Research Article
Meir-Keeler Contractions of Integral Type Are Still Meir-Keeler Contractions

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We prove that the recent fixed point theorem for contractions of integral type due to Branciari is a corollary of the famous Meir-Keeler fixed point theorem. We also prove that Meir-Keeler contractions of integral type are still Meir-Keeler contractions.

1. Introduction

It is well known that the Banach contraction principle [1] is a very useful, simple, and classical tool in nonlinear analysis. Also, this principle has many generalizations; see [2] and others. For example, Meir and Keeler [3] proved the following fixed point theorem.

THEOREM 1.1 (Meir and Keeler [3]). *Let (X, d) be a complete metric space and let T be a Meir-Keeler contraction (MKC) on X, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(x, y) < \varepsilon + \delta \quad \text{implies } d(Tx, Ty) < \varepsilon \tag{1.1}$$

for all $x, y \in X$. Then T has a unique fixed point.

The following is a slight generalization of Theorem 1.1.

THEOREM 1.2 (see Ćirić [4], Jachymski [5], and Matkowski [6, 7]). *Let (X, d) be a complete metric space and let T be a CJMC on X, that is, the following hold:*
   (A3) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \varepsilon + \delta$ implies $d(Tx, Ty) \leq \varepsilon$;*
   (A4) *$x \neq y$ implies that $d(Tx, Ty) < d(x, y)$.*
Then T has a unique fixed point.

See also [8]. Branciari [9] proved the following fixed point theorem.

THEOREM 1.3 (Branciari [9]). *Let (X, d) be a complete metric space and let T be a Branciari contraction on X, that is, there exist $r \in [0, 1)$ and a locally integrable function f from $[0, \infty)$*

into itself such that

$$\int_0^s f(t)dt > 0, \qquad \int_0^{d(Tx,Ty)} f(t)dt \le r \int_0^{d(x,y)} f(t)dt \tag{1.2}$$

for all $s > 0$ and $x, y \in X$. Then T has a unique fixed point.

We note that when f is a constant function, Theorem 1.3 becomes the Banach contraction principle. Also, Theorems 1.1 and 1.2 become the principle when for each $\varepsilon > 0$, we can take $\delta > 0$ such that δ/ε is constant.

It is a natural question whether Theorems 1.1 and 1.3 are independent or not. In this paper, we will prove that Theorem 1.1 includes Theorem 1.3. That is, Branciari contractions are MKC. Moreover, we show that MKC of integral type are still MKC.

2. Main results

In this section, we prove that MKC and CJMC of integral type are still MKC and CJMC, respectively.

THEOREM 2.1. *Let (X, d) be a metric space and let T be a mapping on X. Assume that there exists a function θ from $[0, \infty)$ into itself satisfying the following:*
  (B1) *$\theta(0) = 0$ and $\theta(t) > 0$ for every $t > 0$;*
  (B2) *θ is nondecreasing and right continuous;*
  (B3) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\theta(d(x, y)) < \varepsilon + \delta \quad \text{implies} \quad \theta(d(Tx, Ty)) < \varepsilon \tag{2.1}$$

  for all $x, y \in X$.
Then T is an MKC.

Remark 2.2. The condition of right continuity of θ is essential. However, without assuming this condition, T still has a unique fixed point when X is complete, see Example 2.6 and Theorem 2.7.

Proof. Fix $\varepsilon > 0$. Since $\theta(\varepsilon) > 0$, there exists $\alpha > 0$ such that

$$\theta(d(u, v)) < \theta(\varepsilon) + \alpha \quad \text{implies} \quad \theta(d(Tu, Tv)) < \theta(\varepsilon). \tag{2.2}$$

From the right continuity of θ, there exists $\delta > 0$ such that $\theta(\varepsilon + \delta) < \theta(\varepsilon) + \alpha$. Fix $x, y \in X$ with $d(x, y) < \varepsilon + \delta$. Then we have

$$\theta(d(x, y)) \le \theta(\varepsilon + \delta) < \theta(\varepsilon) + \alpha, \tag{2.3}$$

and hence $\theta(d(Tx, Ty)) < \theta(\varepsilon)$. Therefore $d(Tx, Ty) < \varepsilon$ holds. This completes the proof.
□

Since a function $s \mapsto \int_0^s f(t)dt$ is absolutely continuous, we obtain the following.

COROLLARY 2.3. *Let (X, d) be a metric space and let T be a mapping on X. Let f be a locally integrable function from $[0, \infty)$ into itself satisfying $\int_0^s f(t)dt > 0$ for all $s > 0$. Assume that*

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^{d(x,y)} f(t)dt < \varepsilon + \delta \quad \text{implies} \quad \int_0^{d(Tx,Ty)} f(t)dt < \varepsilon \tag{2.4}$$

for all $x, y \in X$. Then T is an MKC.

