Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 69093, 11 pages
doi:10.1155/2007/69093

*Research Article*

# Generalized Persistency of Excitation

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Received 8 May 2007; Revised 15 September 2007; Accepted 10 October 2007

Recommended by Irena Lasiecka

The paper presents the generalized persistency of excitation conditions. Not only they are valid for much broader range of applications than their classical counterparts but also they elegantly prove the validity of the latter. The novelty and the significance of the approach presented in this publication is due to employing the time averaging technique.



## 1. Introduction

The persistency of excitation conditions appears in numerous applications related to system identification, learning, adaptation, parameter estimation. They guarantee the convergence of the adaptation procedures based on the ideas of gradient and least-squares algorithms. An introduction into this topic can be found, for example, in [1, Chapter 2]. The classical version of the persistency of excitation conditions can be characterized in terms of the asymptotic stability of the linear system

$$\dot{x} = -P(t)x, \quad P(t) = P(t)^T \geq 0, \quad \forall\ t \geq 0. \tag{0.1}$$

Namely, the linear system is uniformly asymptotically stable if $P(t)$ is persistently exciting, that is, there exist positive real numbers $\alpha$, $\delta$ such that

$$\int_t^{t+\delta} P(\tau)d\tau \geq \alpha \cdot I, \quad \forall t \geq 0, \tag{0.2}$$

where $I$ is the identity matrix. It is also assumed that there exists a positive real number $\beta$ such that

$$\int_t^{t+\delta} P(\tau)d\tau \leq \beta \cdot I, \quad \forall t \geq 0. \tag{0.3}$$

The detailed analysis of the conditions (0.2), (0.3) can be found in many publications (see, e.g., [1–8]).

We address (0.2) as classical persistency of excitation conditions. They impose the restrictions that are uniform in time. Moreover, they tacitly demand the exponential convergence of the corresponding adaptation procedures. On the other hand, due to wide range of applications the classical conditions might be a burden for solutions of important problems. This publication presents the new generalized persistency of excitation conditions that do not impose any unnatural (uniform in time, exponential convergence) restrictions. Moreover, the classical version easily follows from the new generalized conditions.

In order to create the generalized version for the persistency of excitation, this paper uses the approach similar in the spirit to the time-averaging developed in [9]. We formulate our new necessary and sufficient conditions for a time-varying system (0.1) to be (in general, nonuniformly) asymptotically stable. Finally, we formulate corollaries of the main result and present examples illustrating the generalized persistency of excitation conditions.

## 1. Preliminaries

Consider a system

$$\dot{x} = -P(t)x, \tag{1.1}$$

where $x \in \mathbb{R}^n, \mathbb{R}^n$: $n$-dimensional linear real space. $P(t)$ is a time-dependent matrix such that

$$P(t) = P(t)^T \geq 0, \quad \forall\, t \in \mathbb{R}, \tag{1.2}$$

where the inequality $P(t) \geq 0$ is understood in the following sense. Given two $n \times n$ symmetric matrices $A$ and $B$, we write

$$A \geq B \tag{1.3}$$

if

$$\langle x, Ax \rangle \geq \langle x, Bx \rangle, \quad \forall\, x \in \mathbb{R}^n. \tag{1.4}$$

Throughout the paper, we assume that $\mathbb{R}^n$ is equipped with the scalar product and $\|x\|$ denotes the magnitude of $x$, that is $\|x\| = \sqrt{\langle x, x \rangle}$, where $\langle x, x \rangle$ is the scalar product of $x$ with itself.

We assume that $P(t)$ is a time-dependent $L_{1,\mathrm{loc}}$-matrix in the following sense. For any real numbers $b > a$ and for any $x,\ y \in \mathbb{R}^n$ we have

$$\int_a^b |\langle y, P(t)x\rangle|\,dt < \infty. \tag{1.5}$$

Consider the initial value problem

$$\begin{aligned}\dot{x}(t) &= -P(t)x(t),\\ x(0) &= x_0,\end{aligned} \tag{1.6}$$

where $P(t) \in L_{1,\mathrm{loc}}$. Its solution is defined to be an $L_{1,\mathrm{loc}}$ vector function $x(t)$ such that for any infinitely differentiable function $\varphi(t) \in C^\infty$ (both $x(t)$ and $\varphi(t)$ take its values from $\mathbb{R}^n$ ) with compact support (that means $\varphi(t) = 0$ outside an interval from $\mathbb{R}$) we have

$$-\int_0^\infty \left\langle \frac{d}{dt}\varphi(t), x(t)\right\rangle dt = \langle \varphi(0), x_0\rangle - \int_0^\infty \langle \varphi(t), P(t)x(t)\rangle\, dt. \tag{1.7}$$

