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Research Article

The Quasimetrization Problem in the (Bi)topological Spaces

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It is our main purpose in this paper to approach the quasi-pseudometrization problem in (bi)topological spaces in a way which generalizes all the well-known results on the subject naturally, and which is close to a "Bing-Nagata-Smirnov style" characterization of quasi-pseudometrizability.

1. Introduction

A *quasi-pseudometric space* is a pair (X,d) where X is a set and d is a mapping from $X \times X$ into the real numbers $\mathbb{R}$ (called a *quasi-pseudometric*) satisfying for all $x,y,z \in X$: (i) $d(x,y) \geq 0$, (ii) $d(x,x) = 0$, (iii) $d(x,y) \leq d(x,z) + d(z,y)$. If d satisfies the additional condition (iv) $d(x,y) = 0$ if and only if $x = y$, then d is called a *quasi-metric* on X. The sets $B(x,r) = \{y \mid d(x,y) < r\}$ constitute a base for a topology τ_d. If d is a quasi-pseudometric on X, then $d^{-1}(x,y) = d(y,x)$ is also a quasi-pseudometric on X. Thus a quasi-pseudometric d determines two topologies, τ_d and $\tau_{d^{-1}}$. We note by $\tau^\star$ the supremum of τ_d and $\tau_{d^{-1}}$.

The quasi-(pseudo)metrization problem for bitopological spaces (X,τ_0,τ_1) is to find necessary and sufficient conditions for $\tau_0 = \tau_d$ and $\tau_1 = \tau_{d^{-1}}$ for some quasi-(pseudo) metric. The problem has been firstly put by Kelly [2] and Lane [3] who give sufficient conditions for a bitopological space to be quasi-pseudometrizable. Patty in [4] states a conjecture which improves the quasi-pseudometrization theorems of Kelly and Lane. In [5], Salbany proves a sufficient condition for quasi-pseudometrizability from which he deduces Patty's conjecture. In [6], Parrek has obtained a necessary and sufficient condition for quasi-metrization of a T_1 bitopological space which generalizes a topological result of Ribeiro [7]. In [8], Raghavan and Reilly make use of the quasi-uniform analogue of the metrization theorem of Alexandroff and Urysohn, and give necessary and sufficient conditions for a pairwise Hausdorff bitopological space to be quasi-metrizable. Romaguera in [9] gives a sufficient condition of quasi-pseudometrization for bitopological spaces and in [10] generalizes the pseudo-metrization problem proved by Guthrie and Henry in [11, 12].

Related to the problem of quasi-pseudometrizability of a bitopological space is that of quasi-pseudometrizability of a topological space. This problem has been firstly put by Wilson in [13], who shows that every second countable T_1 space is quasi-metrizable. After the theorems of Nagata [14], Bing [15], and Smirnov [16] about the pseudo-metrizability of topological spaces, the efforts which have been done during the 60s and in the beginning of 70s intended to give the same kind of theorems concerning quasi-pseudometrizability. In all these cases there are given sufficient conditions for a space to be a quasi-pseudometrizable one and, at the same time, the question is put whether a theorem of Bing-Nagata-Smirnov type is invalid. In this direction, Köfner [17] proves that for a T_1 topological space (X,τ), the existence of a σ-interior preserving base is a necessary and sufficient condition for the non-Archimedean quasi-metrizability of (X,τ). Sion and Zelmer [18] (also Norman [19]) have proved that a topological space (X,τ) is quasi-pseudometrizable if τ has a σ-point finite base. However, as is observed in [18, 19] there are examples of quasi-pseudometrizable spaces which do not have σ-point finite bases.

The quasi-pseudometrization problem is put again in the 90s by Kopperman and Hung. In [20], Kopperman based on a Fox's result, gives a characterization of quasi-pseudometrizable spaces which is closely related to one of the known characterizations of γ-spaces, making use of the cushioned and cocushioned sets' notions. Hung in [21] gives a characterization of quasi-pseudometrizability of a topological space purely in terms of the neighborhood bases.

In this paper, for each regular topological space we characterize the existence of a local uniformity with nested base indexed by an ordinal number κ. For $\kappa = \omega$, this result in the bitopological spaces gives a characterization of the quasi-pseudometrizability equivalent to that of Fox in [22]. Contrary to Fox, our characterization is modelled upon the presentation of the Bing-Nagata-Smirnov's metrization theorem (in the metric case the σ-pairbases we use coincide with the usual σ-locally finite bases which the Bing-Nagata-Smirnov's theorem gives). This allows us to derive all well-known theorems on the subject as immediate corollaries. In topological spaces, this characterization is also situated very close to a "Bing-Nagata-Smirnov's-style" characterization of quasi-pseudometrizability (see [1, Problem O]). More precisely, in the first section we give the solution of the inverse problem raised by Williams (Theorem 1.6). In the second section we present a theorem on the necessary and sufficient conditions for a bitopological space to be quasi-pseudometrizable, as well as an alternative form of that theorem. We also obtain, as immediate corollaries, all the related known results. In the final section, we give necessary and sufficient topological conditions in order that a topological space admits a quasi-pseudometric (see [1, Problem O], [21, page 40]).