Corollary 2.4. *Let X, d, and T satisfy all the assumptions in Theorem 1.3. Then T is an MKC.*

Remark 2.5. That is, Theorem 1.1 includes Theorem 1.3.

Example 2.6. Define a complete metric space (X, d) by $X = [0, \infty)$ and $d(x, y) = x + y$ for $x, y \in X$ with $x \neq y$. Define a mapping T on X and a function θ from $[0, \infty)$ into itself by

$$Tx = \begin{cases} 0 & \text{if } x \le 1, \\ 1 & \text{if } x > 1, \end{cases} \qquad \theta(t) = \begin{cases} t & \text{if } t \le 1, \\ 1 + t & \text{if } t > 1, \end{cases} \tag{2.5}$$

for $x \in X$ and $t \ge 0$. Then all the assumptions of Theorem 2.1 except the right continuity of θ are satisfied. However, T is not an MKC.

Proof. We first show (B3). We note that $\{d(Tx, Ty) : x, y \in X\} = \{0, 1\}$, and hence $\{\theta(d(Tx, Ty)) : x, y \in X\} = \{0, 1\}$. If $\theta(d(Tx, Ty)) = 1$ and $x < y$, then we have $0 \le x \le 1$ and $1 < y$, and hence $\theta(d(x, y)) = \theta(x + y) \ge \theta(y) > 2$. Therefore,

$$\theta(d(Tx, Ty)) \le \frac{1}{2}\theta(d(x, y)) \tag{2.6}$$

holds for all $x, y \in X$. This implies (B3). We next show that T is not an MKC. Put $\varepsilon = 1$ and let $\delta > 0$ be arbitrary. We also put $x = 0$ and $y = 1 + \delta/2$. Then

$$d(x, y) = 1 + \frac{\delta}{2} < \varepsilon + \delta, \qquad d(Tx, Ty) = d(0, 1) = 1 \ge \varepsilon \tag{2.7}$$

hold. We have shown that T is not an MKC. This completes the proof. □

Next, we discuss CJMC.

Theorem 2.7. *Let (X, d) be a metric space and let T be a mapping on X. Assume that there exists a function θ from $[0, \infty)$ into itself satisfying (B1) and the following:*
  (C2) *θ is nondecreasing;*
  (C3) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\theta(d(x, y)) < \varepsilon + \delta$ implies $\theta(d(Tx, Ty)) \le \varepsilon$;*
  (C4) *$x \neq y$ implies $\theta(d(Tx, Ty)) < \theta(d(x, y))$.*
Then T is a CJMC.

Remark 2.8. We do not assume the right continuity of θ. From this, the author thinks that Theorem 1.2 is possibly more natural than Theorem 1.1.

Proof. It is obvious that (C4) implies (A4). We will prove (A3). Fix $\varepsilon > 0$ and put $\beta = \lim_{t \to \varepsilon + 0} \theta(t)$. We consider the following two cases:

(i) $\beta < \theta(\varepsilon + \gamma)$ holds for every $\gamma > 0$;

(ii) there exists $\delta_2 > 0$ such that $\beta = \theta(\varepsilon + \delta_2)$.

In the first case, from (C3), there exists $\alpha > 0$ such that

$$\theta(d(u,v)) < \beta + \alpha \quad \text{implies } \theta(d(Tu,Tv)) \leq \beta. \tag{2.8}$$

We can choose $\delta_1 > 0$ satisfying $\theta(\varepsilon + \delta_1) < \beta + \alpha$. Fix $x, y \in X$ with $d(x,y) < \varepsilon + \delta_1$. Then we have $\theta(d(x,y)) \leq \theta(\varepsilon + \delta_1) < \beta + \alpha$, and hence $\theta(d(Tx,Ty)) \leq \beta$. This implies $d(Tx,Ty) \leq \varepsilon$. In the second case, we also fix $x, y \in X$ with $d(x,y) < \varepsilon + \delta_2$. If $d(Tx,Ty) > \varepsilon$, then we have

$$\beta \leq \theta(d(Tx,Ty)) < \theta(d(x,y)) \leq \beta. \tag{2.9}$$

This is a contradiction. Therefore we obtain $d(Tx,Ty) \leq \varepsilon$. This completes the proof. □

As a direct consequence of Theorem 2.7, we obtain the following.