It is well known (see, e.g., [10, 11]) that the solution $x(t,x_0)$ for (1.6) exists and is unique. Moreover, it is a continuous function of time. Indeed, consider Picard's sequence

$$y_n(t) = x_0 - \int_0^t P(\tau)y_{n-1}(\tau)\,d\tau, \tag{1.8}$$

where $y_0(t) = x_0$ and $n = 1, 2, \ldots$. Since $P(t) \in L_{1,\mathrm{loc}}$, Picard's sequence $\{y_n(t)\}_n$ converges (pointwise) to a continuous function $y(t)$. Let us show that $y(t)$ satisfies (1.7). Integrating by parts

$$-\int_0^\infty \left\langle \frac{d}{dt}\varphi(t), y_n(t)\right\rangle dt \tag{1.9}$$

we obtain

$$-\int_0^\infty \left\langle \frac{d}{dt}\varphi(t), y_n(t)\right\rangle dt = \langle \varphi(0), x_0\rangle - \int_0^\infty \langle \varphi(t), P(t)y_{n-1}(t)\rangle\, dt. \tag{1.10}$$

Since both $y_n(t)$ and $y_{n-1}(t)$ converge to $y(t)$ as $n \to \infty$, we arrive at

$$-\int_0^\infty \left\langle \frac{d}{dt}\varphi(t), y(t)\right\rangle dt = \langle \varphi(t_0), x_0\rangle - \int_0^\infty \langle \varphi(t), P(t), y(t)\rangle\, dt. \tag{1.11}$$

Hence, $y(t) = x(t,x_0)$ is the solution for (1.6) in the sense (1.7).

The goal of this paper is to find necessary and sufficient conditions for the solution $x(t,x_0)$ of the system (1.6) to satisfy

$$\lim_{t\to\infty} x(t,x_0) = 0, \quad \forall\, x_0 \in \mathbb{R}^n, \tag{1.12}$$

and the equilibrium $x = 0$ is stable.

## 2. Necessary and sufficient conditions

Our main goal is to study asymptotic stability of the origin for the system (1.1). Consider a real positive number $S$ and a continuous real function $\omega_S(t)$ such that

$$\begin{aligned} \omega_S(t) > 0, &\quad \text{for } 0 \leq t < S \\ \omega_S(t) = 0, &\quad \text{for } t \geq S. \end{aligned} \tag{2.1}$$

We also assume that $\omega_S(t)$ is differentiable almost everywhere on $\mathbb{R}$. For the sake of brevity, we address $\omega_S(t)$ as a truncation function in the sequel. If the origin for the system (1.1) is asymptotically stable, then

$$\lim_{t\to\infty} \left|\left|x(t,x_0)\right|\right|^2 = 0, \quad \forall\, x_0 \in \mathbb{R}^n. \tag{2.2}$$

Hence, for any fixed real positive number $S$ we have

$$\lim_{t\to\infty} \int_0^S \omega_S(\tau)\left|\left|x(t+\tau,x_0)\right|\right|^2 d\tau = 0, \quad \forall\, x_0 \in \mathbb{R}^n. \tag{2.3}$$

Consider more closely the integral

$$\int_0^S \omega_S(\tau)\left|\left|x(t+\tau,x_0)\right|\right|^2 d\tau. \tag{2.4}$$

It follows from

$$\frac{d}{dt}\left|\left|x(t,x_0)\right|\right|^2 = -2\cdot\langle x(t,x_0), P(t)x(t,x_0)\rangle \leq 0 \tag{2.5}$$

that $\|x(t_1,x_0)\|^2 \geq \|x(t_2,x_0)\|^2$, for $t_2 > t_1$. That means

$$\int_0^S \omega_S(\tau)\cdot\left|\left|x(t+\tau,x_0)\right|\right|^2 d\tau \geq \left|\left|x(t+S,x_0)\right|\right|^2\cdot \int_0^S \omega_S(\tau)d\tau. \tag{2.6}$$

If we find conditions that guarantee

$$\int_0^S \omega_S(\tau)\left|\left|x(t+\tau,x_0)\right|\right|^2 d\tau \longrightarrow 0, \quad \text{as } t \longrightarrow \infty, \tag{2.7}$$

then that will imply the asymptotic stability of the origin for the system (1.1).

