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*Research Article*

# The $k$-Zero-Divisor Hypergraph of a Commutative Ring

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The concept of the zero-divisor graph of a commutative ring has been studied by many authors, and the $k$-zero-divisor hypergraph of a commutative ring is a nice abstraction of this concept. Though some of the proofs in this paper are long and detailed, any reader familiar with zero-divisors will be able to read through the exposition and find many of the results quite interesting. Let $R$ be a commutative ring and $k$ an integer strictly larger than 2. A $k$-uniform hypergraph $H_k(R)$ with the vertex set $Z(R,k)$, the set of all $k$-zero-divisors in $R$, is associated to $R$, where each $k$-subset of $Z(R,k)$ that satisfies the $k$-zero-divisor condition is an edge in $H_k(R)$. It is shown that if $R$ has two prime ideals $P_1$ and $P_2$ with zero their only common point, then $H_k(R)$ is a bipartite (2-colorable) hypergraph with partition sets $P_1 - Z'$ and $P_2 - Z'$, where $Z'$ is the set of all zero divisors of $R$ which are not $k$-zero-divisors in $R$ . If $R$ has a nonzero nilpotent element, then a lower bound for the clique number of $H_3(R)$ is found. Also, we have shown that $H_3(R)$ is connected with diameter at most 4 whenever $x^2 \neq 0$ for all 3-zero-divisors $x$ of $R$. Finally, it is shown that for any finite nonlocal ring $R$, the hypergraph $H_3(R)$ is complete if and only if $R$ is isomorphic to $Z_2 \times Z_2 \times Z_2$.



## 1. Introduction

The notion of a zero-divisor graph $\Gamma(R)$ of a commutative ring $R$ was first introduced by Beck in [1] and was further investigated in [2], where the authors were interested in colorings of $\Gamma(R)$, though their vertex set included the zero element. In [3–9] the authors, using the set of nonzero zero divisors of $R$ as vertex set of $\Gamma(R)$, were interested in examining the interplay between the ring-theoretic properties of $R$ and the graph-theoretic properties of $\Gamma(R)$. In this paper, we extend the concept of a zero-divisor of a commutative ring $R$ to that of a $k$-zero-divisor and investigate the interplay between the ring-theoretic properties of $R$ and the graph-theoretic properties of its associated $k$-uniform hypergraph $H_k(R)$. In this section, we define and study some examples of $k$-zero-divisors and recall some definitions from graph theory. In Section 2, we define and study some basic properties of the $k$-uniform hypergraph $H_k(R)$ and $k$-zero-divisors of a commutative ring $R$. Finally, in the last section, we merely concentrate on the properties of 3-zero-divisor hypergraphs.

*Definition 1.1.* Let $R$ be a commutative ring and $k \geq 2$ a fixed integer. A nonzero nonunit element $a_1$ in $R$ is said to be a $k$-zero-divisor in $R$ if there exist $k-1$ distinct nonunit elements $a_2, a_3, \ldots, a_k$ in $R$ different from $a_1$ such that $a_1 a_2 a_3 \cdots a_k = 0$ and the product of no elements of any proper subset of $A = \{a_1, a_2, \ldots, a_k\}$ is zero.

Clearly, a 2-zero-divisor in $R$ is a zero divisor, but the converse is not true in general. For example, 2 is a zero divisor in $Z_4$, but it is not a 2-zero-divisor.

*Remark 1.2.* In the literature, on zero-divisor graphs, the edges are defined to be between the distinct nonzero zero-divisors in order to construct a graph with no loops. Here, we assume distinctness of the elements in Definition 1.1 for $k$-zero-divisors in order to have a $k$-uniform hypergraph, for any fixed integer $k \geq 3$. Note that the graph constructed by 2-zero-divisors is exactly the same as the zero-divisor graph of a ring.

*Example 1.3.* The element 2 in $Z_{30}$ is a 3-zero-divisor since $2 \cdot 3 \cdot 5 = 0$, and the product of no elements of any proper subset of $\{2, 3, 5\}$ is zero.

By $Z(R, k)$ we denote the set of all $k$-zero-divisors of $R$. It is not difficult to show that the statement "the product of no elements of any proper subset of $A$ is zero" or the statement "the product of no elements of any $(k-1)$-subset of $A$ is zero" can be used in Definition 1.1 equivalently. Clearly, from Definition 1.1, every element of the set $\{a_2, a_3, \ldots, a_k\}$ is a $k$-zero-divisor in $R$. It is clear that every $k$-zero-divisor in $R$ is also a zero divisor in $R$, but, the converse is not true in general. For example, the element 2 is a zero divisor, but not a 3-zero-divisor in $Z_{10}$.

We review some basic graph-theoretic definitions, and for the necessary definitions and notations of hypergraphs, we refer the reader to standard texts of graph theory such as [10]. A hypergraph is a pair $(V, E)$ of disjoint sets, where the elements of $E$ are nonempty subsets (of any cardinality) of $V$. The elements of $V$ are the vertices, and the elements of $E$ are the edges of the hypergraph. The hypergraph $H = (V, E)$ is called $k$-uniform whenever every edge $e$ of $H$ is of size $k$. A $k$-uniform hypergraph $H$ is called complete if every $k$-subset of the vertices is an edge of $H$. The definition of a clique and the clique number of a $k$-uniform hypergraph are taken from [11, 12] as follows.

Let $H$ be a $k$-uniform hypergraph. A subset $A$ of $V(H)$ is called a *clique* of $H$ if every $k$-subset of $A$ is an edge of $H$. The *clique number* of $H$, denoted by $\omega(H)$, is defined to be

$$\omega(H) = \frac{\max\{|A| \mid A \text{ is a clique}\}}{k-1}. \tag{1.1}$$

An $r$-coloring of a hypergraph $H = (V,E)$ is a map $c : V \to \{1,2,\dots,r\}$ such that for every edge $e$ of $H$, there exist at least two vertices $x$ and $y$ in $e$ with $c(x) \neq c(y)$. The smallest integer $r$ such that $H$ has an $r$-coloring is called the chromatic number of $H$ and is denoted by $\chi(H)$. In [11], it is shown that for any $k$-uniform hypergraph $H$, $\chi(H) \geq \lceil \omega(H) \rceil$. A path in a hypergraph $H$ is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \dots, v_k$ such that $v_i$, $v_{i+1}$ is in $e_i$ for all $1 \leq i \leq k-1$. The number of edges of a path is its length. The distance between two vertices $x$ and $y$ of $H$, denoted by $d_H(x,y)$, is the length of the shortest path from $x$ to $y$. If no such path between $x$ and $y$ exists, we set $d_H(x,y) = \infty$. The greatest distance between any two vertices in $H$ is called the diameter of $H$ and is denoted by $\operatorname{diam}(H)$. The hypergraph $H$ is said to be connected whenever $\operatorname{diam}(H) < \infty$. A cycle in a hypergraph $H$ is an alternating sequence of distinct vertices and edges of the form $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_1$ such that $v_i$, $v_{i+1}$ are in $e_i$ for all $1 \leq i \leq k-1$ with $v_k, v_1 \in e_k$. The girth of a hypergraph $H$ containing a cycle, denoted by $\operatorname{gr}(H)$, is the smallest size of the length of cycles of $H$.

