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Research Article

JB^*-Algebras of Topological Stable Rank 1

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In 1976, Kaplansky introduced the class JB^*-algebras which includes all C^*-algebras as a proper subclass. The notion of topological stable rank 1 for C^*-algebras was originally introduced by M. A. Rieffel and was extensively studied by various authors. In this paper, we extend this notion to general JB^*-algebras. We show that the complex spin factors are of tsr 1 providing an example of special JBW^*-algebras for which the enveloping von Neumann algebras may not be of tsr 1. In the sequel, we prove that every invertible element of a JB^*-algebra $\mathcal{J}$ is positive in certain isotope of $\mathcal{J}$; if the algebra is finite-dimensional, then it is of tsr 1 and every element of $\mathcal{J}$ is positive in some unitary isotope of $\mathcal{J}$. Further, it is established that extreme points of the unit ball sufficiently close to invertible elements in a JB^*-algebra must be unitaries and that in any JB^*-algebras of tsr 1, all extreme points of the unit ball are unitaries. In the end, we prove the coincidence between the λ-function and λ_u-function on invertibles in a JB^*-algebra.

1. Introduction

Extending the classical concept of covering dimension (cf. [1]) for compact spaces, Rieffel [2] introduced the concept of *topological stable rank* for C^*-algebras in 1982, which later was identified with the Bass stable rank of a ring (see [3]). The notion of topological stable rank is also related with the notion of real rank of a C^*-algebra, the later notion was introduced jointly by Brown and Pedersen [4]. The C^*-algebras of topological stable rank 1 (tsr 1) have been extensively studied in [2, 3, 5–8], and so forth.

In 1976, Kaplansky introduced the class of JB^*-algebras (originally called Jordan C^*-algebras [9]) which includes all C^*-algebras as a proper subclass. In this paper, we extend the notion of topological stable rank 1 from C^*-algebras to general JB^*-algebras. After setting preliminaries in Section 2, we will show in Section 3 that the complex spin factors are of tsr 1 and that these provide an example of special JBW^*-algebras for which the enveloping von Neumann algebras may not be of tsr 1. Section 4 contains some results about isotopes of JB^*-algebras including that every invertible element of a JB^*-algebra $\mathcal{J}$ is positive in certain isotope of $\mathcal{J}$. Besides various other related results, we prove in Section 5 that any finite-dimensional JB^*-algebra is of tsr 1 and that every element of such a JB^*-algebra $\mathcal{J}$ is positive in some unitary isotope of $\mathcal{J}$; [10, Examples 2.9 and 3.1] show this is not true for infinite-dimensional JB^*-algebras (even for infinite-dimensional C^*-algebras) of tsr 1.

In Section 6, we characterize the extreme points of the unit ball which are unitaries in terms of both distance to invertibles and the spectrum of the extreme point. This would generalize [6, Proposition 3.4]; it may be noted here that the proof we will give is entirely different from the one given for C^*-algebras by Pedersen [6]. By using this result, we will extend one of our previous results appeared in [10] and would also deduce the coincidence $\mathcal{E}(\mathcal{J})_1 = \mathcal{U}(\mathcal{J})$ for any JB^*-algebra $\mathcal{J}$ of tsr 1, where $\mathcal{E}(\mathcal{J})_1$ and $\mathcal{U}(\mathcal{J})$ stand, respectively, for the set of all extreme points of the closed unit ball and the set of all unitaries in the JB^*-algebra $\mathcal{J}$.

Aron and Lohman [11] introduced the λ-function defined on the closed unit ball of a normed linear spaces. Later on, another related function called λ_u-function for C^*-algebras was studied by Pedersen [6]. In the last section, we will prove that the λ-function coincides with the λ_u-function on invertibles in a JB^*-algebra.

2. Preliminaries

A Jordan algebra (cf. [12]) $\mathcal{J}$ with product $\circ$ is called a Banach Jordan algebra if there is a norm $\|\cdot\|$ on $\mathcal{J}$ such that $(\mathcal{J},\|\cdot\|)$ is a Banach space and $\|a\circ b\| \le \|a\|\|b\|$. If, in addition, $\mathcal{J}$ has a unit e with $\|e\| = 1$, then $\mathcal{J}$ is called a unital Banach Jordan algebra. The Jordan product $\circ$ induces another product $\{\cdot,\cdot,\cdot\}$, called the *Jordan triple product*, defined by $\{pqr\} = (p\circ q)\circ r - (p\circ r)\circ q + (q\circ r)\circ p$.

An element x of a Jordan algebra $\mathcal{J}$ with unit e is said to be invertible if there exists an element $x^{-1}\in\mathcal{J}$, called the inverse of x, such that $x\circ x^{-1} = e$ and $x^2\circ x^{-1} = x$. The set of all invertible elements of $\mathcal{J}$ will be denoted by $\mathcal{J}_{\text{inv}}$.

Let $\mathcal{J}$ be a complex Jordan algebra with unit e and let $x\in\mathcal{J}$. As usual, the spectrum of x in $\mathcal{J}$, denoted by $\sigma_{\mathcal{J}}(x)$, is defined by

$$\sigma_{\mathcal{J}}(x) = \{\lambda\in\mathcal{C} : x-\lambda e \text{ is not invertible in } \mathcal{J}\}. \tag{2.1}$$

In the sequel, $\mathcal{J}(x_1,\ldots,x_r)$ denotes the norm closure of the Jordan subalgebra $J(x_1,\ldots,x_r)$ generated by $x_1,\ldots,x_r$ in the Banach Jordan algebra $\mathcal{J}$. The following lemma collects some elementary properties of Banach Jordan algebras which can be easily proved on the pattern of [13] or have appeared in [14].

Lemma 2.1. *Let $\mathcal{J}$ be a Banach Jordan algebra with unit e and $x_1,\ldots,x_r\in\mathcal{J}$.*

(i) *If $J(x_1,\ldots,x_r)$ is an associative subalgebra of $\mathcal{J}$, then $\mathcal{J}(x_1,\ldots,x_r)$ is a commutative Banach algebra.*

(ii) *If $\mathcal{J}$ is unital, then $\mathcal{J}(e,x_1)$ is a commutative Banach algebra.*

(iii) *If* $x \in \mathcal{J}$ *and* $\|x\| < 1$, *then* $(e-x)^{-1} = \sum_{n=0}^{\infty} x^n \in \mathcal{J}(e,x)$.

(iv) *If* K *is a closed Jordan subalgebra of* $\mathcal{J}$ *containing* e, *then* $\sigma_{\mathcal{J}}(x) \subseteq \sigma_K(x)$, *for all* $x \in K$.

(v) *If* K *is a closed Jordan subalgebra of* $\mathcal{J}$ *containing* e *and* $x \in K$ *such that* $\mathscr{C} \setminus \sigma_{\mathcal{J}}(x)$ *is connected, then* $\sigma_{\mathcal{J}}(x) = \sigma_K(x)$.

We will restrict our discussion to a special class of Banach Jordan algebras, called JB^*-algebras, these include all C^*-algebras as a proper subclass (see [9, 15]): a complex Banach Jordan algebra $\mathcal{J}$ with involution $*$ (cf. [13]) is called a JB^*-algebra if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$. It can easily be shown (see [15]) that $\|x^*\| = \|x\|$ for all x in any JB^*-algebra $\mathcal{J}$. An element x of a JB^*-algebra $\mathcal{J}$ is called *selfadjoint* if $x^* = x$.

A closely related class of Banach Jordan algebras called JB-algebras was introduced by Alfsen et al. [16]: a real Banach Jordan algebra $\mathcal{J}$ is called a JB-algebra if $\|x\|^2 = \|x^2\| \le \|x^2 + y^2\|$ for all $x, y \in \mathcal{J}$.

These two classes of algebras are linked as follows.

Theorem 2.2 [9]. (a) *If* $\mathcal{A}$ *is a* JB^**-algebra, then the set of selfadjoint elements of* $\mathcal{A}$ *is a* JB*-algebra.*

(b) *If* $\mathcal{B}$ *is a* JB*-algebra, then under a suitable norm the standard complexification* $\mathcal{C}_{\mathcal{B}}$ *of* $\mathcal{B}$ *is a* JB^**-algebra.*

There are easier subclasses of these algebras (see [16]).

Let $\mathcal{H}$ be any complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the full algebra of bounded linear operators on $\mathcal{H}$.

(a) Any closed selfadjoint complex Jordan subalgebra of $\mathcal{B}(\mathcal{H})$ is called a JC^*-algebra.

(b) Any closed real Jordan subalgebras of selfadjoint operators of $\mathcal{B}(\mathcal{H})$ is called a JC-algebra.

Any JB^*-algebra isometrically $*$-isomorphic to a JC^*-algebra is also called a JC^*-algebra; similarly, any JB-algebra isometrically isomorphic to a JC-algebra is also called a JC-algebra.

Any JC^*-algebra is a JB^*-algebra and a JC-algebra is a JB-algebra but the converses generally are not true (cf. [16]).

Recall (from [12], e.g.) that a Jordan algebra is said to be *special* if it is isomorphic to a Jordan subalgebra of some associative algebra; otherwise, it is called *exceptional.*

Remark 2.3. If a is invertible with inverse b in a Jordan algebra $\mathcal{J}$ with unit e, then by the Shirshov-Cohn theorem [12, page 48], the Jordan subalgebra $J(e,a,b)$ of $\mathcal{J}$ generated by e, a, and b is special and $ab = ba = e$ where ab denotes the underlying associative product of a and b, (see [12, page 51]); so that $J(e,a,b)$ is an associative commutative algebra with unit. On the other hand, if a Jordan subalgebra of an associative algebra has unit e with c, d satisfying $cd = dc = e$, then $c \circ d = e$ and $c^2 \circ d = c$.

It is well known that a JC^*-algebra is a JB^*-algebra and a JC-algebra is a JB-algebra. The converses are not true. Let $\mathcal{K}$ stand for the *Cayley algebra* over the field $\mathfrak{R}$ of all real numbers. It is shown in [16] that the real Jordan algebra M_3^8 of all 3×3 matrices of the form

$$\begin{bmatrix} b_1 & a_3 & \overline{a_2} \\ \overline{a_3} & b_2 & a_1 \\ a_2 & \overline{a_1} & b_3 \end{bmatrix}, \tag{2.2}$$

where $a_1, a_2, a_3 \in \mathcal{K}$, $\overline{a_1}, \overline{a_2}, \overline{a_3}$ are their conjugates and $b_1, b_2, b_3 \in \mathfrak{R}$ with product $x \circ y = (1/2)(xy + yx)m$, xy denotes the "usual" matrix multiplication, is a JB-algebra which is not special.

The following result says that complexification $\mathcal{M}_3^8$ of the JC-algebra M_3^8 is the only exceptional factor representation of JB^*-algebras.