We have to point out that all the quasi-pseudometrization theorems below, which are referred as quasi-metrization theorems, are valid in three forms: for spaces of T_0 form, of T_1 form, and for spaces without any axiom of separation, except the cases when it is explicitly stated that the space is T_1. Throughout the paper the symbols $\mathbb{N}$ and $\mathbb{R}$ are, respectively, used for the sets of all natural numbers and all real numbers. The letter ω will denote the smallest infinite ordinal, which is the order type of the natural numbers and which can even be identified with the set of natural numbers. (If $(K,\leq)$ is a well-ordered set with ordinal number κ, then the set of all ordinals $<\kappa$ is order isomorphic to K. This provides the motivation to define an ordinal as the set of all ordinals less than itself.) The letter $\aleph_0$ denotes the cardinal of the set of natural numbers which is the smallest infinite cardinal. Finally, it is traditional to identify a cardinal number with its initial ordinal. (Each ordinal has an associated cardinal, its cardinality, obtained by simply forgetting the order. Any well-ordered set having that ordinal as its order type has the same cardinality. The smallest ordinal having a given cardinal as its cardinality is called the *initial ordinal* of that cardinal.) Hence, if $\mathcal{A}$ is a collection of families of a space X which has cardinality κ, then we write $\mathcal{A}=\{\mathcal{A}_a \mid a\in\kappa\}$. If (X,τ) is a topological space and $F\subset X$, then $\mathrm{cl}_\tau F$ and $\mathrm{int}_\tau F$ denote the closure and the interior of F in the topology τ, respectively. If $\mathcal{F}=\{F_i \mid i\in I\}$ is any family, then $\mathrm{cl}_\tau\mathcal{F}=\{\mathrm{cl}_\tau F_i \mid i\in I\}$ and $\mathrm{int}_\tau\mathcal{F}=\{\mathrm{int}_\tau F_i \mid i\in I\}$.

Definition 1.1 (see [1, page 162]). A *local quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X\times X$ such that

(i) each member of $\mathcal{U}$ contains the diagonal Δ,

(ii) if $U\in\mathcal{U}$, $x\in X$, then for some $V\in\mathcal{U}$, $(V\circ V)(x)\subseteq U(x)$.

The pair $(X,\mathcal{U})$ is called a *locally quasi-uniform space* and the members of $\mathcal{U}$ are called *entourages*.

A local quasi-uniformity is a *local uniformity* provided that $\mathcal{U}=\mathcal{U}^{-1}$.

A subfamily $\mathcal{B}$ of a local quasi-uniformity $\mathcal{U}$ is a *base* for $\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$. A subfamily S is a subbase for $\mathcal{U}$ if the family of finite intersections of members of S is a base for $\mathcal{U}$. A base $\mathcal{B}=\{B_\lambda \mid \lambda\in\Lambda\}$ of a local quasi-uniformity $\mathcal{U}$ is said to be *decreasing* if $B_\lambda\subseteq B_\mu$ whenever $\lambda,\mu\in\Lambda$ and $\lambda\geq\mu$.

We recall (as in [23, page 441]) that *a local quasi-uniformity* $\mathcal{U}$ *is of cofinality* κ, if κ is the least cardinal κ for which $\mathcal{U}$ has a base of cardinality κ.

We start from a result of J. Williams, which implies the Bing-Nagata-Smirnov metrization theorem.

Theorem 1.2 (see [23, Theorem 2.9]). *The sufficient conditions for a regular space* (X,τ) *to be generated by a local uniformity with a decreasing base indexed by an ordinal number* κ *are the following.*

(i) *There exists a nested collection of families* $\{\mathcal{A}_a \mid a\in\kappa\}$ *such that for any* $a\in\kappa$ *and any subfamily* $\mathcal{B}\subseteq\mathcal{A}_a$, *the sets* $\cap\{B \mid B\in\mathcal{B}\}$ *and* $\cap\{X\setminus\mathrm{cl}_\tau B \mid B\in\mathcal{B}\}$ *are open.*

(ii) $\cup\{\mathcal{A}_a \mid a\in\kappa\}$ *is a base for* τ.