## 2. $k$-zero-divisor hypergraphs

In this section, we define and study some properties of the $k$-uniform hypergraph $H_k(R)$, the $k$-zero-divisors of a commutative ring $R$, and provide some examples.

*Definition 2.1.* A ring $R$ is said to be a $k$-integral domain whenever $Z(R,k)$, the set of all $k$-zero-divisors of $R$, is the empty set.

*Example 2.2.* Let $(R,M)$ be a local ring with maximal ideal $M \neq 0$ such that $M^2 = 0$. Then $R$ is a 3-integral domain which is not an integral domain.

*Example 2.3.* For any integer $k \geq 3$, we have the following results.

(1) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the prime decomposition of $n$, where $p_i \neq p_j$ whenever $i \neq j$ and $1 \leq \alpha_i$ for all $i, j = 1,2,\dots,r$. Then $Z_n$ is a $k$-integral domain whenever $\sum_{i \leq r} \alpha_i \leq k-1$.

(2) Let $n_i = p_{1_i}^{\alpha_{1_i}} p_{2_i}^{\alpha_{2_i}} \cdots p_{r_i}^{\alpha_{r_i}}$ be the prime decomposition of $n_i$ for distinct primes $p_{j_i}$'s and $1 \leq \alpha_{j_i}$ for all $1 \leq i \leq t$ and $j = 1,2,\dots,r$. Then $Z_{n_1} \times Z_{n_2} \times \cdots \times Z_{n_t}$ is a $k$-integral domain whenever

$$\sum_{j \leq r_1} \alpha_{j_1} + \sum_{j \leq r_2} \alpha_{j_2} + \cdots + \sum_{j \leq r_t} \alpha_{j_t} \leq k-1. \tag{2.1}$$

(3) Let $F$ be a field and let $f(x)$ be a polynomial in $F[x]$ such that $f(x) = P_1(x)^{\alpha_1} P_2(x)^{\alpha_2} \cdots P_r(x)^{\alpha_r}$, where $P_i(x) \in F[x]$ are distinct irreducible polynomials and $1 \leq \alpha_i$ for all $1 \leq i \leq r$. Then $F[x]/(f(x))$ is a $k$-integral domain whenever $\sum_{i \leq r} \alpha_i \leq k-1$.

(4) Let $R_i$ be an integral domain for each $i = 1,2,\dots,n$. Then $R = R_1 \times R_2 \times \cdots \times R_n$ is a $k$-integral domain whenever $n \leq k-1$.

By [13], it is true that a nonintegral domain with a finite number of zero divisors is finite. Similarly, we pose the following question for the rings with a finite number of $k$-zero-divisors.

*Question 1.* Does the finiteness of $k$-zero-divisors in a non-$k$-integral domain $R$ imply the finiteness of zero-divisors or, equivalently, finiteness of $R$?

*Definition 2.4.* For any fixed integer $k \geq 3$, an ideal $P$ of a ring $R$ is said to be $k$-prime whenever for any set $A = \{a_1, a_2, \ldots, a_k\}$ of nonzero, distinct, and nonunit elements of $R$, $a_1 a_2 \cdots a_k \in P$ implies that the product of the elements of a proper subset of $A$ is in $P$.

Note that by this definition, every prime ideal of $R$ is a $k$-prime ideal of $R$.

*Example 2.5.* Let $(R_1, M_1)$ and $(R_2, M_2)$ be two local rings with nonzero maximal ideals $M_1$ and $M_2$, respectively. We show that $M_1 \times M_2$ is a 3-prime ideal in $R = R_1 \times R_2$ which is not a prime ideal in $R$. Let $(a_1, b_1)$, $(a_2, b_2)$, and $(a_3, b_3)$ be arbitrary elements in $R_1 \times R_2$, where for each $1 \leq i \leq 3$, $(a_i, b_i)$ is a nonzero nonunit in $R$. Clearly, $(a_1, b_1) \cdot (a_2, b_2) \cdot (a_3, b_3) = (a_1 a_2 a_3, b_1 b_2 b_3) \in M_1 \times M_2$ implies that at least one of the elements $a_i$'s ($b_j$'s) belongs to $M_1(M_2)$ for some $i(j)$ in $\{1,2,3\}$. In this case, there always exists a proper subset of $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ such that the product of its elements belongs to $M_1 \times M_2$. But since $(1,0) \cdot (0,1) \in M_1 \times M_2$ and neither of the elements $(1,0)$ and $(0,1)$ is in $M_1 \times M_2$, then $M_1 \times M_2$ is a 3-prime ideal in $R_1 \times R_2$ which is not a prime ideal in $R$.

The following theorem is similar to the well-known fact on the relationship between prime ideals and integral domains.

Theorem 2.6. *Let $P$ be an ideal in the ring $R$. Then $R/P$ is a $k$-integral domain if $P$ is a $k$-prime ideal.*

The proof follows directly from the definition, and we leave it to the reader.

The converse of the above theorem is not true in general. For example, the ideal $\langle 8 \rangle$ generated by 8 in $Z_{48}$ is not a 3-prime ideal, but $Z_{48}/\langle 8 \rangle$ is a 3-integral domain.

Next, we extend the concept of zero-divisor graph of a commutative ring $R$ to that of a $k$-zero-divisor hypergraph.

*Definition 2.7.* Let $R$ be a commutative ring (with $1 \neq 0$) and let $Z(R,k)$ be the set of all $k$-zero-divisors in $R$. Associate a $k$-uniform hypergraph $H_k(R)$ to $R$ with vertex set $Z(R,k)$, and for distinct elements $x_1, x_2, \ldots, x_k$ in $Z(R,k)$, the set $\{x_1, x_2, \ldots, x_k\}$ is an edge of $H_k(R)$ if and only if $x_1 x_2 \cdots x_k = 0$ and the product of elements of no $(k-1)$-subset of $\{x_1, x_2, \ldots, x_k\}$ is zero.

