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Research Article

Two-Dimensional Electrostatic Problem in a Plane with Earthed Elliptic Cavity due to One or Two Collinear Charged Electrostatic Strips

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A two-dimensional electrostatic problem in a plane with earthed elliptic cavity due to one or two charged electrostatic strips is considered. Using the integral transform technique, each problem is reduced to the solution of triple integral equations with sine kernels and weight functions. Closed-form solutions of the set of triple integral equations are obtained. Also closed-form expressions are obtained for charge density of the strips. Finally, the numerical results for the charge density are given in the form of tables.

1. Introduction

Tranter [1] obtained the closed-form solution to the electrostatic problem of two collinear strips charged to equal and opposite constant potentials. Later on, Srivastava and Lowengrub [2] obtained the closed form solution of the same problem of Tranter [1] with a different method. The advantage of the technique by Srivastava and Lowengrub [2] is that the solution obtained is simpler than that of Tranter [1]. Singh [3] considered the electrostatic field due to two collinear strips charged to equal and opposite constant potentials and lying under the earthed plane and obtained a closed form solution for charge density of the strip. Singh [4] considered the problem of determining the electro static potentials due to two parallel collinear coplanar strips of equal length, charged to equal and opposite constant potential and equidistant from an earthed strip. In recent years, Singh et al. [5] have considered a two-dimensional electrostatic problem due to four collinear and coplanar strips, where the two strips are earthed and the other two are charged to a constant potential. Spence [6] has considered the three-part mixed boundary value problem of electrified disc in a coplanar gap. References of mixed boundary value problems in electrostatics are given in Sneddon [7]. The analysis of this paper can be useful in solving the mixed boundary value problems in electricity and heat conduction.

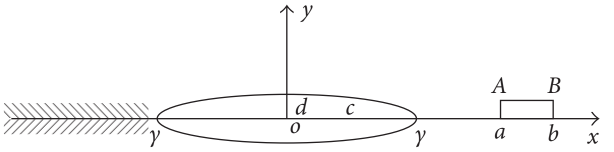

Figure 1.1. One charged strip in a plane with elliptic cavity.

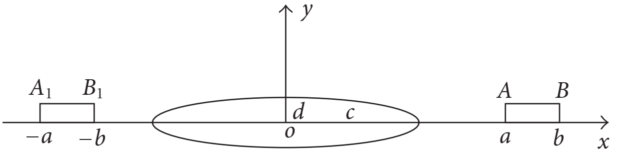

Figure 1.2. Two charged collinear strips in a plane with elliptic cavity.

In this paper, we consider two-dimensional electrostatic problems in a plane with an earthed elliptic cavity and (i) one charged strip of finite length at $y = 0$, $a < x < b$; (ii) two charged strips of finite length at $y = 0$, $a < |x| < b$. The geometry of the problems is shown in Figures 1.1, 1.2. Using the integral transform technique, each problem is reduced into triple integral equations with weight functions.

Closed-form solutions of the triple integral equations are obtained by using the method discussed by Singh [3, 8]. In each problem, we have obtained the closed form expressions for the charge density of the strips. The numerical results are given for the charge density in the form of tables. These types of problems have application in mathematical physics.

As we know, an analytic solution has some advantages over numerical and approximate solutions so that in many cases, analytical solutions in closed form are desired for accurate analysis and design. Moreover, analytical solutions can serve as a benchmark for the purpose of judging the accuracy and efficiency of various numerical and approximate methods.

2. Basic equations

In Cartesian coordinates (x, y), an ellipse centered at the origin is given by the equation

$$\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1. \tag{2.1}$$

We introduce elliptic coordinates (ξ, η), which are defined by

$$x = l\cosh\xi\cos\eta, \qquad y = l\sinh\xi\sin\eta, \tag{2.2}$$

where $\xi \geq 0$, $0 < \eta < 2\pi$, and $l = (c^2 - d^2)^{1/2}$. The ellipse becomes the coordinate line

$$\xi = \gamma = \cosh^{-1}\left(\frac{c}{l}\right), \quad 0 < \eta < 2\pi. \tag{2.3}$$