Definition 1.3 (see [24, page 29]). A collection $\mathcal{C}$ of subsets of a topological space (X,τ) is τ*-interior* (τ*-closure*) *preserving*, provided that if $\mathcal{C}'\subset\mathcal{C}$, then $\mathrm{int}_\tau\cap\{C \mid C\in\mathcal{C}'\}=\cap\{\mathrm{int}_\tau C \mid C\in\mathcal{C}'\}$ ($\mathrm{cl}_\tau(\cup\{C \mid C\in\mathcal{C}'\})=\cup\{\mathrm{cl}_\tau C \mid C\in\mathcal{C}'\}$).

By the previous definition, a collection $\mathscr{C}$ of open subsets is τ-interior (resp., τ-closure) preserving if and only if for each subcollection $\mathscr{C}'$ of $\mathscr{C}$, $\cap\{C \mid C \in \mathscr{C}'\}$ (resp., $\cap\{X \setminus \mathrm{cl}_\tau C \mid C \in \mathscr{C}'\}$) is open. Hence, the condition (i) in the above Williams' theorem is equivalent to being the families $\mathscr{A}_a$ τ-interior preserving, τ-closure preserving.

Conditions (i) and (ii) are necessary ones (see Theorem 1.6) for a regular space to be generated by a local uniformity with a decreasing base. As Williams' theorem indicates, the interior and closure preserving properties are the keys for the Bing-Nagata-Smirnov metrization theorem (see [23, page 443]).

The interior preserving property does not work well for a bitolopogical space [17, Example 1] (example of a quasi-metric space whose topology does not have a σ-interior preserving base) and the quasi-metrization problem fails.

Definition 1.4. A topological space (X,τ) has a *κ-τ-interior preserving, κ-τ-closure preserving base for τ* if and only if there is a nested collection of τ-open families $\{\mathscr{A}_a \mid a \in \kappa\}$ such that

(1) for each $a \in \kappa$, $\mathscr{A}_a$ is a τ-interior preserving, τ-closure preserving family,

(2) $\cup\{\mathscr{A}_a \mid a \in \kappa\}$ is a base for τ.

Remark 1.5. If the index set κ is countable, then the space (X,τ) has a σ-τ-interior preserving, σ-τ-closure preserving base for τ.

Theorem 1.6. *A regular space (X,τ) is generated by a local uniformity with a decreasing base of cofinality κ if and only if τ has a κ-τ-interior-preserving, κ-τ-closure preserving base.*

Proof. The sufficient of the statement is almost as in Theorem 1.2.

We prove the necessity. Suppose that τ is generated by a local uniformity with a countable base ($\kappa = \aleph_0$), thus the space is pseudometrizable (see [23, Corollary 2.6]). Hence, from Nagata-Smirnov's theorem, the space has a σ-locally finite base, say $\{\mathscr{P}_n \mid n \in \omega\}$. If $\mathscr{A}_n = \cup\{\mathscr{P}_m \mid m \leq n\}$, then $\{\mathscr{A}_n \mid n \in \omega\}$ is nested and for a fixed n, $\cap\{A \mid A \in \mathscr{A}_n\}$ and $\cap\{X \setminus clA \mid A \in \mathscr{A}_n\}$ are open. Thus $\{\mathscr{A}_n \mid n \in \omega\}$ is a σ-τ-interior preserving, σ-τ-closure preserving base for τ.

Let, now, (X,τ) be generated by a local uniformity with a decreasing base of cofinality $\kappa > \aleph_0$. By [23, Lemma 2.2, Theorem 2.5] and [25, Theorem 2.1d] we may construct a decreasing family $\{V_a \mid a \in \kappa\}$ of equivalence relations on X which generate the topology of X. For a fixed a as x runs through X, the sets $V_a(x)$ are disjoint (see [25, page 376]) and if $x \neq y$ and $V_a(x) \cap V_a(y) \neq \varnothing$, then $V_a(x) = V_a(y)$. Moreover, every $V_a(x)$ is closed. For the latter: if $t \in \mathrm{cl}_\tau V_a(x)$, then $V_a(t) \cap V_a(x) \neq \varnothing$, hence $V_a(x) = V_a(t)$ and $t \in V_a(x)$. In conclusion the sets $V_a(x)$, where a is fixed and $x \in X$, constitute a partition of X. Let for a fixed a, $\mathscr{V}_a = \{V_a(x) \mid x \in X\}$ be the elements of the corresponding partition. Then, $\mathscr{A}_a = \cup\{\mathscr{V}_\beta \mid \beta \leq a\}$ is a nested collection of families whose union is a base for τ.