Clearly, from the above definition we can conclude that for any $k \geq 3$, $H_k(R)$ is the empty set if and only if $R$ is a $k$-integral domain.

Theorem 2.8. *Let $R$ be a non-$k$-integral domain. If there exist prime ideals $P_1$ and $P_2$ in $R$ such that $P_1 \cap P_2 = \{0\}$, then $\chi(H_k(R)) = 2$.*

*Proof.* Since $P_1 \cap P_2 = \{0\}$, then $P_1 \cup P_2$ is equal to the set of all zero divisors of $R$. On the other hand, since each $k$-zero-divisor is also a zero divisor, each $k$-zero-divisor must belong to the prime ideals $P_1$ or $P_2$. Consider the function $c : V(H_k(R)) \to \{1,2\}$ given by

$$c(x) = \begin{cases} 1, & x \in P_1, \\ 2, & x \in P_2. \end{cases} \tag{2.2}$$

In order to prove that $c$ is a 2-coloring of $H_k(R)$, we need to show that there is no edge $e$ in $H_k(R)$ such that every vertex of $e$ obtains the same color. Without loss of generality, let $e = \{x_1, x_2, \dots, x_k\}$ be an edge of $H_k(R)$ such that $c(x_1) = c(x_2) = \cdots = c(x_k) = 1$. Since $x_1 x_2 \cdots x_k = 0 \in P_2$ and $P_2$ is a prime ideal of $R$, then $x_i \in P_2$ for at least one $1 \le i \le k$, which is a contradiction. Therefore, $\chi(H_k(R)) \le 2$. On the other hand, since $R$ is not a $k$-integral domain, then $H_k(R)$ has at least one edge, which implies that $\chi(H_k(R)) \ge 2$, and the proof is complete. □

*Remark 2.9.* From the above theorem, it is clear that $H_k(R)$ is a bipartite hypergraph with partition sets $V(H_k(R)) \cap P_1$ and $V(H_k(R)) \cap P_2$. Note that in [4], it is shown that for any reduced ring $R$, the zero-divisor graph $\Gamma(R)$ is bipartite if and only if there exist two distinct prime ideals $P_1$ and $P_2$ of $R$ such that $P_1 \cap P_2 = \{0\}$. In addition, if $\Gamma(R)$ is bipartite, then it is a complete bipartite graph.

*Remark 2.10.* By considering the ring $R = Z_2 \times Z_2 \times Z_2$, we see that $\chi(H_3(R)) = 2$. But there are no prime ideals $P_1$ and $P_2$ in $R$ satisfying the condition of Theorem 2.8. Therefore, the converse of Theorem 2.8 is not true in general.

THEOREM 2.11. *Let $R = R_1 \times R_2 \times \cdots \times R_n$, where $R_i$ is an integral domain for each $i = 1, 2, \dots, n$.*

(1) *If $n = k$, then $\chi(H_k(R)) = 2$.*

(2) *If $n = k + t$, then $\chi(H_k(R)) \le 2 + t$ for all $t \ge 0$.*

*Proof.* Let $k = n$. We claim that

$$Z(R,k) = \{(a_1, a_2, \dots, a_k) \mid \text{exactly one of the } a_i\text{'s is zero for } 1 \le i \le k\}. \tag{2.3}$$

It is obvious that any $k$-zero-divisor must have at least one zero component. Let $x_1 = (a_{11}, a_{12}, \dots, a_{1k})$ be a $k$-zero-divisor with at least two zero components. Without loss of generality, assume that $a_{11} = a_{12} = 0$. Consequently, there exist $x_2, x_3, \dots, x_k \in V(H_k(R))$ such that $\{x_1, x_2, \dots, x_k\} \in E(H_k(R))$, where $x_i = (a_{i1}, a_{i2}, \dots, a_{ik})$ for all $1 \le i \le k$. Thus, $\prod_{i \ge 1} a_{ij} = 0$ for each $j \ge 3$. Now since $R_j$ is an integral domain, then for each fixed $j \ge 3$, there exists at least one $i_j$ with $1 \le i \le k$ such that $a_{i_j j} = 0$. Let $I$ be the set of all $i_j$'s such that $a_{i_j j} = 0$ for the smallest $i$ in the set $\{1, 2, \dots, k\}$. Thus, we have $x_1 \prod_{i \in I} x_i = 0$ and since $|I| \le k - 2$, we have a contradiction. Now let $x_1 = (a_1, a_2, \dots, a_k) \in R$ such that exactly one and only one of the components is zero. Without loss of generality, assume that $a_1 = 0$. Let $x_i = (1, 1, \dots, 1, 0, 1, 1, \dots, 1)$, where the $i$th component is the only zero component of $x_i$ for $2 \le i \le k$. It is obvious that $\{x_1, x_2, \dots, x_k\} \in E(H_k(R))$ and the claim is true. Consider the function $c : V(H_k(R)) \to \{1, 2\}$ given by

$$c(x) = \begin{cases} 1 & \text{the first component of } x \text{ is zero,} \\ 2 & \text{otherwise.} \end{cases} \tag{2.4}$$

It is easy to see that $c$ is a 2-coloring of $H_k(R)$, and since $H_k(R)$ has at least one edge, $\chi(H_k(R)) = 2$.

For the proof of part 2, assume $n = k + t$ with $t \ge 0$ a fixed integer. The proof is by induction on $t$. From part 1, the first step of induction for $t = 0$ is true. Now, assume that

$t \geq 1$ and the result is true for $k+t$. Let $c : V(H_k(R_1 \times R_2 \times \cdots \times R_{k+t})) \to \{1,2,\ldots,t+2\}$ be a $t+2$-coloring of $H_k(R_1 \times R_2 \times \cdots \times R_{k+t})$. Consider the function $c' : V(H_k(R_1 \times R_2 \times \cdots \times R_{k+t+1})) \to \{1,2,\ldots,t+3\}$ given by

$$c'(x) = \begin{cases} c(x) & \text{the last component of } x \text{ is zero,} \\ t+3 & \text{otherwise.} \end{cases} \tag{2.5}$$

From this, it is not difficult to show that $c'$ is a $(t+3)$-coloring of $H_k(R_1 \times R_2 \times \cdots \times R_{k+t+1})$, and the proof is complete. □

As a very special case of the above theorem, it is easy to show that the chromatic number of $H_3(Z_2^4)$ and $H_3(Z_2^5)$ is 3. Note that the chromatic number of $H_3(Z_2^5)$ is strictly less than $2+(5-3)$, and the chromatic number of $H_3(Z_2^4)$ equal to 3 shows that the bound is sharp.