Differentiating the integral

$$\int_0^S \omega_S(\tau)\cdot\left|\left|x(t+\tau,x_0)\right|\right|^2 d\tau \tag{2.8}$$

with respect to time yields

$$\frac{d}{dt}\int_0^S \omega_S(\tau)\cdot\left|\left|x(t+\tau,x_0)\right|\right|^2 d\tau = -2\cdot\int_0^S \omega_S(\tau)\cdot\langle x(t+\tau,x_0), P(t+\tau)x(t+\tau,x_0)\rangle d\tau. \tag{2.9}$$

Replacing $P(t+\tau)$ with

$$\frac{d}{d\tau}\int_0^\tau P(t+\theta)d\theta \tag{2.10}$$

and integrating by parts leads us to the following important formula:

$$\frac{d}{dt}\int_0^S \omega_S(\tau)\cdot\|x(t+\tau,x_0)\|^2 d\tau = -2\int_0^S \langle x(t+\tau,x_0), A(t,\tau)x(t+\tau,x_0)\rangle d\tau, \tag{2.11}$$

where

$$A(t,\tau) = -\frac{d}{d\tau}\omega_S(\tau)\cdot\int_0^\tau P(t+\theta)d\theta + \omega_S(\tau)\cdot\frac{d}{d\tau}\left(\int_0^\tau P(t+\theta)d\theta\right)^2. \tag{2.12}$$

Let

$$\lambda_{\min}(t,\tau), \qquad \lambda_{\max}(t,\tau) \tag{2.13}$$

denote minimal and maximal eigenvalues of $A(t,\tau)$. Then, the next theorem gives us necessary and sufficient conditions for the system (1.1) to be (in general nonuniformly) asymptotically stable at the origin. Notice that one is assured by $P(t)\in L_{1,\mathrm{loc}}$ that the integral expressions in the next theorem are well defined.

Theorem 2.1 (generalized persistency of excitation). *If there exist a real number $S>0$ and a truncation function $\omega_S(t)$ such that*

$$\lim_{t\to\infty}\sup\int_0^t\int_0^S \lambda_{\min}(\nu,\tau)d\tau d\nu = \infty, \tag{2.14}$$

*then the system (1.1) is asymptotically stable at the origin. On the other hand, if*

$$\lim_{t\to\infty}\inf\int_0^t\int_0^S \lambda_{\max}(\nu,\tau)d\tau d\nu < \infty, \tag{2.15}$$

*then the system (1.1) is not asymptotically stable at the origin.*

*Proof.* It follows from (2.11) that

$$\begin{aligned}\int_0^S \omega_S(\tau)\cdot\|x(t+\tau,x_0)\|^2 d\tau \le & -\int_0^t\int_0^S \lambda_{\min}(\nu,\tau)\|x(\nu+\tau,x_0)\|^2 d\tau d\nu \\ & + \int_0^S \omega_S(\tau)\|x(\tau,x_0)\|^2 d\tau.\end{aligned} \tag{2.16}$$

Due to monotonicity of $\|x(t,x_0)\|^2$, we have

$$\begin{aligned}\left(\int_0^S \omega_S(\tau)d\tau\right)\cdot\|x(t+S,x_0)\|^2 \le & -\int_0^t\left(\int_0^S \lambda_{\min}(\nu,\tau)d\tau\right)\|x(\nu+S,x_0)\|^2 d\nu \\ & + \left(\int_0^S \omega_S(\tau)d\tau\right)\cdot\|x_0\|^2.\end{aligned} \tag{2.17}$$

It follows from the very well-known Gronwall inequality (see, e.g., [12]) that

$$||x(t+S,x_0)||^2 \le ||x_0||^2 \cdot \exp\left\{ -\frac{\int_0^t \int_0^S \lambda_{\min}(\nu,\tau)d\tau d\nu}{\int_0^S \omega_S(\tau)d\tau} \right\}. \tag{2.18}$$

Thus,

$$\limsup_{t\to\infty} \int_0^t \int_0^S \lambda_{\min}(\nu,\tau)d\tau d\nu = \infty \tag{2.19}$$

implies that the system (1.1) is asymptotically stable.

