1}}, \quad -\triangle z_i(x,t) = \frac{N/A}{(T-t)^{\gamma_i+2\sigma_i}}, \quad i = 1, 2. \tag{5.6}$$

As in [7], we have

$$|\operatorname{div}(|\nabla z_1|^{p-2}\nabla z_1)| \le \frac{N(p-1)(\operatorname{diam}(\Omega))^{p-2}}{A(T-t)^{(\gamma_1+2\sigma_1)(p-1)}} = Q_1. \tag{5.7}$$

In the same way, we have

$$|\operatorname{div}(|\nabla z_2|^{q-2}\nabla z_2)| \le \frac{N(q-1)(\operatorname{diam}(\Omega))^{q-2}}{A(T-t)^{(\gamma_2+2\sigma_2)(q-1)}} = Q_2. \tag{5.8}$$

If $0 \le y_i \le A$, we have $1 \le V(y_i) \le 1 + A/2$ and $V'(y_i) \le 0$, $i = 1, 2$, then

$$\begin{aligned} \int_\Omega z_2^\alpha(x,t)dx &= \frac{1}{(T-t)^{\gamma_2\alpha}} \int_{B(0,R(T-t)^{\sigma_2})} V^\alpha\left(\frac{|x|}{(T-t)^{\sigma_2}}\right) \ge \frac{\widetilde{M_1}}{(T-t)^{\gamma_2\alpha - N\sigma_2}}, \\ \int_\Omega z_1^\beta(x,t)dx &= \frac{1}{(T-t)^{\gamma_1\beta}} \int_{B(0,R(T-t)^{\sigma_1})} V^\beta\left(\frac{|x|}{(T-t)^{\sigma_1}}\right) \ge \frac{\widetilde{M_2}}{(T-t)^{\gamma_1\beta - N\sigma_1}}, \end{aligned} \tag{5.9}$$

where $\widetilde{M_1} = \int_{B(0,R)} V^\alpha(|\xi|)d\xi$, $\widetilde{M_2} = \int_{B(0,R)} V^\beta(|\xi|)d\xi$. Hence,

$$z_{1,t} - \operatorname{div}(|\nabla z_1|^{p-2}\nabla z_1) - \int_\Omega z_2^\alpha dx \le \frac{\gamma_1(1+A/2)}{(T-t)^{\gamma_1+1}} + Q_1 - \frac{\widetilde{M_1}}{(T-t)^{\gamma_2\alpha - N\sigma_2}}, \tag{5.10}$$

$$z_{2,t} - \operatorname{div}(|\nabla z_2|^{q-2}\nabla z_2) - \int_\Omega z_1^\beta dx \le \frac{\gamma_2(1+A/2)}{(T-t)^{\gamma_2+1}} + Q_2 - \frac{\widetilde{M_2}}{(T-t)^{\gamma_1\beta - N\sigma_1}}. \tag{5.11}$$

If $y_i > A$, we have $V(y_i) \leq 1$ and $V'(y_i) \leq -1$, $i = 1,2$, then

$$z_{1,t} - \operatorname{div}\left(|\nabla z_1|^{p-2}\nabla z_1\right) - \int_\Omega z_2^\alpha dx \leq \frac{\gamma_1 - \sigma_1 A}{(T-t)^{\gamma_1+1}} + Q_1, \tag{5.12}$$

$$z_{2,t} - \operatorname{div}\left(|\nabla z_2|^{q-2}\nabla z_2\right) - \int_\Omega z_1^\beta dx \leq \frac{\gamma_2 - \sigma_2 A}{(T-t)^{\gamma_2+1}} + Q_2. \tag{5.13}$$

Since $p,q > 2$ and $\alpha > p-1$, $\beta > q-1$, we can choose $\sigma_1,\sigma_2 > 0$, which is sufficiently small, $\theta > 0$, and

$$\frac{2\sigma_1(p-1)+N\sigma_1}{\alpha-p+1} < \frac{1-2\sigma_1(p-1)}{p-2}, \qquad \frac{2\sigma_2(p-1)+N\sigma_2}{\beta-q+1} < \frac{1-2\sigma_2(q-1)}{q-2}, \tag{5.14}$$

which satisfy

$$0 < \gamma_1 < \frac{1-2\sigma_1(p-1)}{p-2}, \qquad 0 < \gamma_2 < \frac{1-2\sigma_2(q-1)}{q-2}, \tag{5.15}$$

then we have

$$\gamma_2\alpha - N\sigma_2 > \gamma_1 + 1 > (\gamma_1 + 2\sigma_1)(p-1), \qquad \gamma_1\beta - N\sigma_1 > \gamma_2 + 1 > (\gamma_2 + 2\sigma_2)(q-1). \tag{5.16}$$

Select $A > \max\{1, \gamma_1/\sigma_1, \gamma_2/\sigma_2\}$, then for $T > 0$ sufficiently small, (5.10)–(5.13) imply that

$$z_{1,t} - \operatorname{div}\left(|\nabla z_1|^{p-2}\nabla z_1\right) - \int_\Omega z_2^\alpha dx \leq 0, \qquad z_{2,t} - \operatorname{div}\left(|\nabla z_2|^{q-2}\nabla z_2\right) - \int_\Omega z_1^\beta dx \leq 0, \tag{5.17}$$

in which $(x,t) \in \Omega \times (0,T)$.

Since $\phi(0) > 0$ and ϕ are continuous, there exist two positive numbers ρ and ε, such that $\phi(x) \geq \varepsilon$ for all $x \in B(0,\rho) \subset \Omega$. Taking T small enough such that $B(0,RT^{\sigma_i}) \subset B(0,\rho)$ $(i = 1,2)$, and hence $z_i \leq 0$ on $\Omega \times (0,T)$. From (5.5), it follows that $z_1(x,0) \leq M\phi(x)$, $z_2(x,0) \leq M\phi(x)$ for sufficiently large M. By Lemma 4.5, we have $(z_1,z_2) \leq (u,v)$ provided that $(u_0(x),v(0)) \geq (M\phi(x),M\phi(x))$ and (u,v) can exist no later than $t = T$. This shows that (u,v) blows up in finite time for large initial data. □

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