## 3. 3-zero-divisor hypergraphs

In this section, we only focus on some graph-theoretic properties of $H_3(R)$. We show that $H_3(R)$ is connected with diameter at most 4 provided that $x^2 \neq 0$ for all 3-zero-divisors $x$ in $R$. We find a necessary and sufficient condition for its completeness, and we also find a lower bound for its clique number.

Theorem 3.1. *Let $H_3(R)$ be the 3-zero-divisor hypergraph of a ring $R$ such that $x^2 \neq 0$ for every 3-zero-divisor $x \in R$. Then $H_3(R)$ is connected and*

$$\operatorname{diam}(H_3(R)) \leq 4. \tag{3.1}$$

*Proof.* For the proof of the theorem, it is enough to show that for each two edges $e_1 = \{a_1,a_2,a_3\}$ and $e_2 = \{b_1,b_2,b_3\}$ of $H_3(R)$, there exist edges $e_3$ and $e_4$ which satisfy one of the following conditions:

$$e_3 \cap e_1 \neq \varnothing, \qquad e_3 \cap e_2 \neq \varnothing, \tag{$*_1$}$$

or

$$e_3 \cap e_1 \neq \varnothing, \qquad e_4 \cap e_2 \neq \varnothing, \qquad e_4 \cap e_3 \neq \varnothing. \tag{$*_2$}$$

Consequently, for the rest of the proof, we can always assume that $a_i \neq b_j$ and $a_i \neq -b_j$ for all $i, j \in \{1,2,3\}$. Let $G$ be the bipartite graph constructed as follows: $V(G) = e_1 \cup e_2$ and $a_ib_j \in E(G)$ if and only if $a_ib_j = 0$ in the ring $R$.

Suppose $G$ has two isolated vertices, one in $e_1$ and the other in $e_2$. For example, $\deg_G(a_3) = \deg_G(b_3) = 0$. If there exists an element $c \in \{a_1,a_2,b_1,b_2\}$ such that $a_3b_3c = 0$, then $e_3 = \{a_3,b_3,c\}$ satisfies $(*_1)$. Suppose that this is not the case. If $a_3b_3 \notin \{a_1,a_2,b_1,b_2\}$, then $e_3 = \{a_1,a_2,a_3b_3\}$ and $e_4 = \{b_1,b_2,a_3b_3\}$ satisfy $(*_2)$. Otherwise without loss of generality, assume that $a_3b_3 = a_1$. Then $e_3 = \{a_1,b_1,b_2\}$ satisfies $(*_1)$. The rest of our proof depends on the number of edges of $G$.

*Case 1.* Suppose $|E(G)| \leq 2$. Then $G$ has two isolated vertices, one in $e_1$ and the other in $e_2$.

*Case 2.* Suppose $|E(G)| = 3$. We study this case for four different subcases as follows.

*Case 2.1.* Assume the degree of each vertex of $G$ is one and

$$E(G) = \{a_1b_1, a_2b_2, a_3b_3\}. \tag{3.2}$$

Consider the set $\{a_1, a_2b_3, b_1 + b_2\}$. If $a_1 = a_2b_3$, then $a_1b_2 = 0$ is a contradiction. If $a_1 = b_1 + b_2$, then $b_1a_2a_3 = 0$, and $e_3 = \{b_1, a_2, a_3\}$ satisfies $(*_1)$. If $b_1 + b_2 = a_2b_3$, then $a_1b_2a_3 = 0$, and $e_3 = \{a_1, b_2, a_3\}$ satisfies $(*_1)$. Otherwise, $e_3 = \{a_1, a_2b_3, b_1 + b_2\}$ is an edge. Similarly if we consider the set $\{b_1, a_2b_3, a_1 + a_3\}$, then we find an edge $e_3$ which satisfies $(*_1)$ or $e_4 = \{b_1, a_2b_3, a_1 + a_3\}$ is an edge with $e_3$ and $e_4$ satisfying $(*_2)$.

*Case 2.2.* Assume that the degree of exactly one of the vertices of $G$ is one. Without loss of generality, suppose that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_3\}. \tag{3.3}$$

Consider the set $\{a_2, a_3b_1, a_1 + b_3\}$. If $a_2 = a_3b_1$, then $a_1a_2 = 0$ implies a contradiction. If $a_2 = a_1 + b_3$, then $a_2b_2b_1 = 0$, and $e_3 = \{a_2, b_2, b_1\}$ satisfies $(*_1)$. If $a_1 + b_3 = a_3b_1$, then $a_3b_1b_2b_1 = 0$. In this case if $a_3 = b_1b_2$, then $a_1a_3 = 0$, also, $b_1 = b_1b_2$ implies that $b_1b_3 = 0$, which in both cases we have a contradiction. Therefore, $e_3 = \{a_3, b_1b_2, b_1\}$ is an edge which satisfies $(*_1)$. If none of the above conditions holds, then the set $e_3 = \{a_2, a_3b_1, a_1 + b_3\}$ is an edge. Now consider the set $\{b_2, a_3b_1, b_3\}$. Similarly, we find an edge $e_3$ which satisfies $(*_1)$, or $e_4 = \{a_2, a_3b_1, a_1 + b_3\}$ is an edge where $e_3$ and $e_4$ satisfy $(*_2)$.

*Case 2.3.* Let the degree of two vertices of $G$ be two. Without loss of generality, suppose that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_2\}. \tag{3.4}$$

In this case, $\deg_G(a_3) = \deg_G(b_3) = 0$, and the proof is complete.

*Case 2.4.* Assume that the degree of one vertex of $G$ is three. Without loss of generality, suppose

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3\}. \tag{3.5}$$

Suppose that $a_1^2a_2 \neq 0$. Consider the set $\{a_1a_2 - b_1, a_1, a_3\}$. If $a_1a_2 - b_1 = a_1$, then $b_2b_1 = 0$ is a contradiction. If $a_1a_2 - b_1 = a_3$, then $a_3b_3b_2 = 0$, and therefore $e_3 = \{a_3, b_2, b_3\}$ is an edge satisfying $(*_1)$. In the other case, $e_3 = \{a_1a_2 - b_1, a_1, a_3\}$ is an edge. Similarly, if we consider the set $\{a_1a_2 - b_1, b_2, b_3\}$, we will find an edge $e_3$ that satisfies $(*_1)$, or $e_4 = \{a_1a_2 - b_1, b_2, b_3\}$ is an edge with $e_3$ and $e_4$ that satisfy $(*_2)$. Now let $a_1^2a_2 = 0$. Consider the set $\{a_1 - b_1, a_1, a_2\}$. If $a_1 - b_1 = a_2$, then $a_2b_3b_2 = 0$, and therefore $e_3 = \{a_2, b_2, b_3\}$ is an edge satisfying $(*_1)$. In the other case, $e_3 = \{a_1 - b_1, a_1, a_2\}$ is an edge. Similarly, if we consider the set $\{a_1 - b_1, b_2, b_3\}$, we will find a contradiction, or $e_4 = \{a_1a_2 - b_1, b_2, b_3\}$ is an edge with $e_3$ and $e_4$ that satisfy $(*_2)$.