Theorem 2.4 (Gelfand-Neumark theorem [9, 16]). (a) *Let $\mathcal{A}$ be a unital JB-algebra. There is a family G of Jordan homomorphisms $g : \mathcal{A} \to g(\mathcal{A})$ of norm at most one such that*

(i) *for all g in G, either $g(\mathcal{A})$ is a JC-algebra or $g(\mathcal{A}) = M_3^8$;*

(ii) *for all a in $\mathcal{A} \setminus \{0\}$, there exists $g \in G$ such that $g(a) \neq 0$.*

(b) *Let $\mathcal{B}$ be a JB^*-algebra. There is a family F of Jordan $*$-homorphisms $f : \mathcal{B} \to f(\mathcal{B})$ of norm at most one such that*

(i) *for all f in F, either $f(\mathcal{B})$ is a JC^*-algebra or $f(\mathcal{B}) = \mathcal{M}_3^8$;*

(ii) *for all b in $\mathcal{B} \setminus \{0\}$, there exists $f \in F$ such that $f(b) \neq 0$.*

There are identities, called *s-identities*, which are known to hold for all special Jordan algebras but not for all Jordan algebras. For the next result, we need the following s-identity, which is due to Glennie [17]:

$$\begin{aligned} &4\{\{z\{xyx\}z\}y(x \circ z)\} - 2\{z\{x\{y(x \circ z)y\}x\}z\} \\ &\quad = 4\{(x \circ z)y\{x\{zyz\}x\}\} - 2\{x\{z\{y(x \circ z)y\}z\}x\}. \end{aligned} \tag{2.3}$$

Corollary 2.5. *Let $\mathcal{J}$ be a JB^*-algebra with unit e and let a, b be selfadjoint elements of $\mathcal{J}$.*

(a) *If S is a special selfadjoint dense subalgebra of $\mathcal{J}$, then $\mathcal{J}$ is (isometrically and $*$-isomorphically equal to) a JC^*-algebra.*

(b) *$\mathcal{J}(e, a, b)$ is (isometrically and $*$-isomorphically equal to) a JC^*-algebra.*

(c) *If a and b are invertible, then $\mathcal{J}(e, a, b, a^{-1}, b^{-1})$ is (isometrically and $*$-isomorphically equal to) a JC^*-algebra.*

Proof. (a) Suppose that the claim is not true. Then, by part (b) of the previous theorem, there is a Jordan $*$-homomorphism f of $\mathcal{J}$ onto $\mathcal{M}_3^8$. If G is the Glennie's s-identity (*see above*) for M_3^8 and x, y, z are elements of M_3^8 such that $G(x, y, z) \neq 0$, then there exist $p, q, r \in \mathcal{J}$ such that p, q, r are selfadjoint and $f(p) = x$, $f(q) = y$, $f(r) = z$. As S is dense in $\mathcal{J}$ and selfadjoint, there are sequences $\{p_n\}$, $\{q_n\}$, and $\{r_n\}$ of selfadjoint elements of S such that $p_n \to p$, $q_n \to q$ and $r_n \to r$, as $n \to \infty$. As S is special, $0 = G(p_n, q_n, r_n)$ for all $n \in \mathcal{N}$. So by the continuity of the Jordan products, $0 = f(0) = f(\lim_{n \to \infty} G(p_n, q_n, r_n)) = f(g(p, q, r)) = G(f(p), f(q), f(r)) = G(x, y, z) \neq 0$, which is a contradiction.

(b) $J(e,a,b)$ is special by the Shirshov-Cohn theorem, so the result follows by (a).

(c) $J(e,a,b,a^{-1},b^{-1})$ is special by the Shirshov-Cohn theorem with inverses (cf. [18]), so the result follows again by (a). $\square$

Remark 2.6. It follows that if a is a selfadjoint element of a JB^*-algebra $\mathcal{J}$, then, as $J(e,a)$ is a selfadjoint subalgebra, $\mathcal{J}(e,a)$ is a commutative C^*-algebra, so we can apply continuous functional calculus to selfadjoint elements of a JB^*-algebra. Moreover, as $\sigma_{\mathcal{J}(e,a)}(a)$ is simply connected (being a subset of $\mathfrak{R}$ for any selfadjoint element a), $\sigma_{\mathcal{J}(e,a)}(a) = \sigma_{\mathcal{J}}(a)$ by Lemma 2.1.

Analagous to the von Neumann algebras, JBW^*-algebras constitute an important subclass of JB^*-algebras: *A JB^*-algebra is called a JBW^*-algebra if it is a Banach dual space.*

An example of a JBW^*-algebra (which is in fact special) can be constructed as follows.

Example 2.7. Let $\mathcal{H}$ be a real Hilbert space of dimension ≥ 2 with inner product $\langle \cdot, \cdot \rangle$. Let $\mathfrak{R}$ stand for the field of reals. The direct sum $\mathcal{X} := \mathcal{H} \oplus \mathfrak{R}$ is a JB-algebra with product

$$(h,\lambda) \circ (h',\lambda') = (\lambda h' + \lambda' h, \langle h,h' \rangle + \lambda\lambda') \tag{2.4}$$

and norm

$$||(h,\lambda)|| = \|h\| + |\lambda| \tag{2.5}$$

for all $h,h' \in \mathcal{H}$ and $\lambda,\lambda' \in \mathfrak{R}$. This JB-algebra is called a real spin factor and its complexification is a JBW^*-algebra (see [19]). It is interesting to note that the real spin factors are JC-algebras (see [20]) and hence any complex spin factor is JC^*-algebra (see [9, 19], for details). We will denote the element (h,λ) of $\mathcal{X}$ by $h+\lambda I$ and will denote a typical element of the complexification by $x+iy$, where $x = h+\lambda I$ and $y = h' + \lambda' I$ with $h,h' \in \mathcal{H}$ and $\lambda,\lambda' \in \mathfrak{R}$.

3. Topological stable rank 1

In this section, we extend the notion of *topological stable rank 1* from C^*-algebras to general JB^*-algebras of tsr 1. We show that the complex spin factors are of tsr 1 and that these provide an example of special JBW^*-algebras for which the enveloping von Neumann algebras may not be of tsr 1.

Recall (from [2, Proposition 3.1]) that *a C^*-algebra C is of topological stable rank 1 if and only if its invertible elements are norm dense in C*. The C^*-algebra $C_{\mathcal{C}}(X)$ of all complex-valued continuous functions defined on a compact space X of covering dimension 1 or zero (by [2, Proposition 1.7]) and any finite von Neumann algebra are tsr 1 algebras. Indeed, a von Neumann algebra is of tsr 1 if and only if it is finite (by [21, Theorem 5]). It is known (from [6, page 379]) that every von Neumann algebra has real rank zero but infinite topological stable rank unless the algebra is finite. Thus, we take this characterization to define the JB^*-algebras of topological stable rank 1.

A JB^*-algebra $\mathcal{J}$ is said to be of topological stable rank 1 if the set $\mathcal{J}_{\text{inv}}$ of its invertible elements is norm dense in $\mathcal{J}$. We symbolize this by $\text{tsr}(\mathcal{J}) = 1$.

Spin factors. Recall, given any real Hilbert space $\mathcal{H}$ of (algebraic) dimension ≥ 1 with the inner product $\langle\cdot,\cdot\rangle$, what the direct sum $X := \mathcal{H} \oplus \mathfrak{R}$ becomes a JB-algebra, called a real spin factor, under the norm

$$\|h+\lambda I\| = \|h\| + |\lambda| \tag{3.1}$$

and the Jordan product

$$(h+\lambda I)\circ(h'+\lambda' I) = \lambda h' + \lambda' h + (\langle h,h'\rangle + \lambda\lambda')I \tag{3.2}$$

for all $h,h' \in \mathcal{H}$ and $\lambda,\lambda' \in \mathfrak{R}$. Clearly, $0+1I$ is the multiplicative identity in the real spin factor $X := \mathcal{H} \oplus \mathfrak{R}$. The complexification $\mathcal{J}$ of the algebra X is called a complex spin factor. We have already noted above that any complex spin factor is a special JBW^*-algebra. As usual, we write the elements of the complex spin factor $\mathcal{J}$ as sums $x+iy$, where $x,y \in X$. It is well known that real spin factors are JC-algebras (see [20]). Hence, any complex spin factor being the complexification of a JC-algebra is a JC^*-algebra (see [19, 22], for details).

The Jordan multiplication in the complexification of any JB-algebra J is constructed as $(a,b)\circ(c,d) = (a\circ c - b\circ d, a\circ d + b\circ c)$ for any $a,b,c,d \in J$.

Theorem 3.1. *Let $\mathcal{J}$ be a complex spin factor.*

(a) *Every element of $\mathcal{J}$ has spectrum consisting of at most two points.*

(b) *$\mathcal{J}$ is of tsr 1.*

Proof. (a) Let $\mathcal{J}$ be the complexification of a real spin factor $X := \mathcal{H} \oplus \mathfrak{R}$. Let $x = h+0I$, $y = k+0I$ be vectors in X; we identify $h+0I$ with h and $k+0I$ with k. Then $x+iy \in \mathcal{J}$ and so, by the constructions of the Jordan multiplications, we have

$$\begin{aligned}(x+iy)^2 &= (h+ik)\circ(h+ik)\\ &= (h\circ h - k\circ k) + 2ih\circ k\\ &= \big((0+\|h\|^2 I) - (0+\|k\|^2 I)\big) + 2i(0+\langle h,k\rangle I)\\ &= (0+(\|h\|^2 - \|k\|^2)I) + i(0+2\langle h,k\rangle I).\end{aligned} \tag{3.3}$$

Now, since $\sigma_{\mathcal{J}}((x+iy)^2) = \sigma_{\mathcal{J}}((0+(\|h\|^2-\|k\|^2)I) + i(0+2\langle h,k\rangle I))$ being the spectrum of a scalar (in fact, a scalar multiple of the identity $(0+1I)+i(0+0I)$ of $\mathcal{J}$) is singleton, it follows that the spectrum $\sigma_{\mathcal{J}}(h+ik)$ contains at the most two points. Further, for any $h,k \in \mathcal{H}$ and $\alpha,\beta \in \mathfrak{R}$,

$$\begin{aligned}\sigma_{\mathcal{J}}\big((h+\alpha I)+i(k+\beta I)\big) &= \sigma_{\mathcal{J}}\big((h+ik) + ((0+\alpha I)+i(0+\beta I))\big)\\ &= \sigma_{\mathcal{J}}(h+ik) + \sigma_{\mathcal{J}}\big((\alpha+i\beta)((0+1I)+i0)\big)\\ &= \sigma_{\mathcal{J}}\big((h+ik)\big) + \{\alpha+i\beta\}.\end{aligned} \tag{3.4}$$

Thus every element of the complex spin factor $\mathcal{J}$ has spectrum consisting of at most two points.

(b) By the part (a), the spectrum of any element of the spin factor $\mathcal{J}$ contains, at most, two points and hence every element of $\mathcal{J}$ is a norm limit of some sequence of invertible elements of $\mathcal{J}$. Which means that $\mathcal{J}_{\text{inv}}$ is norm dense in $\mathcal{J}$. Thus $\text{tsr}(\mathcal{J}) = 1$. □

With this example, we can deduce that the enveloping von Neumann algebra of a JBW^*-algebra $\mathcal{J}$ need not have the same tsr as $\mathcal{J}$.

Theorem 3.2. *There exists a JBW^*-algebra of tsr 1 which may have enveloping von Neumann algebras of tsr not* 1.

Proof. By Theorem 3.1(b), we know that any complex spin factor is a JBW^*-algebra (see Example 2.7) of tsr 1. By [20, Remark 7.4.4], the C^*-algebra generated by an infinite-dimensional spin factor of countable dimension is known to have factor representations of types I, II, and III. Thus the required result follows. □

4. Isotopes

Let $\mathcal{J}$ be a Jordan algebra and $x \in \mathcal{J}$.The x-homotope of $\mathcal{J}$, denoted by $\mathcal{J}_{[x]}$, is the Jordan algebra consisting of the same elements and linear algebra structure as $\mathcal{J}$ but a different product, denoted by "$\cdot x$", defined by

$$a._x b = \{axb\} \tag{4.1}$$

for all a, b in $\mathcal{J}_{[x]}$.