On the other hand, (2.11) leads us to

$$\begin{aligned} \int_0^S \omega_S(\tau)\cdot ||x(t+\tau,x_0)||^2 d\tau \ge & -\int_0^t \int_0^S \lambda_{\max}(\nu,\tau)||x(\nu+\tau,x_0)||^2 d\tau d\nu \\ & + \int_0^S \omega_S(\tau)\cdot ||x(\tau,x_0)||^2 d\tau. \end{aligned} \tag{2.20}$$

Since the derivative $(d/dt)\|x(t,x_0)\|^2$ is not positive, we have $\|x(\nu+\tau,x_0)\|^2 \le \|x(\nu,x_0)\|^2$ and

$$\begin{aligned} \left(\int_0^S \omega_S(\tau)d\tau\right)\cdot ||x(t,x_0)||^2 \ge & -\int_0^t \left(\int_0^S \lambda_{\max}(\nu,\tau)d\tau\right)||x(\nu,x_0)||^2 d\nu \\ & + \left(\int_0^S \omega_S(\tau)d\tau\right)||x(S,x_0)||^2 \end{aligned} \tag{2.21}$$

for $t \ge S$. After solving this inequality, we obtain

$$||x(t,x_0)||^2 \ge ||x(S,x_0)||^2 \cdot \exp\left\{ -\frac{\int_0^t \int_0^S \lambda_{\max}(\nu,\tau)d\tau d\nu}{\int_0^S \omega_S(\tau)d\tau} \right\} \tag{2.22}$$

for $t \ge S$. Hence, if

$$\liminf_{t\to\infty} \int_0^t \int_0^{\S} \lambda_{\max}(\nu,\tau)d\tau d\nu < \infty, \tag{2.23}$$

then the system (1.1) is not asymptotically stable. □

Notice that we owe the success in proving Theorem 2.1 to the new idea that suggests to consider

$$\int_0^S \omega_S(\tau)\cdot ||x(t+\tau,x_0)||^2 d\tau \tag{2.24}$$

instead of $\|x(t,x_0)\|^2$. This approach seems to have further important consequences not only for control theory but also for studies of general dynamical systems.

After integrating by parts, we have

$$\int_0^S A(t,\tau)d\tau = -\int_0^S \frac{d}{d\tau}\omega_S(\tau)\cdot\left\{\int_0^\tau P(t+\theta)d\theta + \left(\int_0^\tau P(t+\theta)d\theta\right)^2\right\}d\tau. \tag{2.25}$$

Let

$$\gamma_{\min}(t,\tau), \qquad \gamma_{\max}(t,\tau) \tag{2.26}$$

denote minimal and maximal eigenvalues of

$$\int_0^\tau P(t+\theta)d\theta. \tag{2.27}$$

Then, we can reformulate Theorem 2.1 as follows.

COROLLARY 2.2. *If there exist a real number $S > 0$ and a truncation function $\omega_S(t)$ such that*

$$\begin{gathered}\frac{d}{d\tau}\omega_S(\tau) \leq 0, \quad \textit{for } \tau < S, \\ \limsup_{t\to\infty}\left\{-\int_0^t\int_0^S \frac{d}{d\tau}\omega_S(\tau)\big(\gamma_{\min}(\nu,\tau) + (\gamma_{\min}(\nu,\tau))^2\big)d\tau d\nu\right\} = \infty,\end{gathered} \tag{2.28}$$

*then the system (1.1) is asymptotically stable at the origin. On the other hand, if*

$$\liminf_{t\to\infty}\left\{-\int_0^t\int_0^S \frac{d}{d\tau}\omega_S(\tau)\big(\gamma_{\max}(\nu,\tau) + (\gamma_{\max}(\nu,\tau))^2\big)d\tau d\nu\right\} < \infty, \tag{2.29}$$

*then the system (1.1) is not asymptotically stable at the origin.*

*Proof.* Consider the unit eigenvectors $\psi_{\min}(\nu,\tau)$, $\psi_{\max}(\nu,\tau)$ corresponding to minimal and maximal eigenvalues of $\int_0^\tau P(\nu+\theta)d\theta$,

$$\begin{aligned}\gamma_{\min}(\nu,\tau) &= \left\langle \psi_{\min}, \left(\int_0^\tau P(\nu+\theta)d\theta\right)\psi_{\min}\right\rangle, \qquad \langle\psi_{\min}(\nu,\tau),\psi_{\min}(\nu,\tau)\rangle = 1, \\ \gamma_{\max}(\nu,\tau) &= \left\langle \psi_{\max}, \left(\int_0^\tau P(\nu+\theta)d\theta\right)\psi_{\max}\right\rangle, \qquad \langle\psi_{\max}(\nu,\tau),\psi_{\max}(\nu,\tau)\rangle = 1.\end{aligned} \tag{2.30}$$