Corollary 2.9. *Let (X,d) be a metric space and let T be a mapping on X. Let f be a locally integrable function from $[0,\infty)$ into itself satisfying $\int_0^s f(t)dt > 0$ for all $s > 0$. Assume that for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_0^{d(x,y)} f(t)dt < \varepsilon + \delta \quad \text{implies} \quad \int_0^{d(Tx,Ty)} f(t)dt \leq \varepsilon,$$
$$x \neq y \quad \text{implies} \quad \int_0^{d(Tx,Ty)} f(t)dt < \int_0^{d(x,y)} f(t)dt \tag{2.10}$$

for all $x, y \in X$. Then T is a CJMC.

3. Additional result

We finally prove the τ-distance version of Theorem 1.2. In [10], Suzuki introduced the notion of τ-distances.

Definition 3.1 (see [10]). Let (X,d) be a metric space. Then a function p from $X \times X$ into $[0,\infty)$ is called a τ-distance on X if there exists a function η from $X \times [0,\infty)$ into $[0,\infty)$ and the following are satisfied:

(τ1) $p(x,z) \leq p(x,y) + p(y,z)$ for all $x, y, z \in X$;

(τ2) $\eta(x,0) = 0$ and $\eta(x,t) \geq t$ for all $x \in X$ and $t \in [0,\infty)$, and η is concave and continuous in its second variable;

(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply that $p(w,x) \leq \liminf_n p(w, x_n)$ for all $w \in X$;

(τ4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply that $\lim_n \eta(y_n, t_n) = 0$;

(τ5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply that $\lim_n d(x_n, y_n) = 0$.

The metric d is a τ-distance on X. Many useful examples and propositions are stated in [10–15] and references therein. The following is the τ-distance version of Theorem 1.1.

Theorem 3.2 (see [12]). *Let X be a complete metric space with a τ-distance p, and let T be a mapping on X. Suppose that T is a Meir-Keeler contraction with respect to p (p-MKC), that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$p(x,y) < \varepsilon + \delta \quad implies \quad p(Tx,Ty) < \varepsilon \tag{3.1}$$

for all $x, y \in X$. Then T has a unique fixed point z in X. Further, such z satisfies $p(z,z) = 0$ and $\lim_n T^n x = z$ for all $x \in X$.

Using Theorem 3.2, we prove the following theorem.

Theorem 3.3. *Let X be a complete metric space with a τ-distance p, and let T be a mapping on X. Suppose that T is a CJMC with respect to p (p-CJMC), that is, the following hold:*
   (i) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that $p(x,y) < \varepsilon + \delta$ implies $p(Tx,Ty) \le \varepsilon$;*
   (ii) *$p(x,y) > 0$ implies $p(Tx,Ty) < p(x,y)$.*
Then T has a unique fixed point z in X. Further, such z satisfies $p(z,z) = 0$ and $\lim_n T^n x = z$ for all $x \in X$.

Proof. From the assumption, we note that $p(x,y) = 0$ implies $p(Tx,Ty) = 0$. Hence $p(Tx, Ty) \le p(x,y)$ holds for all $x, y \in X$. We will prove that T^2 is a p-MKC. Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that
   (i) $p(u,v) < \varepsilon + \delta$ implies $p(Tu,Tv) \le \varepsilon$.
Fix $x, y \in X$ with $p(x,y) < \varepsilon + \delta$. Then $p(Tx,Ty) \le \varepsilon$. In the case where $p(Tx,Ty) = 0$, we have $p(T^2 x, T^2 y) = 0$. That is, $p(T^2 x, T^2 y) < \varepsilon$. In the other case, where $p(Tx,Ty) > 0$, we have $p(T^2 x, T^2 y) < p(Tx,Ty) \le \varepsilon$. Therefore, T^2 is a p-MKC. By Theorem 3.2, T^2 has a unique fixed point z in X. Further, such z satisfies $p(z,z) = 0$ and $\lim_n T^{2n} x = z$ for all $x \in X$. Since

$$\lim_{n \to \infty} T^{2n} x = z, \qquad \lim_{n \to \infty} T^{2n+1} x = \lim_{n \to \infty} T^{2n} \circ Tx = z, \tag{3.2}$$

we obtain $\lim_n T^n x = z$ for all $x \in X$. We also have

$$z = \lim_{n \to \infty} T^n z = \lim_{n \to \infty} T^{2n+1} z = \lim_{n \to \infty} T \circ T^{2n} z = \lim_{n \to \infty} Tz = Tz. \tag{3.3}$$

That is, z is a fixed point of T. Since z is a unique fixed point of T^2, z is a unique fixed point of T. This completes the proof. □

Remark 3.4. Jachymski [16] proved that if T is a CJMC, then T^2 is an MKC.

We finally point out that p-MKC and p-CJMC of integral type are still p-MKC and p-CJMC, respectively.

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