Lemma 4.1. *If $\mathcal{J}$ is a special Jordan algebra and $a \in \mathcal{J}$, then $\mathcal{J}_{[a]}$ is a special Jordan algebra.*

Proof. As $\mathcal{J}$ is special, $\mathcal{J}$ is isomorphic to a Jordan subalgebra of an associative algebra $\mathcal{A}$. By identifying $\mathcal{J}$ with its image we can assume that $\mathcal{J}$ is a Jordan subalgebra of $\mathcal{A}$. In $\mathcal{A}$, define a new product by $x \cdot y = xay$. Then $(\mathcal{A}, \cdot)$ is an associative algebra and $x \cdot_a y = \{xay\} = (1/2)(xay + yax) = (1/2)(x \cdot y + y \cdot x)$. □

If $x \in \mathcal{J}_{\text{inv}}$, then x acts as the unit for the homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$.

If $\mathcal{J}$ is a unital Jordan algebra and $x \in \mathcal{J}_{\text{inv}}$, then by x-isotope of $\mathcal{J}$, denoted by $\mathcal{J}^{[x]}$, we mean the x^{-1}-homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$. We denote the multiplication "$\cdot x^{-1}$" of $\mathcal{J}^{[x]}$ by "$\circ_x$". $\{\cdot,\cdot,\cdot\}_x$, y^{-1_x} will stand for the Jordan triple product and the multiplicative inverse (if exists) of y in the isotope $\mathcal{J}^{[x]}$, respectively.

Let $\mathcal{J}$ be a Jordan algebra with unit e. Let $x \in \mathcal{J}$ and let $a \in \mathcal{J}_{\text{inv}}$. We define the linear operator U_{x_a} on $\mathcal{J}$ as follows:

$$U_{x_a} z = \{xzx\}_a \quad \forall z \in \mathcal{J}. \tag{4.2}$$

This reduces to the well-known operator U_x for $a = e$. We will frequently use some basic properties of the operator U_x, particularly, in relation to invertibility as given in famous the Jacobson theorem (cf. [12]).

Part (ii) of the next lemma gives the invariance of the set of invertible elements in a unital Jordan algebra on passage to any of its isotopes.

Lemma 4.2. *For any invertible element a in a unital Jordan algebra $\mathcal{J}$,*

(i) $U_{x_a} = U_x U_{a^{-1}}$, *for all* $x \in \mathcal{J}$;

(ii) $\mathcal{J}_{inv} = \mathcal{J}_{inv}^{[a]}$;

(iii) $y^{-1_a} = \{ay^{-1}a\}$, *for all* $y \in \mathcal{J}_{inv}$.

Proof. We recall the following Jordan identity from [12, page 57]:

$$2\{\{xyz\}yx\} - \{\{xyx\}yz\} = \{x\{yzy\}x\}. \tag{4.3}$$

Taking $y = a^{-1}$, we get $2\{xa^{-1}\{xa^{-1}z\}\} - \{\{xa^{-1}x\}a^{-1}z\} = \{x\{a^{-1}za^{-1}\}x\}$. So, for any $x,z \in \mathcal{J}^{[a]}$, $\{xzx\}_a = 2x \circ_a (x \circ_a z) - (x \circ_a x) \circ_a z = 2\{xa^{-1}\{xa^{-1}z\}\} - \{\{xa^{-1}x\}a^{-1}z\} = \{x\{a^{-1}za^{-1}\}x\}$. This means that $U_{x_a}z = U_x U_{a^{-1}}z$ for all $x,z \in \mathcal{J}$. Hence, $U_{x_a} = U_x U_{a^{-1}}$ for all $x \in \mathcal{J}$.

(ii) By part (i), $U_{x_a} = U_x U_{a^{-1}}$ for all $x \in \mathcal{J}$ and for all $a \in \mathcal{J}_{\text{inv}}$. Hence, by the Jacobson theorem, the product operator U_{x_a} is invertible if and only if the operator U_x is invertible. Thus, again by the Jacobson theorem, x is invertible in $\mathcal{J}^{[a]}$ if and only if x is invertible in the original algebra $\mathcal{J}$. This establishes the required equality in part (ii).

(iii) By part (ii), y has its inverse y^{-1_a} in the isotope $\mathcal{J}^{[a]}$. Since by part (i) $U_{y_a} = U_y U_{a^{-1}}$, it follows by the Jacobson theorem and Jordan identity $x \circ (x^{-1} \circ y) = x^{-1} \circ (x \circ y)$ that $y^{-1_a} = U_{y_a^{-1_a}} y$ (as a is the unit in the isotope $\mathcal{J}^{[a]}$) $= U_{y_a}^{-1} y = (U_y U_{a^{-1}})^{-1} y = U_{a^{-1}}^{-1} U_y^{-1} y = U_a U_{y^{-1}} y = \{ay^{-1}a\}$. □

Lemma 4.3. *Let $\mathcal{J}$ be a unital JB^*-algebra and $a \in \mathcal{J}_{inv}$. Then the operation $*_a$ given by $x^{*_a} = \{ax^*a\}$ satisfies $\{vy^{*_a}w\}_a = \{vy^*w\}$ for all $v,w,y \in \mathcal{J}$.*

Proof.

Step 1. By Lemma 4.2(i), for any $x,y \in \mathcal{J}^{[a]}$, $\{xyx\}_a = \{x\{a^{-1}ya^{-1}\}x\}$. Hence, $\{xy^{*_a}x\}_a = \{x\{a^{-1}y^{*_a}a^{-1}\}x\} = \{x\{a^{-1}\{ay^*a\}a^{-1}\}x\} = U_x U_{a^{-1}} U_a y^* = U_x y^*$ by the Jacobson theorem. Thus $\{xy^{*_a}x\}_a = \{xy^*x\}$ for all $x,y \in \mathcal{J}$.

Step 2. We linearize the above equation by taking $x = v + w$ and using the well-known linearity and symmetry (in the outer variables) of the Jordan triple products, we get that $\{vy^{*_a}v\}_a + 2\{vy^{*_a}w\}_a + \{wy^{*_a}w\}_a = \{vy^{*_a}v\}_a + \{wy^{*_a}v\}_a + \{vy^{*_a}w\}_a + \{wy^{*_a}w\}_a = \{(v+w)y^{*_a}(v+w)\}_a = \{xy^{*_a}x\}_a = \{xy^*x\} = \{(v+w)y^*(v+w)\} = \{vy^*v\} + \{wy^*v\} + \{vy^*w\} + \{wy^*w\} = \{vy^*v\} + 2\{vy^*w\} + \{wy^*w\}$, so again by (i), we conclude $\{vy^{*_a}w\}_a = \{vy^*w\}$, which is as (ii). □

Let $\mathcal{J}$ be a JB^*-algebra. $u \in \mathcal{J}$ is called *unitary* if $u^* = u^{-1}$, the inverse of u. In such a case, $\|u\| = 1$. The set of all unitary elements of $\mathcal{J}$ will be denoted by $\mathcal{U}(\mathcal{J})$.

The next result is well known due to Braun, Kaup and Upmeier [23, 24].It states that the isotope of a JB^*-algebra determined by any uniatry element is itself a JB^*-algebra; such isotopes are called uniatry isotope.

Theorem 4.4 (unitary isotopes [23, 24]). *Let u be a unitary element of the JB^*-algebra $\mathcal{J}$. Then the isotope $\mathcal{J}^{[u]}$ is a JB^*-algebra having u as its unit with respect to the original norm and the involution $*_u$ given by $x^{*_u} = \{ux^*u\}$.*

Thus, for any unitary element u of JB^*-algebra $\mathcal{J}$, $\operatorname{tsr}(\mathcal{J}) = 1$ if and only if $\operatorname{tsr}(\mathcal{J}^{[u]}) = 1$, by Lemma 4.2.

We now turn to show that $\mathcal{U}(\mathcal{J}) = \mathcal{U}(\mathcal{J}^{[u]})$ for unitary isotopes. To do this, we need the following lemma.

Lemma 4.5. *Let $\mathcal{J}$ be a JB^*-algebra with unit e. Then $u \in \mathcal{U}(\mathcal{J}) \Rightarrow e \in \mathcal{U}(\mathcal{J}^{[u]})$. Moreover, $\mathcal{J}^{[u][e]} = \mathcal{J}$.*

Proof. Let $u \in \mathcal{U}(\mathcal{J})$. We note that $e^{*_u} = \{ue^*u\} = \{ueu\} = u \circ u$. Since $u^* = u^{-1}$, by using the Jordan identity $x \circ (x^{-1} \circ y) = x^{-1} \circ (x \circ y)$, we get that $e \circ_u e^{*_u} = \{eu^*(u \circ u)\} = u^* \circ (u \circ u) = u^{-1} \circ (u \circ u) = u \circ (u^{-1} \circ u) = u$, which is the unit of $\mathcal{J}^{[u]}$. It is well known that any Jordan algebra is integral power associative, so that $(e \circ_u e) \circ_u e^{*_u} = \{\{eu^*e\}u^*(u \circ u)\} = (u \circ u) \circ (u^* \circ u^*) = (u \circ u) \circ (u^{-1} \circ u^{-1}) = u^{2-2} = e$. Hence $e \in \mathcal{U}(\mathcal{J}^{[u]})$. Thus the unitary isotope $\mathcal{J}^{[u][e]}$ of the isotope $\mathcal{J}^{[u]}$ is well defined.

Now, for any $x, y \in \mathcal{J}$, we see that $x \circ_{u_e} y = \{xe^{*_u}y\}_u = \{xe^*y\} = x \circ_e y$ by part (ii) of Lemma 4.3, where $\circ_{u_e}$ denotes the Jordan product of the isotope $\mathcal{J}^{[u][e]}$. Again by Lemma 4.3, $x^{*_{u_e}} = \{ex^{*_u}e\}_u = \{ex^*e\} = x^{*_e}$ for all $x \in \mathcal{J}$. Therefore, the product and the involution of $\mathcal{J}^{[u][e]}$ coincide, respectively, with the product and the involution of $\mathcal{J}^{[e]}$. Hence, $\mathcal{J}^{[u][e]} = \mathcal{J}^{[e]}$, as the JB^*-algebras. Thus $\mathcal{J}^{[u][e]} = \mathcal{J}$ since it is easy to see that $\mathcal{J}^{[e]} = \mathcal{J}$, as JB^*-algebras. □

Next result is an important tool for our subsequent work.

Theorem 4.6 (unitaries). *For any unitary element u in the JB^*-algebra $\mathcal{J}$,*

$$\mathcal{U}(\mathcal{J}) = \mathcal{U}(\mathcal{J}^{[u]}). \tag{4.4}$$

Proof. Let $v \in \mathcal{U}(\mathcal{J})$. Since v is invertible in $\mathcal{J}$, its inverse v^{-1_u} exists in the isotope $\mathcal{J}^{[u]}$ by Lemma 4.2(ii). By Lemma 4.2(iii), $v^{-1_u} = \{uv^{-1}u\} = \{uv^*u\} = v^{*_u}$. We conclude that

(i)

$$\mathcal{U}(\mathcal{J}) \subseteq \mathcal{U}(\mathcal{J}^{[u]}). \tag{4.5}$$

For the other inclusion, we note that the identity e being a unitary in $\mathcal{J}$ is a unitary in $\mathcal{J}^{[u]}$ by (i). Hence, again by (i),

(ii)

$$\mathcal{U}(\mathcal{J}^{[u]}) \subseteq \mathcal{U}(\mathcal{J}^{[u][e]}). \tag{4.6}$$

By Lemma 4.5, $\mathcal{J}^{[u][e]} = \mathcal{J}$. Thus (ii) together with (i) gives the required equality $\mathcal{U}(\mathcal{J}^{[u]}) = \mathcal{U}(\mathcal{J})$. □

Next, we need to show that *every unitary isotope of a JC^*-algebra is again a JC^*-algebra*. This involves the following lemma on the construction of an associative unitary isotope of a C^*-algebra.