Then, taking into account that

$$\left\langle \frac{d}{d\tau}\psi_{\min},\psi_{\min}\right\rangle = 0, \qquad \left\langle \frac{d}{d\tau}\psi_{\max},\psi_{\max}\right\rangle = 0, \tag{2.31}$$

we obtain

$$\begin{aligned}\left\langle \psi_{\min}, \frac{d}{d\tau}\left(\int_0^\tau P(\nu+\theta)d\theta\right)^2\psi_{\min}\right\rangle &= \frac{d}{d\tau}\left\langle \psi_{\min}, \left(\int_0^\tau P(\nu+\theta)d\theta\right)^2\psi_{\min}\right\rangle, \\ \left\langle \psi_{\max}, \frac{d}{d\tau}\left(\int_0^\tau P(\nu+\theta)d\theta\right)^2\psi_{\max}\right\rangle &= \frac{d}{d\tau}\left\langle \psi_{\max}, \left(\int_0^\tau P(\nu+\theta)d\theta\right)^2\psi_{\max}\right\rangle.\end{aligned} \tag{2.32}$$

Consequently,

$$\langle \psi_{\min}, A(\nu,\tau)\psi_{\min}\rangle = -\frac{d}{d\tau}\omega_S(\tau)\gamma_{\min}(\nu,\tau) + \omega_S(\tau)\frac{d}{d\tau}(\gamma_{\min}(\nu,\tau))^2, \tag{2.33}$$

$$\langle \psi_{\max}, A(\nu,\tau)\psi_{\max}\rangle = -\frac{d}{d\tau}\omega_S(\tau)\gamma_{\max}(\nu,\tau) + \omega_S(\tau)\frac{d}{d\tau}(\gamma_{\max}(\nu,\tau))^2. \tag{2.34}$$

Let $\lambda_{\min}(\nu,\tau)$ and $\lambda_{\max}(\nu,\tau)$ denote minimal and maximal eigenvalues for $A(\nu,\tau)$, respectively. Then, after integrating by parts,

$$\int_0^S \lambda_{\min}(\nu,\tau)d\tau = -\int_0^S \frac{d}{d\tau}\omega_S(\tau)\left(\gamma_{\min}(\nu,\tau) + (\gamma_{\min}(\nu,\tau))^2\right)d\tau \tag{2.35}$$

follows from (2.33) and

$$\int_0^S \lambda_{\max}(\nu,\tau)d\tau = -\int_0^S \frac{d}{d\tau}\omega_S(\tau)\left(\gamma_{\max}(\nu,\tau) + (\gamma_{\max}(\nu,\tau))^2\right)d\tau \tag{2.36}$$

follows from (2.34). □

If

$$\omega_S(\tau) = \begin{cases} (S-\tau), & \text{for } \tau < S, \\ 0, & \text{for } \tau \geq S, \end{cases} \tag{2.37}$$

then we obtain the following important corollary of Theorem 2.1.

Corollary 2.3. *If there exists a real number $S > 0$ such that*

$$\limsup_{t\to\infty} \int_0^t \int_0^S \gamma_{min}(\nu,\tau)d\tau d\nu = \infty, \tag{2.38}$$

*then the system (1.1) is asymptotically stable at the origin.*

Though Corollary 2.3 gives us only a sufficient condition of (in general nonuniform) asymptotic stability for the system (1.1), its simple form makes it valuable for practical applications.

At the conclusion of this section, we present a simple and elegant proof for the classical persistency of excitation conditions.

Corollary 2.4 (classical persistency of excitation). *If there exist real numbers $\alpha > 0$, $\delta > 0$ such that*

$$\int_0^\delta P(t+s)ds \geq \alpha I, \qquad \forall\, t \geq 0, \tag{2.39}$$

*then the system (1.1) is asymptotically stable.*

*Proof.* It follows from (2.39) that for the minimal eigenvalue $\gamma_{\min}(\nu,\tau)$ from Corollary 2.3 we have

$$\gamma_{\min}(\nu,\tau) \geq \alpha, \qquad \text{for } \tau \geq \delta. \tag{2.40}$$