*Case 3.* Suppose $|E(G)| = 4$. We study this case using four different subcases as follows.

*Case 3.1.* Assume the degree of one vertex of $G$ is three. Without loss of generality, suppose that

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3, a_2b_3\}. \tag{3.6}$$

Consider the set $\{a_3b_1, a_2, a_1 + b_3\}$. If $a_3b_1 = a_2$, then $a_3b_3b_1 = 0$, and therefore $e_3 = \{a_3, b_1, b_3\}$ is an edge satisfying $(*_1)$. If $a_3b_1 = a_1 + b_3$, then $a_1^2 = 0$ is a contradiction. If $a_2 = a_1 + b_3$, then $b_3^2 = 0$ is a contradiction. In the other case, $e_3 = \{a_3b_1, a_2, a_1 + b_3\}$ is an edge. Similarly, if we consider the set $\{a_3b_1, b_2, b_3\}$, we will find an edge $e_3$ that satisfies $(*_1)$, or $e_4 = \{a_3b_1, b_2, b_3\}$ is an edge with $e_3$ and $e_4$ that satisfy $(*_2)$.

*Case 3.2.* Assume that the degree of four vertices of $G$ is two. Without loss of generality, suppose that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}. \tag{3.7}$$

In this case, $\deg_G(a_3) = \deg_G(b_3) = 0$, and the proof is complete.

*Case 3.3.* Let the degree of three vertices of $G$ be two. Suppose without loss of generality that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_2, a_2b_3\}. \tag{3.8}$$

Consider the set $\{a_3b_3, a_1, a_2\}$. If $a_3b_3 = a_1$ or $a_2$, then $a_3b_3b_2 = 0$, and therefore $e_3 = \{a_3, b_2, b_3\}$ is an edge that satisfies $(*_1)$. In the other case, $e_3 = \{a_3b_3, a_1, a_2\}$ is an edge. Similarly, if we consider the set $\{a_3b_3, b_1, b_2\}$, we will find an edge $e_3$ that satisfies $(*_1)$, or $e_4 = \{a_3b_3, b_1, b_2\}$ is an edge with $e_3$ and $e_4$ that satisfy $(*_2)$.

*Case 3.4.* Assume that the degree of two vertices of $G$ is two. In this case, there might be two different nonisomorphic cases. Without loss of generality, for one case we can assume that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_2, a_3b_3\}, \tag{3.9}$$

and in the other case

$$E(G) = \{a_1b_1, a_1b_2, a_2b_3, a_3b_3\}. \tag{3.10}$$

In the first case, consider the set $\{a_3b_1, a_2, a_1 + b_2\}$. If $a_3b_1 = a_2$, then $a_3b_1b_2 = 0$, and therefore $e_3 = \{a_3, b_1, b_2\}$ is an edge that satisfies $(*_1)$. If $a_3b_1 = a_1 + b_2$, then $a_1^2 = 0$ is a contradiction. Also, $a_2 = a_1 + b_2$ implies that $b_2^2 = 0$, which is a contradiction. In the other case, $e_3 = \{a_1 + b_2, a_2, b_1a_3\}$ is an edge. Similarly, if we consider the set $\{a_3b_3, b_1, a_1 + b_2\}$, we will find an edge $e_3$ that satisfies $(*_1)$, or $e_4 = \{a_3b_3, b_1, a_1 + b_2\}$ is an edge with $e_3$ and $e_4$ that satisfy $(*_2)$.

Similarly, for the second case, by considering the sets $\{a_1 + b_1, a_2, a_3\}$ and $\{a_1 + b_1, b_2, b_3\}$, we find an edge $e_3$ that satisfies $(*_1)$, or two edges $e_3$ and $e_4$ that satisfy $(*_2)$.

*Case 4.* Suppose $|E(G)| = 5$. We continue our investigation for five different nonisomorphic subcases as follows.

*Case 4.1.* Without loss of generality, we can assume that

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3, a_2b_1, a_2b_2\}. \tag{3.11}$$

Consider the set $\{a_3b_3, a_2, a_1 + b_2\}$. If $a_3b_3 = a_2$, then $a_3b_3b_2 = 0$, and therefore $e_3 = \{a_3, b_2, b_3\}$ is an edge that satisfies $(*_1)$. If $a_3b_3 = a_1 + b_2$, then $a_1^2 = 0$, which is a contradiction. If $a_1 + b_2 = a_2$, then $b_1b_2 = 0$ is a contradiction. In the other case, $e_3 = \{a_1 + b_2, a_2, a_3b_3\}$ is an edge. Similarly, if we consider the set $\{a_3b_3, b_1, b_2\}$, we will find an edge $e_3$ which satisfies $(*_1)$, or $e_4 = \{a_3b_3, b_1, b_2\}$ is an edge with $e_3$ and $e_4$ that satisfy $(*_2)$.

*Case 4.2.* Assume without loss of generality that

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3, a_2b_1, a_3b_2\}. \tag{3.12}$$

Consider the set $\{a_1 + b_1, a_2, b_2\}$. If $a_1 + b_1 = a_2$, then $b_1^2 = 0$ is a contradiction. If $a_1 + b_1 = b_2$, then $a_1^2 = 0$ implies a contradiction. In the other case, $e_3 = \{a_1 + b_2, a_2, a_3b_3\}$ is an edge that satisfies $(*_1)$.

*Case 4.3.* Assume without loss of generality that

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3, a_2b_1, a_3b_1\}. \tag{3.13}$$

Consider the set $\{a_1 + b_1, a_2, b_2\}$. If $a_1 + b_1 = a_2$, then $b_1^2 = 0$ is a contradiction. If $a_1 + b_1 = b_2$, then $a_2a_3b_2 = 0$, and $e_3 = \{a_2, b_2, a_3\}$ is an edge which satisfies $(*_1)$. In the other case, $e_3 = \{a_1 + b_1, a_2, b_2\}$ is an edge that satisfies $(*_1)$.