Lemma 4.7. *Let u be a unitary operator in $\mathcal{B}(\mathcal{H})$, the C^*-algebra of all bounded linear operators on an arbitrarily fixed complex Hilbert space $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})_u$ denote the underlying normed linear space of the $\mathcal{B}(\mathcal{H})$. Then $\mathcal{B}(\mathcal{H})_u$ is a unital C^*-algebra under the new* *multiplication "$\bullet$" and involution "$\dagger_u$" defined for $x, y \in \mathcal{B}(\mathcal{H})_u$ by*

$$x \bullet y = xu^* y, \qquad x^{\dagger_u} = ux^* u, \tag{4.7}$$

where "$*$" stands for the usual involution on $\mathcal{B}(\mathcal{H})$ and the juxtaposition xy of x, y denotes their ordinary operator product.

Proof. As noted in the proof of Lemma 4.1, $(\mathcal{B}(\mathcal{H})_u, \bullet)$ is an associative algebra. Note that u is the unit in $(\mathcal{B}(\mathcal{H})_u, \bullet)$ because $u \bullet x = uu^{-1}x = x = x \bullet u$ for all $x \in \mathcal{B}(\mathcal{H})_u$.

Further, $u^{\dagger_u} = uu^* u = u$ so that $\|u^{\dagger_u}\| = \|u\| = 1$. Moreover,

$$\begin{aligned} (x+y)^{\dagger_u} &= u(x+y)^* u = ux^* u + uy^* u = x^{\dagger_u} + y^{\dagger_u}, \\ (\lambda x)^{\dagger_u} &= u(\lambda x)^* u = \overline{\lambda}(ux^* u) = \overline{\lambda} x^{\dagger_u}, \\ (x \bullet y)^{\dagger_u} &= u(xu^* y)^* u = uy^* ux^* u = uy^* uu^* ux^* u = y^{\dagger_u} \bullet x^{\dagger_u}, \\ x^{\dagger_u^{\dagger_u}} &= u(ux^* u)^* u = x. \end{aligned} \tag{4.8}$$

Finally,

$$\begin{aligned} \|x \bullet y\| &= \|xu^* y\| \le \|x\| \|u^*\| \|y\| = \|x\| \|y\|, \\ \|x^{\dagger_u} \bullet x\| &= \|ux^* x\| = \|x^* x\| = \|x\|^2 \end{aligned} \tag{4.9}$$

since u is a unitary operator in $\mathcal{B}(\mathcal{H})$. □

Theorem 4.8. *With the above notations and assumptions, one has*

$$(\mathcal{B}(\mathcal{H})^{[u]}, \circ_u, *_u) = (\mathcal{B}(\mathcal{H})_u, \odot_u, \dagger_u) \tag{4.10}$$

as JC^*-algebras, where the multiplications "$\circ_u$" and "$\odot_u$" are defined by

$$x \circ_u y = \{xu^* y\}, \qquad x \odot_u y = \frac{1}{2}(x \bullet y + y \bullet x) \tag{4.11}$$

on $\mathcal{B}(\mathcal{H})^{[u]}$ and $\mathcal{B}(\mathcal{H})_u$, respectively.

Proof. By Theorem 4.4 and Lemma 4.1, the isotope $(\mathcal{B}(\mathcal{H})^{[u]}, \circ_u, *_u)$ is a JC^*-algebra. Since (by Lemma 4.7) $(\mathcal{B}(\mathcal{H})_u, \bullet, \dagger_u)$ is a C^*-algebra, $(\mathcal{B}(\mathcal{H})_u, \odot_u, \dagger_u)$ is JC^*-algebra.

Moreover, for any operators x, y in $\mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} x \circ_u y = \{xu^* y\} &= \frac{1}{2}(xu^* y + yu^* x) = \frac{1}{2}(x \bullet y + y \bullet x) = x \odot_u y, \\ x^{*_u} &= \{ux^* u\} = ux^* u = x^{\dagger_u}. \end{aligned} \tag{4.12}$$

□

An element x of a JB^*-algebra $\mathcal{J}$ is positive in $\mathcal{J}$ if $x^* = x$ and its spectrum $\sigma_{\mathcal{J}}(x)$ is contained in the set of nonnegative real numbers. Our next aim is to develop another tool for the sequel: every invertible element of a JB^*-algebra $\mathcal{J}$ is positive in certain isotope of $\mathcal{J}$.

Lemma 4.9. *Let $\mathcal{J}$ be a JC^*-algebra embedded in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and let $x \in \mathcal{J}$ be invertible with $\|x\| \leq 1$. Then $x^{*^{-1}}(x^*x)^{1/2} \in \mathcal{J}$.*

Proof. Let $\mathcal{J}$ be embedded (isometrically and $*$-isomorphically) into the JC^*-algebra $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, under the ordinary involution and the Jordan multiplication induced by the usual operator multiplication. Considering x as an invertible element and e as the identity element of the JC^*-algebra $\mathcal{B}(\mathcal{H})$, we have by Remark 2.3 that $xx^{-1} = e = x^{-1}x$, where juxtaposition of x and x^{-1} denotes their usual product as operators. That is, x is an invertible element of the C^*-algebra $\mathcal{B}(\mathcal{H})$. Therefore, x has the standard polar decomposition $x = u|x|$ in $\mathcal{B}(\mathcal{H})$ with the unitary and positive operators $u = x|x|^{-1}$, $|x| = (x^*x)^{1/2}$, respectively.

Observe that $u = x|x|^{-1} = x(x^*x)^{-1/2} = x(x^*x)^{-1}(x^*x)^{1/2} = x^{*^{-1}}(x^*x)^{1/2}$.

(i) That is,

$$u = x^{*^{-1}}(x^*x)^{1/2}. \tag{4.13}$$

Considering x in $\mathcal{B}(\mathcal{H})$, we have $\{xx^*x\} = 2(x \circ x^*) \circ x - (x \circ x) \circ x^* = (xx^* + x^*x) \circ x - (1/2)(x^2x^* + x^*x^2) = (1/2)(2xx^*x + x^*x^2 + x^2x^*) - (1/2)(x^2x^* + x^*x^2) = xx^*x$, so for any positive integer m,

$$x^{*^{-1}}(x^*x)^m = (x^{*^{-1}}x^*)(\overbrace{xx^*x\cdots xx^*x}^{2m-1 \text{ factors}}) = \{x\{x^*\{x\{\cdots\{aba\}\cdots\}x\}x^*\}x\}, \tag{4.14}$$

where $a^* = b$ and $b = x^*$ if m is even, otherwise $b = x$. So $x^{*^{-1}}(x^*x)^m \in \mathcal{J}$ for all $m = 0, 1, 2, \ldots$. Hence, we have

(ii)

$$x^{*^{-1}}P(x^*x) \in \mathcal{J} \tag{4.15}$$

for any polynomial $P : [0,1] \to \mathbb{C}$.

By the famous Stone-Weierstrass theorem, there exists a sequence (P_n) in $\mathbf{C}_{\mathbb{C}}[0,1]$ such that P_n is a polynomial and (P_n) converges uniformly to $g \in \mathbf{C}_{\mathbb{C}}[0,1]$ given by $g(t) = t^{1/2}$. Hence, by the standard functional calculus for the selfadjoint operators,

(iii)

$$\lim_{n\to\infty} P_n(x^*x) = (x^*x)^{1/2} \tag{4.16}$$

as $\|x\| \leq 1$. Since $\mathcal{J}$ is norm closed, it follows from (i)–(iii) that

$$u = x^{*^{-1}}(x^*x)^{1/2} = x^{*^{-1}} \lim_{n\to\infty} P_n(x^*x) = \lim_{n\to\infty} x^{*^{-1}}P_n(x^*x) \in \mathcal{J}. \tag{4.17}$$

We conclude that $x^{*^{-1}}(x^*x)^{1/2} = u \in \mathcal{J}$. □

We next show that this element is invariant under unitary isotopes.

Lemma 4.10. *Let $\mathcal{J}$ be a JC^*-algebra embedded in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and let $x \in \mathcal{J}$ be invertible with $\|x\| \leq 1$. If $u \in \mathcal{J}$ is unitary and $\mathcal{B}(\mathcal{H})_u$ is the unital C^*-algebra* *under the multiplication "$\bullet$" and involution "$\dagger_u$" defined (as in Lemma 4.7) by $x \bullet y = xu^* y$ and $x^{\dagger_u} = ux^* u$, then*

$$(x^{\dagger_u})^{-1} \bullet (x^{\dagger_u} \bullet x)^{1/2} = (x^*)^{-1} (x^* x)^{1/2}. \tag{4.18}$$

Proof. x is invertible in $\mathcal{J}$ and hence in $\mathcal{J}^{[u]}$ by Lemma 4.2. Hence x is invertible in $\mathcal{B}(\mathcal{H})_u$ by Lemma 4.1 and Remark 2.3.

For any positive integer m, we observe that

$$\begin{aligned}
&(x^{\dagger_u})^{-1} \bullet (x^{\dagger_u} \bullet x)^m \\
&\quad = \overbrace{x \bullet x^{\dagger_u} \bullet x \bullet x^{\dagger_u} \bullet \cdots \bullet x^{\dagger_u} \bullet x}^{(2m-1)\text{ factors}} \quad (\text{since } \bullet \text{ is associative in } \mathcal{B}(\mathcal{H})) \\
&\quad = xu^* ux^* uu^* xu^* ux^* uu^* \cdots u^* ux^* uu^* x \quad (\text{by the constructions of } \bullet \text{ and } \dagger_u) \\
&\quad = \overbrace{xx^* xx^* \cdots x^* x}^{(2m-1)\text{ factors}} \quad (\text{since } u^* = u^{-1}) = (x^*)^{-1} (x^* x)^m.
\end{aligned} \tag{4.19}$$

That is,

$$(x^{\dagger_u})^{-1} \bullet (x^{\dagger_u} \bullet x)^m = (x^*)^{-1} (x^* x)^m, \quad \forall m = 0, 1, 2, \ldots. \tag{4.20}$$

Therefore, for any polynomial $P : [0,1] \to \mathbb{C}$, we have

(i)

$$(x^*)^{-1} P(x^* x) = (x^{\dagger_u})^{-1} \bullet P(x^{\dagger_u} \bullet x). \tag{4.21}$$

By the Stone-Weierstrass theorem, there exists a sequence (P_n) in $\mathbf{C}_{\mathbb{C}}[0,1]$ such that P_n is a polynomial and (P_n) converges uniformly to $g \in \mathbf{C}_{\mathbb{C}}[0,1]$ given by $g(t) = t^{1/2}$. Hence, by the standard functional calculus for the selfadjoint operators,

(ii)

$$\lim_{n \to \infty} P_n(x^* x) = (x^* x)^{1/2}, \tag{4.22}$$

(iii)

$$\lim_{n \to \infty} P_n(x^{\dagger_u} \bullet x) = (x^{\dagger_u} \bullet x)^{1/2}. \tag{4.23}$$

Hence, by (i)–(iii) and norm continuity of the multiplication involved, we get

$$\begin{aligned}
(x^{\dagger_u})^{-1} \bullet (x^{\dagger_u} \bullet x)^{1/2} &= (x^{\dagger_u})^{-1} \bullet \lim_{n \to \infty} P_n(x^{\dagger_u} \bullet x) \\
&= \lim_{n \to \infty} (x^{\dagger_u})^{-1} \bullet P_n(x^{\dagger_u} \bullet x) = \lim_{n \to \infty} (x^*)^{-1} P_n(x^* x) \\
&= (x^*)^{-1} \lim_{n \to \infty} P_n(x^* x) = (x^*)^{-1} (x^* x)^{1/2}.
\end{aligned} \tag{4.24}$$

Thus $(x^{\dagger_u})^{-1} \bullet (x^{\dagger_u} \bullet x)^{1/2} = (x^*)^{-1} (x^* x)^{1/2}$. □

The following result extends Lemma 4.9 to general JB^*-algebra.