In elliptic coordinates, the electrostatic potential function V satisfies the differential equation

$$\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} = 0. \tag{2.4}$$

3. Boundary conditions and solution of problem (2.1)

Due to the geometric symmetry, the problem reduces to finding a function $V(\xi,\eta)$ satisfying (2.4) in the region $\gamma < \xi \leq \infty$, $0 \leq \eta \leq \pi$ subject to the conditions

$$\begin{aligned} V(\xi,\pi) &= 0, \quad \xi > \gamma, \\ V(\gamma,\eta) &= 0, \quad 0 < \eta < \pi, \end{aligned} \tag{3.1}$$

$$\begin{aligned} V(\xi,0) &= \Delta(\xi), \quad \alpha < \xi < \beta, \\ \left.\frac{\partial V(\xi,\eta)}{\partial \eta}\right|_{\eta=0} &= 0, \quad \gamma < \xi < \alpha,\ \beta < \xi, \end{aligned} \tag{3.2}$$

where

$$\alpha = \cosh^{-1}\left(\frac{a}{l}\right), \qquad \beta = \cosh^{-1}\left(\frac{b}{l}\right). \tag{3.3}$$

We can easily find the solution of Laplace equation (2.4) in the form

$$V(\xi,\eta) = \int_0^\infty \frac{\sinh[u(\pi-\eta)]}{\sinh(u\pi)} f(u) \sin[u(\xi-\gamma)]\,du, \tag{3.4}$$

which satisfies the boundary conditions (3.1) identically and the remaining conditions (3.2) lead to the following triple integral equations:

$$\begin{aligned} \int_0^\infty f(u)\sin[u(\xi-\gamma)]\,du &= \Delta(\xi), \quad \alpha < \xi < \beta, \\ \int_0^\infty u\coth(\pi u) f(u)\sin[u(\xi-\gamma)]\,du &= 0, \quad \gamma < \xi < \alpha,\ \beta < \xi, \end{aligned} \tag{3.5}$$

for the determination of $f(u)$.

On introducing $x_1 = \xi - \gamma$, $a_1 = \alpha - \gamma$, $b_1 = \beta - \gamma$, the above equations (3.5) reduce to the following integral equations:

$$\int_0^\infty f(u)\sin(ux_1)\,du = \Delta(x_1+\gamma), \quad a_1 < x_1 < b_1, \tag{3.6}$$

$$\int_0^\infty u\coth(\pi u) f(u)\sin(u\pi)\,du = 0, \quad 0 < x_1 < a_1,\ b_1 < x_1 < \infty. \tag{3.7}$$

Assuming

$$\int_0^\infty \coth(\pi u) u f(u) \sin(u x_1)\, du = \frac{\pi}{2} R(x_1), \quad a_1 < x_1 < b_1, \tag{3.8}$$

we find its inverse Fourier sine transform as

$$u f(u) \coth(\pi u) = \int_{a_1}^{b_1} R(t) \sin(ut)\, dt. \tag{3.9}$$

Substituting from (3.9) into (3.6), interchanging the order of integrations and using the following integral from Gradshteyn and Ryzhik (see [9, 4.117(2), page 516]):

$$\int_0^\infty u^{-1} \tanh(u\pi) \sin(ut) \sin(u x_1)\, du = \frac{1}{2} \log \left| \frac{\sinh(x_1/2) + \sinh(t/2)}{\sinh(x_1/2) - \sinh(t/2)} \right|, \tag{3.10}$$

we find that

$$\int_{a_1}^{b_1} R(t) \log \left| \frac{\sinh(x_1/2) + \sinh(t/2)}{\sinh(x_1/2) - \sinh(t/2)} \right| dt = 2\Delta(x_1 + y), \quad a_1 < x_1 < b_1. \tag{3.11}$$

Differentiating both sides of the above equation with respect to x_1, we find that

$$\int_{a_1}^{b_1} \frac{R(t) \sinh(t/2)\, dt}{\cosh(t) - \cosh(x_1)} = \frac{\Delta'(x_1 + y)}{\cosh(x_1/2)} = p_1(x_1) \text{ (say)}, \quad a_1 < x_1 < b_1, \tag{3.12}$$

where prime denotes the derivative with respect to x_1. Making use of a suitable Tricomi theorem given by Singh [3], we find that