*Case 4.4.* Without loss of generality, we can assume that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_1, a_2b_2, a_3b_3\}. \tag{3.14}$$

Consider the set $\{a_3 + b_1, a_1, b_3\}$. If $a_3 + b_1 = a_1$ or $a_3 + b_1 = b_3$, then $a_1a_2b_3 = 0$, and $e_3 = \{a_1, a_2, b_3\}$ is an edge that satisfies $(*_1)$. In the other case, $e_3 = \{a_3 + b_1, a_1, b_3\}$ is an edge which satisfies $(*_1)$.

*Case 4.5.* Assume without loss of generality that

$$E(G) = \{a_1b_1, a_1b_2, a_2b_2, a_2b_3, a_3b_3\}. \tag{3.15}$$

Consider the set $\{a_1 + b_2, a_2, b_1\}$. If $a_1 + b_2 = a_2$, then $b_2^2 = 0$. If $a_1 + b_2 = b_1$, then $a_1^2 = 0$, which is a contradiction. Therefore $e_3 = \{a_1 + b_2, a_2, b_1\}$ is an edge that satisfies $(*_1)$.

*Case 5.* Suppose $|E(G)| = 6$. We study three different nonisomorphic subcases as follows.

*Case 5.1.* Without loss of generality, we can assume that

$$E(G) = \{a_1b_1, a_1b_2, a_1b_3, a_2b_1, a_2b_2, a_3b_1\}. \tag{3.16}$$

Consider the sets $\{a_1+b_1,a_2,a_3\}$ and $\{a_1+b_1,b_2,b_3\}$. If $a_1+b_1=a_2$, then $b_1b_2=0$. If $a_1+b_1=a_3$, then $b_1^2=0$. Also, $a_1+b_1=b_2$ or $a_1+b_1=b_3$ implies that $a_1^2=0$, and in either case, we have a contradiction. Therefore, $e_3=\{a_1+b_1,a_2,a_3\}$ and $e_4=\{a_1+b_1,b_2,b_3\}$ are two edges that satisfy $(*_2)$.

*Case 5.2.* Without loss of generality, we can assume that

$$E(G)=\{a_1b_1,a_1b_2,a_1b_3,a_2b_1,a_2b_2,a_3b_3\}. \tag{3.17}$$

Consider the set $\{a_1+b_3,a_3,b_1\}$. If $a_1+b_3=a_3$, then $b_3^2=0$. Also $a_1^2=0$ whenever $a_1+b_3=b_1$, which is a contradiction. Therefore, $e_3=\{a_1+b_3,a_3,b_1\}$ is an edge that satisfies $(*_1)$.

*Case 5.3.* Assume without loss of generality that

$$E(G)=\{a_1b_1,a_1b_3,a_2b_1,a_2b_2,a_3b_2,a_3b_3\}. \tag{3.18}$$

In this case, similar to the above subcase, $e_3=\{a_1+b_3,a_3,b_1\}$ is an edge which satisfies $(*_1)$.

*Case 6.* Suppose that $7\le|E(G)|\le 9$. In this case, there always exist two vertices with degree three, one from $e_1$ and the other from $e_2$. Let $d_G(a_1)=d_G(b_1)=3$. Consider the sets $\{a_1+b_1,a_2,a_3\}$ and $\{a_1+b_1,b_2,b_3\}$. If $a_1+b_1=a_2$ or $a_3$, then $b_1^2=0$; and if $a_1+b_1=b_2$ or $b_3$, then $a_1^2=0$, which is a contradiction in all cases. Therefore, $e_3=\{a_1+b_1,a_2,a_3\}$ and $e_4=\{a_1+b_1,b_2,b_3\}$ are two edges that satisfy $(*_2)$. □

*Remark 3.2.* From the above theorem and the fact that

$$\operatorname{gr}(H_3(R))\le 2\operatorname{diam}(H_3(R))+1, \tag{3.19}$$

we can conclude that the diameter and girth of any hypergraph $H_3(R)$ containing a cycle and satisfying the conditions in the above theorem are bounded by 4 and 9, respectively. Note that a similar result for a zero-divisor graph $\Gamma(R)$ is studied in [5, 8, 9, 14] as follows.

(1) $\Gamma(R)$ is connected and $\operatorname{diam}(\Gamma(R))\le 3$.

(2) If $\Gamma(R)$ contains a cycle, then $\operatorname{gr}(\Gamma(R))\le 4$.

Lemma 3.3. *Let $R$ be a finite ring with $|R|\ge 4$. Then $R\cong Z_2\times Z_2$, or there exist two distinct elements $x$ and $y$ in $R-\{0,1\}$ such that $xy\ne 0$.*

*Proof.* For the case $|R|=4$, it is clear that $R$ is isomorphic to either $Z_2\times Z_2$, $Z_2[x]/\langle x^2\rangle$ or $Z_4$, which implies the desired result. Next, we study the case for $|R|\ge 5$ by a contrary method. Suppose $R-\{0,1\}=\{a_1,a_2,\dots,a_m\}$, $m\ge 3$, and $a_ia_j=0$ for all $1\le i\ne j\le m$. It is clear that $a_2+1$ is different from 0 and 1. Otherwise, $a_1=0$ or $a_2=0$, which is a contradiction to the choice of $a_1$ and $a_2$. If $a_2+1\ne a_1$, then $a_1(a_2+1)=0$, and we have $a_1=0$, which is a contradiction. Thus, $a_2+1=a_1$. Similarly, $a_1a_3=0$, and $a_3+1=a_1$ implies that $a_3=a_2$, which is a contradiction. □

In the next theorem, we give a necessary and sufficient condition for a hypergraph $H_3(R)$ to be complete. In the process of the following proof, we consider the obvious fact that $H_3(Z_2 \times Z_2 \times Z_2)$ has only one edge, and necessarily it is a complete hypergraph. Note that for a detailed study of the completeness of a zero-divisor graph $\Gamma(R)$, the reader is referred to [5].

Theorem 3.4. *Let $R$ be a finite nonlocal ring. Then $H_3(R)$ is complete if and only if $R = Z_2 \times Z_2 \times Z_2$.*

*Proof.* The sufficient part of the theorem is trivial, because $H_3(R)$ has only one edge, and therefore is complete whenever $R = Z_2 \times Z_2 \times Z_2$. Suppose that $H_3(R)$ is complete. It is a well-known fact that any finite ring $R$ is isomorphic to the product of local rings. Thus, assume that $R = R_1 \times R_2 \times \cdots \times R_n$, where each $R_i$ is a local ring for all $i = 1,2,\ldots,n$. Now, we study the following cases for different values of $n$.