LEMMA 4.11. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in \mathcal{J}$ be invertible. Then $\mathcal{J}(e,x,x^*, x^{-1},x^{*^{-1}})$ is a JC^*-algebra. If $\mathcal{J}(e,x,x^*,x^{-1},x^{*^{-1}})$ is considered as a subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ in which x has a standard polar decomposition $x = u|x|$, where $|x| = (x^*x)^{1/2}$ is positive and $u = x|x|^{-1}$ is a unitary, then $u \in \mathcal{J}(e,x,x^*,x^{-1},x^{*^{-1}}) \subseteq \mathcal{J}$.*

Proof. Without loss of generality, we may assume $\|x\| = 1$ since if $y = x/\|x\|$, then $y|y|^{-1} = (x/\|x\|)((x^*x)^{1/2}/\|x\|)^{-1} = x(x^*x)^{-1/2} = u$.

By Corollary 2.5, the norm closed Jordan subalgebra $\mathcal{J}(e,x,x^*,x^{-1},x^{*^{-1}})$, generated by x, x^*, their inverses, and the identity element e, is a JC^*-algebra.

Now, considering the JC^*-algebra $\mathcal{J}(e,x,x^*,x^{-1},x^{*^{-1}})$ embedded in the $\mathcal{B}(\mathcal{H})$, we get, by Lemma 4.9, that $x^{*^{-1}}(x^*x)^{1/2} \in \mathcal{J}(e,x,x^*,x^{-1},x^{*^{-1}})$. It follows that $u = x|x|^{-1} \in \mathcal{J}$. $\square$

Our final result in this section shows that an invertible is positive in a unitary isotope, where the unitary comes from the polar decomposition of the invertible.

THEOREM 4.12 (positivity of invertibles). *Every invertible element x of the JB^*-algebra $\mathcal{J}$ is positive (in fact, positive invertible) in the isotope $\mathcal{J}^{[u]}$ of $\mathcal{J}$, where $u \in \mathcal{U}(\mathcal{J})$ and is given by the usual polar decomposition $x = u|x|$ of x considered as an operator in some $\mathcal{B}(\mathcal{H})$.*

Proof. Since $x \in \mathcal{J}_{\text{inv}}$, the norm closed Jordan subalgebra $\mathcal{J}(e,x,x^*,x^{-1},x^{*^{-1}})$ of the JB^*-algebra $\mathcal{J}$, generated by the identity element e, x, x^* and their inverses, is a unital JC^*-algebra by Corollary 2.5.

Let the JC^*-algebra $\mathcal{J}(e,x,x^*,x^{-1},x^{*^{-1}})$ be embedded in the JC^*-algebra $(\mathcal{B}(\mathcal{H}),\circ,*)$, for some Hilbert space $\mathcal{H}$. Then x considered as an invertible operator in the C^*-algebra $\mathcal{B}(\mathcal{H})$ (see Remark 2.3) has the standard polar decomposition $x = u|x|$, where $|x| = (x^*x)^{1/2}$ is a positive operator and

(i)

$$u = x(x^*x)^{-1/2} = (x^*)^{-1}(x^*x)^{1/2} \tag{4.25}$$

is a unitary operator. By Lemma 4.11, $\mathcal{J}(e,x,x^*,x^{-1},x^{*^{-1}})^{[u]}$ is a unitary isotope of $\mathcal{J}(e,x, x^*,x^{-1},x^{*^{-1}})$. Moreover, $\mathcal{J}(e,x,x^*,x^{-1},x^{*^{-1}})^{[u]}$ is a JC^*-subalgebra of the u-isotope $\mathcal{B}(\mathcal{H})^{[u]}$ of the JC^*-algebra $(\mathcal{B}(\mathcal{H}),\circ,*)$ by Lemma 4.1.

By Lemma 4.2, $\mathcal{B}(\mathcal{H})^{[u]}_{\text{inv}} = \mathcal{B}(\mathcal{H})_{\text{inv}}$ as JC^*-algebras. But x is invertible in the JC^*-algebra $(\mathcal{B}(\mathcal{H}),\circ,*)$. Therefore, x is invertible in the JC^*-algebra $(\mathcal{B}(\mathcal{H})^{[u]},\circ_u,*_u)$. Hence x is invertible in the JC^*-algebra $(\mathcal{B}(\mathcal{H})_u,\odot_u,\dagger_u)$ by Theorem 4.8. So it follows by Remark 2.3 that x is invertible in the C^*-algebra $(\mathcal{B}(\mathcal{H})_u,\bullet,\dagger_u)$ (see Lemma 4.7).

Now, x being an invertible in the C^*-algebra $(\mathcal{B}(\mathcal{H})_u,\bullet,\dagger_u)$ has the standard polar decomposition, namely,

(ii)

$$x = w \bullet (x^{\dagger_u} \bullet x)^{1/2}, \tag{4.26}$$

where $(x^{\dagger_u} \bullet x)^{1/2}$ is the positive square root of $(x^{\dagger_u} \bullet x)$ in $(\mathcal{B}(\mathcal{H})_u,\bullet,\dagger_u)$ and w is its unitary element given by

(iii)

$$w = x \bullet (x^{\dagger_u} \bullet x)^{-1/2} = (x^{\dagger_u})^{-1} \bullet (x^{\dagger_u} \bullet x)^{1/2}. \tag{4.27}$$

By Lemma 4.10, we already know that
(iv)

$$(x^{\dagger_u})^{-1} \bullet (x^{\dagger_u} \bullet x)^{1/2} = (x^*)^{-1}(x^*x)^{1/2}. \tag{4.28}$$

This equation (iv) together with equations (i) and (iii) give

$$w = (x^{\dagger_u})^{-1}(x^{\dagger_u} \bullet x)^{1/2} = (x^*)^{-1}(x^*x)^{1/2} = u. \tag{4.29}$$

This together with equation (ii) gives

$$x = w \bullet (x^{\dagger_u} \bullet x)^{1/2} = u \bullet (x^{\dagger_u} \bullet x)^{1/2} = uu^*(x^{\dagger_u} \bullet x)^{1/2} = (x^{\dagger_u} \bullet x)^{1/2}. \tag{4.30}$$

Thus, x is a positive element of the C^*-algebra $(\mathcal{B}(\mathcal{H})_u, \bullet, \dagger_u)$. That is, $x^{\dagger_u} = x$ and $\sigma_{(\mathcal{B}(\mathcal{H})_u, \bullet, \dagger_u)}(x) \subseteq (0,\infty)$.

Let $\mathcal{C}(u,x)$ be the norm closed subalgebra of the C^*-algebra $(\mathcal{B}(\mathcal{H})_u, \bullet, \dagger_u)$, generated by its identity element u and x. Then $\mathcal{C}(u,x)$ is indeed a commutative C^*-subalgebra of $(\mathcal{B}(\mathcal{H})_u, \bullet, \dagger_u)$ since $x^{\dagger_u} = x$. Hence, $\mathcal{C}(u,x)$ being commutative coincides with the JC^*-subalgebra $\mathcal{C}(u,x)^{[u]}$ of the JC^*-algebra $\mathcal{J}(u,x)^{[u]}$ generated by u and x. Recall that $\mathcal{J}(u,x)^{[u]}$ is a JC^*-subalgebra of the u-isotope $(\mathcal{B}(\mathcal{H})^{[u]}, \circ_u, *_u)$ of the JC^*-algebra $(\mathcal{B}(\mathcal{H}), \circ, *)$. But, by Theorem 4.8,

$$(\mathcal{B}(\mathcal{H})^{[u]}, \circ_u, *_u) = (\mathcal{B}(\mathcal{H})_u, \odot_u, \dagger_u), \tag{4.31}$$

as JC^*-algebras. Hence, by [13, Theorem 11.29], we conclude that

$$(0,\infty) \supseteq \sigma_{(\mathcal{B}(\mathcal{H})_u, \bullet, \dagger_u)}(x) = \sigma_{\mathcal{C}(u,x)}(x) = \sigma_{\mathcal{C}(u,x)^{[u]}}(x) \supseteq \sigma_{\mathcal{J}(u,x)^{[u]}}(x). \tag{4.32}$$

Of course, $\mathcal{J}(u,x)^{[u]}$ is a JC^*-subalgebra of the u-isotope $\mathcal{J}^{[u]}$. It follows that

$$(0,\infty) \supseteq \sigma_{\mathcal{J}(u,x)^{[u]}}(x) \supseteq \sigma_{\mathcal{J}^{[u]}}(x). \tag{4.33}$$

Also note that

$$x^{*_u} = \{ux^*u\} = ux^*u = x^{\dagger_u} = x. \tag{4.34}$$

Thus, x is positive (and invertible as well, by Lemma 4.2) in the unitary isotope $\mathcal{J}^{[u]}$ of the original JB^*-algebra $\mathcal{J}$. □

5. Finite-dimensional JB^*-algebras

The first result in this section gives a sufficient condition for a JB^*-algebra to be of tsr 1.

Theorem 5.1. *A unital JB^*-algebra is of tsr 1 if every element is selfadjoint in some of its unitary isotopes.*

The proof follows immediately from [10, Theorem 2.6].

In this section, we will prove a partial converse to the above theorem that every finite-dimensional JB^*-algebra is of tsr 1 and that every element of such an algebra is positive in some unitary isotope of the algebra. Example 2.9 of [10] shows that this is not true in the infinite-dimensional case even for C^*-algebras of tsr 1.

We know from Theorem 4.12 that every invertible element of a JB^*-algebra $\mathcal{J}$ is selfadjoint in a unitary isotope of $\mathcal{J}$. The first result in this section describes some of the points on the boundary of the invertibles.

Theorem 5.2. (1) *Suppose that $\mathcal{J}$ is a JB^*-algebra. Suppose that $x \in \operatorname{bdry}(\mathcal{J}_{inv})$ (the boundary of invertibles) and there exist a sequence (x_n) in $\mathcal{J}_{inv}$ such that $x_n \to x$ and a subsequence (u_{n_k}) of unitaries in the polar decomposition of x_{n_k}'s (via Lemma 4.11) is Cauchy. Then x is selfadjoint in some unitary isotope of $\mathcal{J}$.*

(2) *If $\mathcal{J}$ is a finite-dimensional JB^*-algebra with identity e, then every element y in $\mathcal{J}$ is the norm limit of a sequence of invertibles such that the corresponding sequence of unitaries (obtained via Lemma 4.11) has a convergent subsequence, hence y is selfadjoint in some of the unitary isotopes of $\mathcal{J}$.*

(3) *A finite-dimensional JB^*-algebra is of tsr 1.*

Proof. (1) Since (u_{n_k}) is Cauchy and since $\mathcal{U}(\mathcal{J})$ is closed (as noted above), there exists a unitary $u \in \mathcal{U}(\mathcal{J})$ such that $u_{n_k} \to u$. By Theorem 4.12, each x_{n_k} being invertible is selfadjoint in the isotope $\mathcal{J}^{[u_{n_k}]}$. Hence x is selfadjoint in the isotope $\mathcal{J}^{[u]}$ because $\{ux^*u\} = \{\lim u_{n_k} \lim x^*_{n_k} \lim u_{n_k}\} = \lim\{u_{n_k} x^*_{n_k} u_{n_k}\} = \lim x_{n_k} = x$ since the involution and Jordan triple product in any JB^*-algebra are norm continuous (see [25]).

(2) Now, let $\mathcal{J}$ be a finite-dimensional JB^*-algebra with identity e. Let $y \in \mathcal{J}$. In view of Theorem 4.12, the result is clear for the invertible elements.