$$\begin{aligned} R(t) = &-\frac{2\cosh(t/2)}{\pi^2} \left(\frac{\cosh(t) - \cosh(a_1)}{\cosh(b_1) - \cosh(t)} \right)^{1/2} \\ &\times \int_{a_1}^{b_1} \left(\frac{\cosh(b_1) - \cosh(y)}{\cosh(y) - \cosh(a_1)} \right)^{1/2} \frac{\sinh(y) p(y)\, dy}{\cosh(y) - \cosh(t)} \\ &+ \frac{2C_1 \cosh(t/2)}{\left[(\cosh(t) - \cosh(a_1)) (\cosh(b_1) - \cosh(t)) \right]^{1/2}}, \quad a_1 < t < b_1, \end{aligned} \tag{3.13}$$

where C_1 is an arbitrary constant. If $\Delta(x_1)$ is constant such that

$$\Delta(x_1 + y) = \Delta_1 \text{ (constant)}, \tag{3.14}$$

we find that

$$p(x_1) = 0, \tag{3.15}$$

and from (3.13), we find that

$$R(t) = \frac{2C_1 \cosh(t/2)}{\left[(\cosh(t) - \cosh(a_1)) (\cosh(b_1) - \cosh(t)) \right]^{1/2}}, \quad a_1 < t < b_1. \tag{3.16}$$

Substituting the value of $R(t)$ from (3.16) into (3.11) and using the integral

$$\begin{aligned}\int_{a_1}^{b_1} &\frac{\cosh(t/2)\log\left|\left(\sinh(x_1/2)+\sinh(t/2)\right)/\left(\sinh(x_1/2)-\sinh(t/2)\right)\right|dt}{\left[\left(\cosh(t)-\cosh(a_1)\right)\left(\cosh(b_1)-\cosh(t)\right)\right]^{1/2}} \\ &= \frac{\pi}{\sinh(b_1/2)}K\left(\frac{\sinh(a_1/2)}{\sinh(b_1/2)}\right), \quad a_1 < t < b_1,\end{aligned} \tag{3.17}$$

we find that

$$C_1 = \frac{\Delta_1}{\pi K(\delta)}\sinh\left(\frac{b_1}{2}\right), \tag{3.18}$$

where

$$\delta = \frac{\sinh(a_1/2)}{\sinh(b_1/2)}, \tag{3.19}$$

and $K()$ is the complete integral defined in Gradshteyn and Ryzhik (see [9, page 905]).
From (3.16) and (3.18), we find that

$$R(t) = \frac{\Delta_1\cosh(t/2)\sinh(b_1/2)}{\pi K(\delta)\left[\left(\cosh^2(t/2)-\cosh^2(a_1/2)\right)\left(\cosh^2(b_1/2)-\cosh^2(t/2)\right)\right]^{1/2}}, \quad a_1 < t < b_1. \tag{3.20}$$

The charge density of the strip is defined by the relation

$$\begin{aligned}\sigma_1 &= \frac{-1}{4l\pi\sinh(\xi)}\frac{\partial V(\xi,\eta)}{\partial\eta}\bigg|_{\eta=0} \\ &= \frac{1}{4l\pi\sinh(\xi)}\int_0^\infty uf(u)\coth(u\pi)\sinh\left[u(\xi-\gamma)\right]du, \quad \alpha<\xi<\beta,\ \eta=0.\end{aligned} \tag{3.21}$$

The above equation can be written in the form

$$\begin{aligned}\sigma_1 &= \frac{R(x_1)}{8\pi\sinh(\xi)l} \\ &= \frac{\Delta_1\sinh((\beta-\gamma)/2)\cosh((\xi-\gamma)/2)}{8\pi\sinh(\xi)K(\delta)\left[\left(\sinh^2((\xi-\gamma)/2)-\sinh^2(a_1/2)\right)\left(\sinh^2(b_1/2)-\sinh^2((\xi-\gamma)/2)\right)\right]^{1/2}l}, \\ &\qquad a<x<b,\ y=0,\end{aligned} \tag{3.22}$$

where

$$\xi = \cosh^{-1}\left(\frac{x}{l}\right), \qquad a_1 = \cosh^{-1}\left(\frac{a}{l}\right)-\gamma, \qquad b_1 = \cosh^{-1}\left(\frac{b}{l}\right)-\gamma. \tag{3.23}$$

Equation (3.22) represents the expression for the charge density at $y=0$, $a<x<b$, whose numerical values are given in Table 3.1.