*Case 1.* Suppose $n \geq 4$. It is clear that $e_1 = \{x_1,x_2,x_3\}$ and $e_2 = \{y_1,y_2,y_3\}$ with

$$\begin{aligned} &x_1 = (1,1,0,0,\ldots,0), \quad &&x_2 = (1,0,1,0,\ldots,0), \quad &&x_3 = (0,1,1,0,\ldots,0),\\ &y_1 = (1,0,0,1,\ldots,0), \quad &&y_2 = (1,1,0,0,\ldots,0), \quad &&y_3 = (0,1,0,1,\ldots,0) \end{aligned} \tag{3.20}$$

are two edges of $H_3(R)$. Clearly, $H_3(R)$ is not complete since $\{x_1,x_2,y_1\}$ is not an edge of $H_3(R)$.

*Case 2.* Let $R = R_1 \times R_2 \times R_3$. Without loss of generality, suppose that $|R_1| \geq 3$. Let $x \in R_1 - \{0,1\}$. Obviously, $\{(1,1,0),(1,0,1),(0,1,1)\} \in E(H_3(R))$ and $\{(x,1,0),(1,0,1),(0,1,1)\} \in E(H_3(R))$. But $\{(x,1,0),(1,0,1),(1,1,0)\} \notin E(H_3(R))$, which implies that $H_3(R)$ is not complete. Hence, we can conclude that $|R_i| \leq 2$ and $R = Z_2 \times Z_2 \times Z_2$.

*Case 3.* Let $R = R_1 \times R_2$. If $H_3(R)$ does not have any vertices, we do not have anything to prove. Therefore, first we assume that $|R_i| \geq 4$ for each $1 \leq i \leq 2$ and investigate the following subcases.

*Case 3.1.* The square of one of the components of some 3-zero-divisor of $R$ is zero. Let $(a,b)$ be a 3-zero-divisor in $R$ with $a^2 = 0$ and let $e = \{(a,b),(c,d),(f,g)\}$ be an edge of $H_3(R)$. Since $Z_2 \times Z_2$ is not a local ring, by Lemma 3.3 there exist distinct elements $x$ and $y$ in $R_2 - \{0,1\}$ such that $xy \neq 0$. Now, from the fact that $\{(a,1),(a,x),(1,0)\}$ and $\{(a,1),(a,y),(1,0)\}$ are in $E(H_3(R))$ and $\{(a,x),(a,y),(a,1)\} \notin E(H_3(R))$, we can conclude that $H_3(R)$ is not complete.

*Case 3.2.* The square of none of the components of any 3-zero-divisor of $R$ is zero. Suppose that $e = \{(a_1,b_1),(a_2,b_2),(a_3,b_3)\}$ is an edge of $H_3(R)$. In this case, there always exists $i \in \{1,2,3\}$, say $i = 1$, such that $a_1a_2 \neq 0$ and $a_1a_3 \neq 0$, or similarly, $b_1b_2 \neq 0$ and $b_1b_3 \neq 0$. Otherwise, the product of two elements of $e$ will be zero, which contradicts the definition for $e$ to be an edge in $H_3(R)$. Without loss of generality, we assume that $a_1a_2 \neq 0$ and $a_1a_3 \neq 0$. By using Lemma 3.3, similar to Case 3.1, there exist distinct elements $x$ and $y$ in $R_2 - \{0,1\}$ such that $xy \neq 0$. Since $\{(a_1,0),(a_2,x),(a_3,1)\}$ and $\{(a_1,0),(a_2,y),(a_3,1)\}$ are the edges of $H_3(R)$, and $\{(a_2,x),(a_2,y),(a_3,1)\}$ is not an edge of $H_3(R)$, then $H_3(R)$ is not complete.

Next, we assume that the size of one of the rings $R_i$'s is 2, where $i = 1,2$. Without loss of generality, assume that $R_2 = Z_2$. It is clear that $R$ does not have any 3-zero-divisors whenever $R_1$ is an integral domain. Thus, $R_1$ has at least four elements. Obviously, the edges of $H_3(R)$ cannot be different from the following forms:

$$\{(a,0),(b,0),(c,0)\}, \qquad \{(a,1),(b,0),(c,0)\}, \qquad \{(a,1),(b,1),(c,0)\}. \tag{3.21}$$

*Case 3.3.* Let $H_3(R)$ have an edge of the form $\{(a,0),(b,0),(c,0)\}$.

Then $\{(a,1),(b,0),(c,0)\} \in E(H_3(R))$, $\{(a,0),(b,1),(c,0)\} \in E(H_3(R))$, and $\{(a,0),(b,0),(c,1)\} \in E(H_3(R))$. In this case, the completeness of $H_3(R)$ implies that $\{(a,1),(b,1),(c,1)\} \in E(H_3(R))$, which is a contradiction.

*Case 3.4.* Suppose $\{(a,1),(b,0),(c,0)\}$ is an edge of $H_3(R)$. Therefore, $b\neq c$, $ab\neq 0$, $ac\neq 0$, and $bc\neq 0$. In this subcase, we study two different cases:

(a) The first components of two elements of $\{(a,1),(b,0),(c,0)\}$ are equal. For example, assume $a=b$. Thus, $\{(a^2,1),(1,0),(c,1)\} \in E(H_3(R))$ whenever $a^2 \neq c$. In this case, $c\neq 1$, and the completeness of $H_3(R)$ implies that $\{(a,1),(1,0),(c,0)\} \in E(H_3(R))$, which contradicts $ac\neq 0$.

On the other hand if $a^2=c$, we have $a^4=0$, which implies $a^3\neq a$. Therefore, $\{(a,1),(a^3,1),(1,0)\}$ is an edge of $H_3(R)$, which contradicts $ac\neq 0$.

(b) Let $a\neq b$ and $a\neq c$. In this case, $\{(a,1),(b,1),(c,0)\}$ and $\{(a,1),(b,0),(c,1)\}$ are in $E(H_3(R))$. Consequently, the completeness of $H_3(R)$ implies that $\{(a,1),(b,1),(c,1)\} \in E(H_3(R))$, which is a contradiction.