Hence, if we take $S > \delta$, then

$$\int_0^t \int_0^S \gamma_{\min}(\nu,\tau)d\tau d\nu \geq \int_0^t \int_\delta^S \gamma_{\min}(\nu,\tau)d\tau d\nu \geq (S-\delta)t\alpha \tag{2.41}$$

and asymptotic stability follows from

$$\lim_{t\to\infty} \sup \int_0^t \int_0^S \gamma_{\min}(\nu,\tau)d\tau d\nu \geq \lim_{t\to\infty}(S-\delta)t\alpha = \infty. \tag{2.42}$$

□

There is a common belief that the classical persistency of excitation is a very hard condition to check in practice. However, a simplified version of the generalized persistency of excitation condition presented in Theorem 2.1 can be effectively verified with the help of the famous Gershgorin circle theorem [13, 14]. Let $p_{ij}(t)$ denote the element of the matrix $P(t)$ from $i$th row and $j$th column. Then, the following statement is true.

Theorem 2.5. *If*

$$\lim_{t\to\infty} \int_0^t \min_i \left( p_{ii}(\nu) - \sum_{j\neq i} |\, p_{ij}(\nu)\,| \right) d\nu = \infty, \tag{2.43}$$

*then the system (1.1) is asymptotically stable at the origin.*

*Proof.* Due to Gershgorin circle theorem, the eigenvalue $\gamma_{\min}(\nu,\tau)$ satisfies the inequality

$$\min_i \left( \int_0^\tau p_{ii}(\nu+\theta)d\theta - \sum_{j\neq i} \left| \int_0^\tau p_{ij}(\nu+\theta)d\theta \right| \right) \leq \gamma_{\min}(\nu,\tau) \tag{2.44}$$

which leads us to

$$\int_0^S \int_0^\tau \min_i \left( p_{ii}(\nu+\theta) - \sum_{j\neq i} |\, p_{ij}(\nu+\theta)\,| \right) d\theta d\tau \leq \int_0^S \gamma_{\min}(\nu,\tau)d\tau. \tag{2.45}$$

After integrating this inequality with respect to $\nu$ and using Fubini theorem, we obtain

$$\int_0^S \int_0^\tau \int_0^t \min_i \left( p_{ii}(\nu+\theta) - \sum_{j\neq i} |\, p_{ij}(\nu+\theta)\,| \right) d\nu d\theta d\tau \leq \int_0^t \int_0^S \gamma_{\min}(\nu,\tau)d\tau d\nu. \tag{2.46}$$

Hence, due to the condition (2.43) we have

$$\lim_{t\to\infty} \sup \int_0^t \int_0^S \gamma_{\min}(\nu,\tau)d\tau d\nu = \infty \tag{2.47}$$

and by Corollary 2.3 the system (1.1) is asymptotically stable at the origin. □

## 3. Examples

Theorem 2.1 and its corollaries find many important applications. This section illustrates how one can use them in order to verify the persistency of excitation conditions.

Consider the system (see also [6] for a similar example)

$$\dot{x} = -\begin{pmatrix} \Xi(t) & 0 \\ 0 & 1-\Xi(t) \end{pmatrix} x, \tag{3.1}$$

where $\Xi(t)$ is a characteristic function of a closed subset $C \subset \mathbb{R}$,

$$\Xi(t) = \begin{cases} 1, & \text{for } t \in C, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

Consider the sequence of real numbers $\{a_n\}_{n=0}^{\infty}$ defined as

$$a_{n+1} = (n+1) + a_n, \tag{3.3}$$

where $a_0 = 0$. If we define the closed subset $C$ as

$$C = \{t \in \mathbb{R}; \exists\, n > 1 \text{ such that } a_n \leq t \leq a_n + 1\}, \tag{3.4}$$

then the classical condition (0.2) is not valid. However, the system is persistently exciting due to Corollary 2.3. It is worth mentioning here that the system in question is nonuniformly asymptotically stable. That is why the classical persistency of excitation condition fails in this example.

In the literature, it is implicitly or explicitly assumed that the matrix $P(t)$ is bounded while often, it is only necessary for $P(t)$ to be locally integrable. In other words, one can omit the condition (0.3) and still have the system with persistency of excitation. However, this system in general is not uniformly asymptotically stable. To illustrate this statement consider the system

$$\dot{x} = -\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} x. \tag{3.5}$$

The condition (0.3) is not valid and no time rescaling can bring this system into a form where both (0.2) and (0.3) are satisfied. However, the system is persistently exciting due to Theorem 2.5.

## Acknowledgments

The author is very grateful to the referee who made many important suggestions that allowed to improve the formulation of the main result and the style of the paper. In particular, Theorem 2.5 was added as a response to the referee remarks.

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