Now, suppose that $y \in \mathcal{J} \setminus \mathcal{J}_{\text{inv}}$. Since the algebra $\mathcal{J}$ is finite-dimensional, the spectrum of y is a finite set. Hence, $y_n =_{\text{def}} y + (1/n)e$ is invertible for all but finite integral values of n. Clearly, the sequence (y_n) is norm convergent to y. By Lemma 4.11, there exist unitaries $v_n \in \mathcal{J}$, for all such integral values of n, so that each invertible y_n is positive and hence selfadjoint in the isotope $\mathcal{J}^{[v_n]}$. Thus, by the construction of the involution "$*_{v_n}$" (as given in Theorem 4.4),

(i)

$$\{v_n y_n^* v_n\} = y_n. \tag{5.1}$$

We know that the set of unitaries in any finite-dimensional JB^*-algebra being closed and bounded is sequentially compact. Hence, there exists a subsequence (v_{n_k}) of (v_n) that converges to a unitary $v \in \mathcal{J}$, in the norm topology. As noted above, the involution and Jordan triple product in any JB^*-algebra both are norm continuous, so (i) gives that

$$\begin{aligned} y &= \lim_{n_k\to\infty} y_{n_k} = \lim_{n_k\to\infty} \{v_{n_k} y^*_{n_k} v_{n_k}\} \\ &= \Big\{\lim_{n_k\to\infty} v_{n_k} \Big(\lim_{n_k\to\infty} y_{n_k}\Big)^* \lim_{n_k\to\infty} v_{n_k}\Big\} \\ &= \{v y^* v\} = y^{*_v}. \end{aligned} \tag{5.2}$$

Thus, y is selfadjoint in the unitary isotope $\mathcal{J}^{[v]}$.

(3) As seen in the above proof of part (2), every element of a finite-dimensional JB^*-algebra is a norm limit of invertible elements and so the set of invertible elements is norm dense in the algebra. □

Remark 5.3. The sequence (u_n) of unitaries obtained by the corresponding polar decompositions (as in Lemma 4.11) of a convergent sequence (x_n) of invertibles may not be Cauchy even in finite-dimensional case, for example, let $x = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and let $x_n = ((-1)^n/n)\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ for $n = 1,2,3,\ldots$. Then $x_n \to x$ as $n \to \infty$. Clearly, the polar decomposition of x_n, for each n, is $x_n = u_n|x_n|$, where the positive part of x_n and unitaries are given, respectively, by $|x_n| = (1/n)\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $u_n = (-1)^n\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Since

$$||u_m - u_n|| = |(-1)^m - (-1)^n| = \begin{cases} 0 & \text{if } m-n \text{ is a multiple of 2;} \\ 2 & \text{otherwise,} \end{cases} \tag{5.3}$$

so (u_n) is not a Cauchy sequence.

Remark 5.4. From [10, Example 3.1], we know that even for JB^*-algebras of tsr 1 (as the algebra $\mathscr{C}_{\mathscr{A}}(S^1)$ is of tsr 1 by [2, Example 4.13]) not every element on $\operatorname{bdry}(\mathscr{J}_{\text{inv}})$ is selfadjoint in some (unitary) isotope. Therefore, the sequence of unitaries (obtained via Lemma 4.11) by the polar decompositions of invertibles may not even have Cauchy subsequence.

We can in fact say more in the finite-dimensional case. To do so we will need the following lemmas.

Lemma 5.5. *Let $(\Upsilon, \geq)$ be a directed set and let (y_α) and (z_α), where $\alpha \in \Upsilon$, be nets in a JBW^*-algebra $\mathscr{J}$ such that (y_α) is bounded and weak*-converges to y, and z_α converges in norm topology to z. Then $y_\alpha \circ z_\alpha$ converges to $y \circ z$ in the weak*-topology.*

Proof. We have to show that $f(y_\alpha \circ z_\alpha - y \circ z) \to 0$ for all $f \in \mathscr{J}_*$, where $\mathscr{J}_*$ denotes the predual of $\mathscr{J}$. Let $f \in \mathscr{J}_*$. Then

(i)

$$\begin{aligned} |f(y_\alpha \circ z_\alpha - y \circ z)| &= |f(y_\alpha \circ z_\alpha - y_\alpha \circ z) + f(y_\alpha \circ z - y \circ z)| \\ &\leq ||y_\alpha|| \|f\| ||z_\alpha - z|| + |f((y_\alpha - y) \circ z)| \\ &\leq M\|f\| ||z_\alpha - z|| + |f(T_z(y_\alpha - y))|, \end{aligned} \tag{5.4}$$

where M is the bound of the net (y_α). As $y_\alpha \to y$ in the *weak**-topology and T_z is *weak**-continuous, $T_z(y_\alpha - y) \to 0$ in *weak**-topology. Therefore, $f(T_z(y_\alpha - y)) \to 0$. As $\|z_\alpha - z\| \to 0$, $\lim f(y_\alpha \circ z_\alpha - y \circ z) = 0$ by the inequalities (i). □

Lemma 5.6. *Let $(\Upsilon, \geq)$ be a directed set and let (w_α), (v_α), and (y_α), where $\alpha \in \Upsilon$, be nets in a JBW^*-algebra $\mathscr{J}$ such that (w_α), (v_α), and (y_α) are all bounded and (y_α) is weak*-convergent to y and w_α, v_α converge to w, v in norm topology, respectively. Then $\{w_\alpha v_\alpha y_\alpha\}$ converges to $\{wvy\}$ in weak*-topology.*

Proof. For all $\alpha \in \Upsilon$,

$$\{w_\alpha v_\alpha y_\alpha\} = w_\alpha \circ (v_\alpha \circ y_\alpha) - v_\alpha \circ (w_\alpha \circ y_\alpha) + y_\alpha \circ (v_\alpha \circ w_\alpha). \tag{5.5}$$

As $v_\alpha \circ w_\alpha \to v \circ w$ in norm topology, $y_\alpha \circ (v_\alpha \circ w_\alpha)$ converges to $y \circ (v \circ w)$ and $w_\alpha \circ y_\alpha$ converges to $w \circ y$ in *weak**-topology by Lemma 5.5. Also note that (by our hypothesis about boundedness of $\{y_\alpha\}$ and $\{w_\alpha\}$) there exists number K such that $\|w_\alpha \circ y_\alpha\| \le K$ for all $\alpha \in \Upsilon$. Repeating the application of Lemma 5.5, we get following convergence in *weak**-topology

$$v_\alpha \circ (w_\alpha \circ y_\alpha) \longrightarrow v \circ (w \circ y). \tag{5.6}$$

Similarly,

$$w_\alpha \circ (v_\alpha \circ y_\alpha) \longrightarrow w \circ (v \circ y). \tag{5.7}$$

We conclude that

$$\{w_\alpha v_\alpha y_\alpha\} \longrightarrow w \circ (v \circ y) - v \circ (w \circ y) + y \circ (v \circ w) = \{wvy\} \tag{5.8}$$

in the *weak**-topology. □

Although we do not require it, under the hypothesis of the previous lemma it follows that $\{w_\alpha y_\alpha v_\alpha\} \to \{wyv\}$ in the *weak**-topology.

The next theorem gives a sufficient condition for a selfadjoint element of any JB^*-algebra $\mathcal{J}$ to be positive in some unitary isotope of $\mathcal{J}$. Using this sufficient condition of positivity, we will deduce that every element of any finite-dimensional JB^*-algebra is positive in some unitary isotope of the algebra.

Theorem 5.7. *Suppose that there are a sequence (x_n) in the JB^*-algebra $\mathcal{J}$ and a sequence (u_n) in $\mathcal{U}(\mathcal{J})$ such that $x_n \to x \in \mathcal{J}$ and $u_n \to u \in \mathcal{U}(\mathcal{J})$ as $n \to \infty$. If each x_n is positive in the u_n-isotope $\mathcal{J}^{[u_n]}$ (so that x is selfadjoint in the u-isotope $\mathcal{J}^{[u]}$, see the proof of Theorem 5.2(1)), then x is positive in the $\mathcal{J}^{[u]}$.*

Proof. Let each x_n be positive in the u_n-isotope $\mathcal{J}^{[u_n]}$. Then x is selfadjoint in the u-isotope $\mathcal{J}^{[u]}$. Let $\lambda > 0$. Then it is sufficient to prove that $-\lambda$ is not in the spectrum of x relative to the isotope $\mathcal{J}^{[u]}$. For this we proceed as follows.

Since x_n is positive in the isotope $\mathcal{J}^{[u_n]}$ and since $\lambda > 0$, we see that $x_n + \lambda u_n$ has a multiplicative inverse y_n (say) in the isotope $\mathcal{J}^{[u_n]}$, for each n. Then, by the functional calculus for selfadjoint elements,

(i)

$$||y_n|| \le \frac{1}{\lambda} \tag{5.9}$$

because $1/\lambda < \|y_n\|$ gives $\lambda > \|y_n\|^{-1} = \|(x_n + \lambda u_n)^{-1}\|^{-1} = \inf\{\rho : \rho \in \sigma_{\mathcal{J}^{[u_n]}}(x_n + \lambda u_n)\} = \inf\{\rho + \lambda : \rho \in \sigma_{\mathcal{J}^{[u_n]}}(x_n)\} \ge \lambda$, which is a contradiction.

Now, since y_n is the inverse of $x_n + \lambda u_n$ in the JB^*-algebra $\mathcal{J}^{[u_n]}$, we have (see Section 1, for details)

(ii)

$$x_n+\lambda u_n = ((x_n+\lambda u_n)\circ_{u_n}(x_n+\lambda u_n))\circ_{u_n} y_n = \{\{(x_n+\lambda u_n)u_n^*(x_n+\lambda u_n)\}u_n^* y_n\}, \tag{5.10}$$

(iii)

$$u_n = (x_n+\lambda u_n)\circ_{u_n} y_n = \{(x_n+\lambda u_n)u_n^* y_n\}. \tag{5.11}$$

Let $\mathcal{B}_{1/\lambda}$ denote the closed ball with center 0 and radius $1/\lambda$ in the Banach second dual $(\mathcal{J}^{[u]})^{**}$, a JBW^*-algebra (see details in [22]), of the JB^*-algebra $\mathcal{J}^{[u]}$ (recall that the Banach space structure of any unitary isotope of a JB^*-algebra coincides, by its construction, with the original one). Then, by the well-known Banach-Alaoglu theorem (cf. [13]), $\mathcal{B}_{1/\lambda}$ is *weak**-compact. Hence, by (i), there exists a *weak**-limit y (say) of a subnet (y_α) of the sequence (y_n).

Now, taking the *weak**-limits of the corresponding subnets of (ii) and (iii), we get by Lemma 5.6 that

(iv)

$$x+\lambda u = \{\{(x+\lambda u)u^*(x+\lambda u)\}u^* y\} = ((x+\lambda u)\circ_u(x+\lambda u))\circ_u y, \tag{5.12}$$

(v)

$$u = \{(x+\lambda u)u^* y\} = (x+\lambda u)\circ_u y, \tag{5.13}$$

respectively.

From equations (iv) and (v), we deduce that $-\lambda$ is not in the spectrum of x relative to the double dual of the JB^*-algebra $\mathcal{J}^{[u]}$. It follows (from Lemma 2.1) that $-\lambda \notin \sigma_{\mathcal{J}^{[u]}}(x)$. □

Remark 5.8. As seen above that $x+\lambda u$ is invertible in $\mathcal{J}^{[u]}$ for $\lambda<0$, we get $x+\lambda u\in\mathcal{J}_{\mathrm{inv}}$ by Lemma 4.2. Let $(x+\lambda u)^{-1}$ denote the inverse of $x+\lambda u$ in $\mathcal{J}$. Then $\{u(x+\lambda u)^{-1}u\}\in\mathcal{J}^{[u]}$. Moreover, by using the Jordan identities,

$$\{ab^{-1}\{ba^{-1}b\}\} = b, \qquad \{\{ab^{-1}a\}b^{-1}\{ba^{-1}b\}\} = a, \tag{5.14}$$

we get that

$$(x+\lambda u)\circ_u\{u(x+\lambda u)^{-1}u\} = u, \qquad (x+\lambda u)^2\circ_u\{u(x+\lambda u)^{-1}u\} = x+\lambda u. \tag{5.15}$$

Hence $\{u(x+\lambda u)^{-1}u\}$ being the unique inverse of $x+\lambda u$ in the isotope $\mathcal{J}^{[u]}$ coincides with y in the proof of Theorem 5.7.