*Case 3.5.* Let all the edges of $H_3(R)$ be of the form $\{(a,1),(b,1),(c,0)\}$. Assume that $\{(a,1),(b,1),(c,0)\}$ and $\{(a',1),(b',1),(c',0)\}$ are two edges of $H_3(R)$. Therefore, by the completeness of $H_3(R)$, one of the sets

$$\{(a,1),(b,1),(a',1)\}, \qquad \{(a,1),(b,1),(b',1)\}, \qquad \{(a,1),(c,0),(c',0)\} \tag{3.22}$$

should be an edge of $H_3(R)$. This is a contradiction to the definition of an edge or to Case 3.4. Now, we can conclude that $H_3(R)$ has only one edge of the form $\{(a,1),(b,1),(c,0)\}$, where $ac\neq 0$ and $bc\neq 0$. Furthermore, if $ab\neq 0$, then $\{(a,1),(b,0),(c,0)\}$ is an edge of $H_3(R)$, which is a contradiction. Thus, $ab=0$. Consequently, $c\neq 1$ implies that $\{(a,1),(b,1),(c,0)\}$, $\{(a,1),(b,1),(1,0)\}$ and $\{(a,1),(b,1),(-1,0)\}$ are edges in $H_3(R)$, which is a contradiction. Hence, we can conclude that $\{(a,1),(b,1),(1,0)\}$ is the only edge of $H_3(R)$ and $1=-1$ in $R_1$. Next, we show that $a^2=a$ and $b^2=b$. Since $\{(a,1),(b,1),(a+1,0)\}$ is not an edge in $H_3(R)$, $ba=0$, and $b\neq 0$, then $b(a+1)\neq 0$, and we must have $a(a+1)=0$, which implies that $a^2=a$. By a similar argument, we can conclude that $b^2=b$. Suppose $x\in R_1-\{0,1,a,b\}$. Since $\{(a,1),(b,1),(x,0)\}$ is not an edge of $H_3(R)$, then $ax=0$ or $bx=0$. Without loss of generality, suppose that $ax=0$. Now, since $b+x\neq b$, $\{(a,1),(b+x,1),(1,0)\}$ is not an edge of $H_3(R)$. Therefore, $b+x=0$ or $b+x=a$. If $b+x=0$, we have $b=x$, which is a contradiction. Let $b+x=a$. Then $x=b+a$, and therefore $a(b+a)=0$, which implies that $a=a^2=0$, a contradiction. Thus, $\{0,1,a,b\}$ are the only elements of $R_1$. Since $R_1$ is a local ring with 4 elements, then $R_1=Z_4$ or $R_1=Z_2[x]/\langle x^2\rangle$. In either case, $R=R_1\times Z_2$ does not have any edges, and $H_3(R)$ is not complete.

Finally, since the proof of the case $R_2 = Z_3$ is similar to the above argument, we leave the rest of the proof to the reader. □

*Remark 3.5.* Bounds for $\omega(\Gamma(R))$ are given by using nilpotent elements of $R$ as studied in [6] as follows. Let $R$ be a commutative ring and $0 \neq x \in \text{nil}(R)$, and let $n$ be the least positive integer such that $x^n = 0$.

(1) If $n = 2t$, then $\omega(\Gamma(R)) \geq 2^t - 1$.

(2) If $n = 2t+1$, then $\omega(\Gamma(R)) \geq 2^t$.

Similarly, in the next theorem, we give a lower bound for the clique number of $H_3(R)$ using the index of nilpotence as studied in [6] for a zero-divisor graph $\Gamma(R)$.

Theorem 3.6. *Let $x$ be an element of a commutative ring $R$ such that $x^n = 0$ and $x^{(n-1)} \neq 0$. Then*

$$\omega(H_3(R)) \geq \begin{cases} 2^{2t-2} & \text{if } n = 3t, \\ \dfrac{2^{2t-1}+1}{2} & \text{otherwise.} \end{cases} \tag{3.23}$$

*Proof.* For $n = 3t$, the set

$$A = \{x^t(1 + a_1x + a_2x^2 + \cdots + a_{2t-1}x^{2t-1}) \mid a_i \in \{0,1\},\ 1 \leq i \leq 2t-1\} \tag{3.24}$$

is a clique of size $2^{2t-1}$.

Similarly, for $n = 3t+1$ and $n = 3t+2$, the set

$$A = \{x^{t+1}(1 + a_1x + a_2x^2 + \cdots + a_{2t-1}x^{2t-1}) \mid a_i \in \{0,1\},\ 1 \leq i \leq 2t-1\} \cup \{x^t\} \tag{3.25}$$

is a clique of size $2^{2t-1}+1$. □

Theorem 3.7. *For any integer $m \geq 3$, there exists an integer $n$ such that*

$$\omega(H_3(Z_2^n)) \geq \frac{m}{2}, \tag{3.26}$$

*where $Z_2^n = Z_2 \times Z_2 \times \cdots \times Z_2$ ($n$ times).*

*Proof.* For $m = 3$, it is clear that the set $\{(1,1,0),(1,0,1),(0,1,1)\}$ is a clique of size 3 in $H_3(Z_2^3)$. Suppose that $\{a_1,a_2,\ldots,a_m\}$ is a clique of size $m$ in $H_3(Z_2^{n'})$. Let $n = n' + m$. We define $b_i$ in $H_3(Z_2^n)$ to be the $n$-tuple whose first $n'$ components are exactly $a_i$ and all the other components are 0, except the $(n'+i)$th component, which is 1 for all $1 \leq i \leq m$. Let $b_{m+1}$ be the $n$-tuple whose first $n'$ components are 0 and all the other $m$ components are 1. Now, it is easy to see that $\{b_1,b_2,\ldots,b_{m+1}\}$ is a clique of size $m+1$ in $H_3(Z_2^n)$. Note that $n$ satisfies the recursion relation $x_m = x_{m-1} + m - 1$, where $m \geq 4$ and $x_3 = 3$. □

The following corollary is an immediate consequence of the above theorem.

Corollary 3.8. *The chromatic number of $H_3(Z_2^n)$ goes to infinity as n approaches infinity. That is,*

$$\lim_{n\mapsto\infty}\chi(H_3(Z_2^n)) = \infty. \tag{3.27}$$

We conclude this section by posing a question on the isomorphism of the rings of 3-zero-divisor hypergraphs. In [6], it is shown that for any finite reduced commutative rings $A$ and $B$ which are not fields, then $\Gamma(A) \cong \Gamma(B)$ as graphs if and only if $A \cong B$ as rings. Furthermore, in [3], this result is generalized to the case that if $A$ is a finite reduced ring which is not isomorphic to $Z_2 \times Z_2$ or $Z_6$ with $B$ a ring such that $\Gamma(A) \cong \Gamma(B)$, then $A \cong B$. Also, in [7], it is shown that $A$ and its total quotient ring $T(A)$ have isomorphic zero-divisor graphs.

*Question 2.* Let $A$ and $B$ be two commutative rings. Under what condition(s) does the isomorphism of $H_3(A)$ and $H_3(B)$ imply the isomorphism of $A$ and $B$?

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