Table 3.1. Numerical results for problem (2.1).

$c = 0.5$, $d = 0.2$, $b = 1$, $a = 0.6$	
x	$\dfrac{\sigma_1}{\Delta_1}$
0.7	0.2852
0.75	0.2243
0.8	0.1944
0.85	0.1822
0.9	0.1869

4. Boundary conditions and solution of problem (2.2)

Since the configuration to be investigated in problem (2.2) is symmetric with respect to x and y axes, we require to find an electrostatic function $V(\xi,\eta)$ which is harmonic in the region $\gamma < \xi < \infty$, $0 < \eta < \pi/2$ and satisfies the conditions

$$\left.\frac{\partial V(\xi,\eta)}{\partial \eta}\right|_{\eta=\pi/2} = 0, \quad \xi > \gamma, \tag{4.1}$$

$$V(\gamma,\eta) = 0, \quad 0 < \eta < \frac{\pi}{2}, \tag{4.2}$$

$$V(\xi,0) = V_0(\xi), \quad \alpha < \xi < \beta, \tag{4.3}$$

$$\left.\frac{\partial V(\xi,\eta)}{\partial \eta}\right|_{\eta=0} = 0, \quad \gamma < \xi < \alpha,\ \beta < \xi. \tag{4.4}$$

Suitable solution of (4.4) can be written in the form

$$V(\xi,\eta) = \int_0^\infty \frac{A(u)\cosh[u(\pi/2-\eta)]}{\cosh(\pi u/2)}\sin[(\xi-\gamma)u]\,du, \tag{4.5}$$

which satisfies conditions (4.1) and (4.2), and the conditions (4.3) and (4.4) give rise to the following integral equations:

$$\int_0^\infty A(u)\sin(ux_1)\,du = V_0(x_1+\gamma), \quad a_1 < x_1 < b_1, \tag{4.6}$$

$$\int_0^\infty uA(u)\tanh\left(\frac{u\pi}{2}\right)\sin(ux_1)\,du = 0, \quad 0 < x_1 < a_1,\ b_1 < x_1 < \infty, \tag{4.7}$$

for the determination of $A(u)$. By assuming

$$\int_0^\infty uA(u)\tanh\left(\frac{u\pi}{2}\right)\sin(ux_1)\,du = R_0(x_1), \quad a_1 < x_1 < b_1, \tag{4.8}$$

and using (4.7), we find that

$$uA(u)\tanh\left(\frac{u\pi}{2}\right) = \frac{2}{\pi}\int_{a_1}^{b_1} R_0(t)\sin(ut)\,dt. \tag{4.9}$$

Substituting from (4.9) into (4.6), interchanging the order of integrations and using the following integral from Gradshteyn and Ryzhik (see [9, 4.116(3), page 516]):

$$\int_0^\infty u^{-1}\coth\left(\frac{u\pi}{2}\right)\sin(ut)\sin(ux_1)\,du = \frac{1}{2}\log\left|\frac{\tanh x_1+\tanh t}{\tanh x_1-\tanh t}\right|, \tag{4.10}$$

we find that

$$\frac{1}{\pi}\int_{a_1}^{b_1} R_0(t)\left|\frac{\tanh x_1+\tanh t}{\tanh x_1-\tanh t}\right|dt = V_0(x_1+y), \quad a_1<x_1<b_1. \tag{4.11}$$

Differentiating both sides of the above equation with respect to x_1, we obtain

$$\frac{1}{\pi}\int_{a_1}^{b_1}\frac{2R_0(t)\tanh(t)dt}{\tanh^2(t)-\tanh^2(x_1)} = \frac{V_0'(x_1+y)}{\operatorname{sech}^2 x_1} = p(x_1)\ (\text{say}), \quad a_1<x_1<b_1, \tag{4.12}$$

where prime denotes the derivative with respect to x_1. Using a suitable Tricomi theorem given by Singh [3], we find that