Theorem 5.9. *Every element of any finite-dimensional JB^*-algebra $\mathcal{J}$ is positive in some unitary isotope of $\mathcal{J}$.*

The proof follows from Theorems 4.12, 5.2, and 5.7.

6. Coincidence of $\mathscr{E}(\mathscr{J})$ with $\mathscr{U}(\mathscr{J})$

Pedersen proved in [6] that any extreme point of the closed unit ball of a C^*-algebra having the distance to the invertible elements strictly less than 1 is a unitary, and as an immediate consequence to this result, he got the coincidence of the set of extreme points of the closed unit ball and the set of all unitaries in a C^*-algebra of topological stable rank 1 (see [6, Proposition 3.3, Corollary 3.4]). In this section, we will extend these results for general JB^*-algebra. It should be noted that our approach in proving these results for JB^*-algebras is entirely different from the one given in [6] for C^*-algebras. We will give some characterizations of invertible extreme points of the closed unit ball $(\mathscr{J})_1$ of any JB^*-algebra $\mathscr{J}$. From these, we will immediately get the coincidence of $\mathscr{E}(\mathscr{J})$ with $\mathscr{U}(\mathscr{J})$ in tsr 1 case.

Let $\mathscr{J}$ be a unital JB^*-algebra. We define the number $\alpha(x)$, for all $x \in \mathscr{J}$, called the *distance from x to the invertibles*, by $\alpha(x) = \operatorname{dist}(x, \mathscr{J}_{\text{inv}}) = \inf\{\|x - v\| : v \in \mathscr{J}_{\text{inv}}\}$.

Remark 6.1. Let $u \in \mathscr{U}(\mathscr{J})$. By Lemma 4.2 and Theorem 4.4, we have for any unitary u and $x \in \mathscr{J}$ that $\operatorname{dist}(x, \mathscr{J}_{\text{inv}}) = \operatorname{dist}(x, \mathscr{J}^{[u]}_{\text{inv}})$. Thus, $\alpha(x)$ is invariant in any unitary isotope of $\mathscr{J}$.

We start with some elementary properties of $\alpha(x)$.

Lemma 6.2. *Let $\mathscr{J}$ be a JB^*-algebra with unit e and let $x \in \mathscr{J}$. Then*
(i) $\alpha(x) = \alpha(x^*)$;
(ii) $\alpha(rx) = |r|\alpha(x)$ *for all* $r \in \mathbb{C}$;
(iii) $\alpha(x) \leq \|x\|$;
(iv) $|\alpha(x) - \alpha(y)| \leq \|x - y\|$ *for all* $y \in \mathscr{J}$;
(v) *if* $|r| < \alpha(x)$, *then* $r \in \sigma_{\mathscr{J}^{[u]}}(x)$ *for all* $u \in \mathscr{U}(\mathscr{J})$.

Proof. (i) Since the involution $*$ preserves the invertibility and is isometric (see Section 1), $\operatorname{dist}(x, \mathscr{J}_{\text{inv}}) = \operatorname{dist}(x^*, \mathscr{J}_{\text{inv}})$.

(ii) This is clear as $y \in \mathscr{J}_{\text{inv}}$ if and only if $ry \in \mathscr{J}_{\text{inv}}$ for all $r \neq 0$. However, $\alpha(0) = 0$ so $\alpha(rx) = |r|\alpha(x)$ for all $r \in \mathbb{C}$.

(iii) We observe that $(1/n)e \in \mathscr{J}_{\text{inv}}$ and $\|x - (1/n)e\| \leq \|x\| + 1/n$ for any positive integer n. Hence $\alpha(x) = \operatorname{dist}(x, \mathscr{J}_{\text{inv}}) \leq \|x\|$.

(iv) By definition of $\alpha(x)$, for each $n \in \mathbb{N}$, there exists an element $x_n \in \mathscr{J}_{\text{inv}}$ such that $\|x - x_n\| \leq \alpha(x) + 1/n$. Hence $\alpha(y) \leq \|y - x_n\| \leq \|y - x\| + \|x - x_n\| \leq \|x - y\| + \alpha(x) + 1/n$ for all $n \in \mathbb{N}$; so that $\alpha(y) \leq \|x - y\| + \alpha(x)$. Interchanging x and y, we get $\alpha(x) \leq \|x - y\| + \alpha(y)$. Thus, $|\alpha(x) - \alpha(y)| \leq \|x - y\|$ for all $y \in \mathscr{J}$.

(v) Suppose that $|r| < \alpha(x)$ and $r \notin \sigma_{\mathscr{J}^{[u]}}(x)$ for some $u \in \mathscr{U}(\mathscr{J})$. Then $x - ru \in \mathscr{J}^{[u]}_{\text{inv}}$. By Lemma 4.2, we have $\mathscr{J}^{[u]}_{\text{inv}} = \mathscr{J}_{\text{inv}}$, so that $x - ru \in \mathscr{J}_{\text{inv}}$. However, $\|x - (x - ru)\| = |r| < \alpha(x)$, which is a contradiction. □

Theorem 6.3. *Let x, y be elements of a unital JB^*-algebra $\mathscr{J}$. Let $\epsilon > 0$ and let $\|x - y\| \leq \alpha(x) + \epsilon$. Suppose that there is a unitary $v \in \mathscr{J}$ such that y is positive invertible in the isotope $\mathscr{J}^{[v]}$. If $u = y/\|y\| + i(v - (y/\|y\|)^2)^{1/2}$ where the square and square root are operations in $\mathscr{J}^{[v]}$, then u is unitary in $\mathscr{J}$ and $-\alpha(x) - \epsilon \notin \sigma_{\mathscr{J}^{[u]}}(x)$.*

Proof. Let $z = \|y\|^{-1}y$. Then u is clearly a unitary in the isotope $\mathcal{J}^{[v]}$ so $u \in \mathcal{U}(\mathcal{J})$, by Theorem 4.6.

Suppose that $-\alpha(x) - \epsilon \in \sigma_{\mathcal{J}^{[u]}}(x)$. Let $\mathcal{J}(u,x)$ denote the norm closed Jordan subalgebra generated by x and the identity u in the isotope $\mathcal{J}^{[u]}$. Then $\mathcal{J}(u,x)$ is a commutative Banach algebra and $-\alpha(x) - \epsilon \in \sigma_{\mathcal{J}^{[u]}}(x) \subseteq \sigma_{\mathcal{J}(u,x)}(x)$ by Lemma 2.1. Hence, by [13, Theorem 11.5], there exists a multiplicative linear functional f on the Banach algebra $\mathcal{J}(u,x)$ such that $f(x) = -\alpha(x) - \epsilon$ and $f(u) = 1$. As $\|f\| \le 1$ (see [13, Theorem 10.7]), by the Hahn-Banach theorem there exists g in the Banach dual $(\mathcal{J}^{[u]})^*$ such that $g = -f$ on $\mathcal{J}(u,x)$ and $\|g\| = \|f\|$ (≤ 1).

Let $g(z) = a + ib$ with $a, b \in \mathfrak{R}$. We observe that $|g(u) - g(z)| = |g(u - z)| \le \|u - z\| = \|(v - z^2)^{1/2}\| < 1$ since y (and hence z) is positive invertible in $\mathcal{J}^{[v]}$ and $\|z\|^2 \le 1$. Hence, $|g(u) - g(z)| < 1$. If $a \ge 0$, then $|g(u) - g(z)| = |-f(u) - g(z)| = |-1 - (a + ib)| = ((1 + a)^2 + b^2)^{1/2} \ge 1$, which is a contradiction to the previously observed inequality. This forces us to take $a < 0$. Thus $\mathrm{Re}(g(z)) < 0$. As $z = \|y\|^{-1}y$ and g is linear, $\mathrm{Re}(g(y)) < 0$.

Now, since $\mathrm{Re}(g(y)) < 0$ and since $g(x) = -f(x) = \alpha(x) + \epsilon > 0$, we get that

$$|g(x - y)| = |\mathrm{Re}(g(y)) + i\,\mathrm{Im}(g(y)) - g(x)| \ge |\mathrm{Re}(g(y)) - g(x)| > \alpha(x) + \epsilon. \tag{6.1}$$

However,

$$|g(x - y)| \le \|g\|\|x - y\| \le \alpha(x) + \epsilon. \tag{6.2}$$

This is a contradiction. Hence $-\alpha(x) - \epsilon \notin \sigma_{\mathcal{J}^{[u]}}(x)$. □

Corollary 6.4. *Let $\mathcal{J}$ be a unital JB^*-algebra, and let $x \in \mathcal{J}$. Let $S = \cap\{\sigma_{\mathcal{J}^{[u]}}(x) : u \in \mathcal{U}(\mathcal{J})\}$. Then $-\alpha(x) \in \mathrm{bdry}(S)$.*

Proof. Let $\epsilon > 0$. Then there exists $y \in \mathcal{J}_{\mathrm{inv}}$ such that $\|x - y\| < \alpha(x) + \epsilon$. By Theorem 4.12, y is positive in some unitary isotope $\mathcal{J}^{[v]}$. Then, by Theorem 6.3, $-\alpha(x) - \epsilon \notin S$. However, by Lemma 6.2(v), $\{\lambda \in \mathscr{C} : |\lambda| < \alpha(x)\} \subseteq S$. Hence, $-\alpha(x) \in \mathrm{bdry}(S)$. □

Corollary 6.5. *If x is an element of a unital JB^*-algebra $\mathcal{J}$ with $\|x - y\| < 1$ for some $y \in \mathcal{J}_{inv}$, then $\sigma_{\mathcal{J}^{[u]}}(x) \neq \{c \in \mathbb{C} : |c| \le 1\}$ for some $u \in \mathcal{U}(\mathcal{J})$.*

Proof. By Theorem 6.3, for any $\epsilon > 0$ there is a unitary $u \in \mathcal{U}(\mathcal{J})$ such that $-\alpha(x) - \epsilon \notin \sigma_{\mathcal{J}^{[u]}}(x)$. Since $\|x - y\| < 1$ with $y \in \mathcal{J}_{\mathrm{inv}}$, $\alpha(x) < 1$, so that $-\alpha(x) - \epsilon \in \{c \in \mathscr{C} : |c| \le 1\}$ for sufficiently small ϵ. This gives the required result. □

Now, we turn to the case when x is some extreme point of the unit ball $(\mathcal{J})_1$.

Lemma 6.6. *Let $\mathcal{J}$ be a JC^*-algebra, and let $x \in \mathscr{E}(\mathcal{J})_1$ with $\alpha(x) < 1$. Then $x \in \mathcal{U}(\mathcal{J})$.*

Proof. Since $\alpha(x) < 1$, there exists some $z \in \mathcal{J}_{\mathrm{inv}}$ such that $\|x - z\| < 1$. Considering $\mathcal{J}$ as a JC^*-algebra of bounded linear operators on some Hilbert space, we have $x^*xx^* = x^*$ since $x \in \mathscr{E}(\mathcal{J})_1$ where the juxtaposition of operators denotes the usual product of the operators. Then

$$\begin{aligned} ||x^*xz^{-1}(e - xx^*)|| &= ||x^*xz^{-1}(e - xx^*) - x^*zz^{-1}(e - xx^*)|| \\ &\le ||x^*||\|x - z\|||z^{-1}(e - xx^*)||. \end{aligned} \tag{6.3}$$

Now, if $xx^* \neq e$, then this in the presence of the above inequality implies $\|x^*xz^{-1}(e - xx^*)\| < \|z^{-1}(e - xx^*)\|$ and hence $0 \neq (e - x^*x)z^{-1}(e - xx^*)$, which is a contradiction to a famous characterization of extreme points (cf. [23, 26]). Thus $xx^* = e$. Similarly $x^*x = e$. Hence, $x \in \mathcal{U}(\mathcal{J})$. □

Now, we are in position to give the following important characterization of the invertible extreme points of the closed unit ball of a unital JB^*-algebra. This generalizes [6, Proposition 3.3].

Theorem 6.7. *Let $\mathcal{J}$ be a JB^*-algebra with identity element e. Then, for any $x \in \mathcal{E}(\mathcal{J})_1$, the following conditions are equivalent:*

(i) $x \in \mathcal{U}(\mathcal{J})$;

(ii) $\sigma_{\mathcal{J}}(x) \neq \{c \in \mathcal{C} : |c| \leq 1\}$;

(iii) $\alpha(x) < 1$.

Proof. (i) ⇒ (ii): if $x \in \mathcal{U}(\mathcal{J})$, then $\sigma_{\mathcal{J}}(x)$ is contained in the unit circle, hence $\sigma_{\mathcal{J}}(x) \neq \{c \in \mathcal{C} : |c| \leq 1\}$.

(ii)⇒(iii): if $\sigma_{\mathcal{J}}(x) \neq \{c \in \mathcal{C} : |c| \leq 1\}$, then there exists $c \in \mathcal{C} \setminus \sigma_{\mathcal{J}}(x)$ such that $|c| < 1$ (for otherwise, $\{c \in \mathcal{C} : |c| < 1\} \subseteq \sigma_{\mathcal{J}}(x)$ together with the compactness of the spectrum implies $\sigma_{\mathcal{J}}(x) = \{c \in \mathcal{C} : |c| \leq 1\}$) so that $x - ce$ is invertible. Hence, $\|x - (x - ce)\| = |c| < 1$ gives the inequality $\alpha(x) < 1$.

(iii)⇒(i): let $\alpha(x) < 1$. Then, by definition of the distance $\alpha(x)$, there exists at least one $y \in \mathcal{J}_{\text{inv}}$ such that $\|x - y\| < 1$. Hence, by Corollary 6.5, $\sigma_{\mathcal{J}^{[u]}}(x) \neq \{c \in \mathcal{C} : |c| \leq 1\}$ for some $u \in \mathcal{U}(\mathcal{J})$. Then there is some c in $\mathcal{C} \setminus \sigma_{\mathcal{J}^{[u]}}(x)$ with $|c| < 1$ such that $z =_{\text{defn}} x - cu$ is invertible in $\mathcal{J}^{[u]}$. Let $\mathcal{J}(u, z, z^{*_u}, z^{-1_u}, z^{*_u^{-1_u}})$ denote the JB^*-subalgebra of $\mathcal{J}^{[u]}$ generated by z, z^{*_u} (the adjoint of z in the isotope $\mathcal{J}^{[u]}$), their inverses and the identity element u in $\mathcal{J}^{[u]}$. By Corollary 2.5, $\mathcal{J}(u, z, z^{*_u}, z^{-1_u}, z^{*_u^{-1_u}})$ is a JC^*-algebra. We observe that both $x = z + cu$ and $x^{*_u} = z^{*_u} + \bar{c}u$ are elements of the JC^*-algebra $\mathcal{J}(u, z, z^{*_u}, z^{-1_u}, z^{*_u^{-1_u}})$, $x \in \mathcal{E}(\mathcal{J}(u, z, z^{*_u}, z^{-1_u}, z^{*_u^{-1_u}}))_1$ and z is invertible in $\mathcal{J}(u, z, z^{*_u}, z^{-1_u}, z^{*_u^{-1_u}})$ such that $\|x - z\| = |c| < 1$. Hence, by Lemma 6.6, x is a unitary in $\mathcal{J}(u, z, z^{*_u}, z^{-1_u}, z^{*_u^{-1_u}})$ and hence in the isotope $\mathcal{J}^{[u]}$. Thus, $x \in \mathcal{U}(\mathcal{J})$ by Theorem 4.6. □

Remark 6.8. A result similar to the one given above in Theorem 6.7 for C^*-algebras has been obtained by Berntzen (see [27, Proposition 4.3]).

As a first consequence of the above result, we extend our previous result appeared as in [10, Example 2.8].

Corollary 6.9. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in \mathcal{J}$ with $\|x - v\| < 1$ for some nonunitary extreme point $v \in (\mathcal{J})_1$. Then x is not selfadjoint in any unitary isotope of $\mathcal{J}$.*

Proof. Suppose that x is selfadjoint in the isotope $\mathcal{J}^{[u]}$, for some $u \in \mathcal{U}(\mathcal{J})$. Then $x + (i/n)u$ is invertible in $\mathcal{J}^{[u]}$. Moreover, from the hypothesis, we have $\|x + (i/n)u - v\| < 1$, for sufficiently large n. So, by Theorem 6.7, v must be a unitary in the isotope $\mathcal{J}^{[u]}$. Hence, by Theorem 4.6 we get $v \in \mathcal{U}(\mathcal{J})$, which is a contradiction. □

Next, we identify the extreme points of the unit ball of a tsr 1 algebra with its unitary elements.

Corollary 6.10. *Let $\mathcal{J}$ be a unital JB^*-algebra of tsr 1. Then*

$$\mathcal{E}(\mathcal{J})_1 = \mathcal{U}(\mathcal{J}). \tag{6.4}$$

Hence, the distance from any $x \in \mathcal{J}$ to $\mathcal{E}(\mathcal{J})_1$ is $dist(x, \mathcal{U}(\mathcal{J}))$.

Proof. Since $\operatorname{tsr}(\mathcal{J}) = 1$, $\mathcal{J}_{\text{inv}}$ is norm dense in $\mathcal{J}$ such that $\operatorname{dist}(x, \mathcal{J}_{\text{inv}}) = 0$ for all $x \in \mathcal{J}$. So that $\alpha(x) < 1$ for all $x \in \mathcal{E}(\mathcal{J})_1$. Hence, by Theorem 6.7, $\mathcal{E}(\mathcal{J})_1 \subseteq \mathcal{U}(\mathcal{J})$. The reverse inclusion is true for any JB^*-algebra since every unitary in $\mathcal{J}$ is an extreme point of the closed unit ball in $\mathcal{J}$. □

Remark 6.11. The converse of the above result is false even for the C^*-algebra $\mathcal{C}_{\mathbb{C}}(X)$, of all complex-valued continuous functions on some nonempty compact Hausdorff space X with supremum norm. $\mathcal{E}(\mathcal{C}_{\mathbb{C}}(X))_1 = \mathcal{U}(\mathcal{C}_{\mathbb{C}}(X))$ (see [28], e.g.) for all $\mathcal{C}_{\mathbb{C}}(X)$ while the topological stable rank of $\mathcal{C}_{\mathbb{C}}(X)$, may not be 1, in general (see [2, Proposition 1.7]).

7. Coincidence of λ-function with λ_u-function

In [11], Aron and Lohman introduced a geometric function, called the λ-function for normed spaces. Another related function, namely, the λ_u-function defined on the unit ball was originally introduced for C^*-algebras by Pedersen [6]. Present author [29] initiated a study of these functions for general JB^*-algebras and gave the computation of $\lambda_u(x)$ for invertibles. In this section, coincidence between the two functions on invertible elements of a JB^*-algebra is established.

Definition 7.1. Let $\mathcal{J}$ be a unital JB^*-algebra. Then the λ-function and λ_u-function are defined on $(\mathcal{J})_1$, respectively, by

$$\begin{aligned} \lambda(x) &= \sup\{\alpha \in [0,1] : x = \alpha v + (1-\alpha)b,\ v \in \mathcal{E}(\mathcal{J})_1,\ b \in (\mathcal{J})_1\}, \\ \lambda_u(x) &= \sup\{0 \le \alpha \le 1 : x = \alpha v + (1-\alpha)y \text{ with } v \in \mathcal{U}(\mathcal{J}),\ y \in (\mathcal{J})_1\}. \end{aligned} \tag{7.1}$$

Remark 7.2. Since $\mathcal{U}(\mathcal{J}) \subseteq \mathcal{E}(\mathcal{J})_1$, we get $\lambda_u(x) \le \lambda(x)$ for all $x \in (\mathcal{J})_1$. Further, both the functions coincide whenever $\mathcal{U}(\mathcal{J}) = \mathcal{E}(\mathcal{J})_1$. In particular, $\lambda_u = \lambda$ if $\operatorname{tsr}(\mathcal{J}) = 1$ by Corollary 6.10.

In [29], present author proved various results on λ_u-function including the one given below.

Lemma 7.3. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in (\mathcal{J})_1$ be invertible. Then $1/2 < \lambda_u(x) = (1/2)(1 + \|x^{-1}\|^{-1})$ and there exist unitaries u_1, u_2 in $\mathcal{U}(\mathcal{J})$ such that $x = \lambda_u(x)u_1 + (1 - \lambda_u(x))u_2$.*

For the proof, see [29, Corollary 6.10 and its proof].

The following theorem gives the coincidence of the λ- and λ_u-functions on the invertible elements of the closed unit ball.

Theorem 7.4. *Let $\mathcal{J}$ be a unital JB^*-algebra and let $x \in (\mathcal{J})_1$ be invertible. Then $\lambda_u(x) = \lambda(x)$.*

Proof. We already know (see Remark 7.2) that $\lambda_u(x) \leq \lambda(x)$ for all $x \in \mathcal{J}$. Since x is invertible, we have from Lemma 7.3 that

(i)

$$\lambda_u(x) = \frac{1}{2}\left(1 + \|x^{-1}\|^{-1}\right) > \frac{1}{2}. \tag{7.2}$$

If $\lambda_u(x) < \lambda(x)$, then (by a property of the supremum $\lambda(x)$) there is a pair $(v,b) \in \mathcal{E}(\mathcal{J})_1 \times (\mathcal{J})_1$ such that

(ii)

$$x = \lambda_o v + (1 - \lambda_o)b \quad \text{with}\ \lambda(x) \geq \lambda_o > \lambda_u(x) > \frac{1}{2} \tag{7.3}$$

by (i). Setting $y = \lambda_o^{-1}x$, we have $y \in \mathcal{J}_{\text{inv}}$ and $\|v - y\| < \|(1 - \lambda_o^{-1})b\| \leq \lambda_o^{-1} - 1 < 1$ by (ii). Hence, by Theorem 6.7, the extreme point v is in $\mathcal{U}(\mathcal{J})$. This is a contradiction, so $\lambda_u(x) \geq \lambda(x)$. $\square$

Remark 7.5. For any invertible element x of the unit ball of a JB^*-algebra, we have by Lemma 7.3 that $\lambda_u(x) \in \{0 \leq \alpha \leq 1 : x = \alpha v + (1 - \alpha)y \text{ with } v \in \mathcal{U}(\mathcal{J}),\ y \in (\mathcal{J})_1\}$ and hence, $\lambda(x) \in \{\lambda \in [0,1] : x = \lambda v + (1 - \lambda)b : v \in \mathcal{E}(\mathcal{J})_1,\ b \in (\mathcal{J})_1\}$ by Theorem 7.4. Unfortunately, this is not true for noninvertibles even in C^*-algebras (see [6, Proposition 6.2]).

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