$$\begin{aligned} R_0(t) = {} & -\frac{\operatorname{sech}^2(t)}{\pi}\left(\frac{\tanh^2(t)-\tanh^2(a_1)}{\tanh^2(b_2)-\tanh^2(t)}\right)^{1/2} \\ & \times\int_{a_1}^{b_1}\left(\frac{\tanh^2(b_1)-\tanh^2(x_1)}{\tanh^2(x_1)-\tanh^2(a_1)}\right)^{1/2}\frac{2\tanh(x_1)\operatorname{sech}^2(x_1)p(x_1)dx_1}{\tanh^2(x_1)-\tanh^2(t)} \\ & +\frac{C_2\operatorname{sech}^2(t)}{\left[\left(\tanh^2(t)-\tanh^2(a_1)\right)\left(\tanh^2(b_1)-\tanh^2(t)\right)\right]^{1/2}}, \quad a_1<t<b_1, \end{aligned} \tag{4.13}$$

where C_2 is an arbitrary constant. If we assume that $V_0(x_1+y)=\Delta_0$ (constant), then we find that

$$p(x_1) = 0, \tag{4.14}$$

$$R_0(t) = \frac{C_2\operatorname{sech}^2(t)}{\left[\left(\tanh^2(t)-\tanh^2(a_1)\right)\left(\tanh^2(b_1)-\tanh^2(t)\right)\right]^{1/2}}. \tag{4.15}$$

Substituting the value of $R_0(t)$ from (4.15) into (4.11) and using the integral

$$\begin{aligned} & \int_{a_1}^{b_1}\frac{\operatorname{sech}^2 t\log\left|\left(\tanh(x_1)+\tanh(t)\right)/\left(\tanh(x_1)-\tanh(t)\right)\right|dt}{\left[\left(\tanh^2(t)-\tanh^2(a_1)\right)\left(\tanh^2(b_1)-\tanh^2(t)\right)\right]^{1/2}} \\ & \quad = \frac{\pi}{\tanh(b_1)}K\left(\frac{\tanh a_1}{\tanh b_1}\right), \qquad a_1<x_1<b_1, \end{aligned} \tag{4.16}$$

we obtain

$$C_2 = \frac{\Delta_0\tanh(b_1)}{K\left(\tanh a_1/\tanh b_1\right)}, \tag{4.17}$$

Table 4.1. Numerical results for problem (2.2).

$c = 0.5$, $d = 0.2$, $b = 1$, $a = 0.6$	
x	$\frac{\sigma_1}{\Delta_0}$
0.7	0.4152
0.75	0.3204
0.8	0.2717
0.85	0.2487
0.9	0.2489

where $K()$ is the complete integral defined in Gradshteyn and Ryzhik (see [9, page 905]). From (4.15) and (4.17), we find that

$$R_0(t) = \frac{\operatorname{sech}^2(t)\tanh(b_1)\Delta_0}{K(\tanh a_1/\tanh b_1)\left[(\tanh^2(t)-\tanh^2(a_1))(\tanh^2(b_1)-\tanh^2(t))\right]^{1/2}}, \quad a<t<b. \tag{4.18}$$

The charge density is given by

$$\begin{aligned}\sigma_1 &= \frac{-1}{\sinh(\xi)l}\frac{\partial V(\xi,\eta)}{\partial\eta}\bigg|_{\eta=0}\\ &= \frac{1}{\sinh(\xi)l}\int_0^\infty A(u)\tanh\left(\frac{\pi u}{2}\right)\sin\left[u(\xi-y)\right]du = \frac{R_0(x_1)}{4\pi\sinh(\xi)l}\\ &= \frac{\operatorname{sech}^2(x_1)\tanh(b_1)\Delta_0}{4\pi\sinh(\xi)K(\delta_1)\left[(\tanh^2 x_1-\tanh^2 a_1)(\tanh^2 b_1-\tanh^2 x_1)\right]^{1/2}}, \quad a_1<x_1<b_1,\ y=0,\end{aligned} \tag{4.19}$$

where

$$\delta_1 = \frac{\tanh(a_1)}{\tanh(b_1)}. \tag{4.20}$$

The above result may be written in the following form:

$$\sigma_1 = \frac{\operatorname{sech}^2(\xi-y)\tanh(\beta-y)\Delta_0}{4\pi\sinh(\xi)K(\delta_1)\left[(\tanh^2(\xi-y)-\tanh^2 a_1)(\tanh^2 b_1-\tanh^2(\xi-y))\right]^{1/2}l}, \quad a<x<b, \quad y=0. \tag{4.21}$$

The numerical values of the charge density σ_1 are given in Table 4.1.

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