It remains to prove that, for each $a \in A$, the family $\mathscr{A}_a$ is τ-interior preserving, τ-closure preserving. Indeed, suppose that $\beta \leq a < \kappa$, $V_\beta(x), V_a(y) \in \mathscr{A}_a$, and $V_\beta(x) \cap V_a(y) \neq \varnothing$. Using the fact that $\{V_a \mid a \in \kappa\}$ is a decreasing family of equivalence relations, we see that $V_\beta(x) \cap V_\beta(y) \neq \varnothing$, so that $V_\beta(x) = V_\beta(y)$. Hence, $V_\beta(x) \cap V_a(y) = V_\beta(y) \cap V_a(y) = V_a(y)$. The rest is obvious, since $\{V_a(x) \mid x \in X\}$ is a partition of X. $\square$

Amongst the propositions which aim to the point, the more closer to our interests is the following theorem of Köfner.

THEOREM 1.7 (see [17, Proposition 1]). *In a T_1 space, the existence of a σ-interior preserving base is equivalent to the admission of a non-Archimedean quasimetric.*

As we have said above, the example of Köfner in [17, Example 1] shows that the previous theorem (of Köfner as well) does not solve the quasi-metrization problem.

We proceed to the second section, firstly giving a definition and two lemmas.

Definition 1.8 (see [26, Definition 0.2]). A *quasi-semiuniformity* on a set X is a filter $\mathcal{V}$ on $X \times X$ such that for each $V \in \mathcal{V}$,

$$\Delta(X) = \{(x,x) \mid x \in X\} \subseteq V. \tag{1.1}$$

In the case where the family $\mathcal{N}_x = \{V(x) \mid V \in \mathcal{V}\}$ for every x constitutes a neighborhood system of x for a topology τ on X, we will call the quasi-semiuniformity, *topological quasi-semiuniformity*.

LEMMA 1.9. *Let (X,τ_0,τ_1) be a bitopological space and for each $c \in \{0,1\}$, let $\mathcal{B}_c$ be a collection of $\tau_{1-c} \times \tau_c$-open neighborhoods of the diagonal such that for any $x \in X$ and any τ_c-neighborhood M_c of x, there are τ_c-neighborhood N_c of x and $V_c \in \mathcal{B}_c$ with $V_c(N_c) \subseteq M_c$. Then $\mathcal{B}_c$ is a subbase for a local quasi-uniformity which generates τ_c.*

Proof. We prove it for $c = 0$. Suppose that $\mathcal{B}_0$ consists of $\tau_1 \times \tau_0$-neighborhoods of the diagonal and for a τ_0-neighborhood $U(x)$ of x there is another $W(x)$ (U and $W \in \mathcal{B}_0$) and $V \in \mathcal{B}_0$ such that $V(W(x)) \subseteq U(x)$. Then $[(W \cap V) \circ (W \cap V)](x) \subseteq V(W(x)) \subseteq U(x)$. □

LEMMA 1.10. *A topological quasi-semiuniformity finer than a local quasi-uniformity and generating the same topology with it is a local quasi-uniformity as well.*

Proof. Let $\mathcal{U}$ be a local quasi-uniformity on a set X. Suppose that $\mathcal{V}$ is a topological quasi-semiuniformity on X which is finer than $\mathcal{U}$ and generates the same topology with it. Then given $x \in X$ and $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ such that $U(x) \subseteq V(x)$, whilst there is $U_1 \in \mathcal{U}$ such that $U_1^2(x) \subseteq U(x)$. Since $\mathcal{V}$ is finer than $\mathcal{U}$, there is $V_1 \in \mathcal{V}$ such that $V_1 \subseteq U_1$, hence $V_1^2(x) \subseteq V(x)$. □

2. The quasi-metrizability in bitopological spaces

We firstly introduce some new notions referring to a topological space (X,τ).

Definition 2.1. Let (X,τ) be a topological space. A pair family $(\mathcal{A},\mathcal{A}^\star) = \{(A_i,A_i^\star) \mid i \in I\}$ of pairs of subsets of X is said to be an open pair family, if for any $i \in I$, A_i, $A_i^\star$ are open and $A_i \cap A_i^\star \neq \emptyset$. Such a pair family is said to be

(1) *enclosing* if for any $i \in I$, $A_i \subseteq A_i^\star$ (see [20, Definition 1.4]);

(2) *pairbase for τ* if for each $x \in X$ and each $A \in n_x$ (n_x is the τ-neighborhood filter of x), there exists $(A_i,A_i^\star) \in (\mathcal{A},\mathcal{A}^\star)$ such that $x \in A_i \subseteq A_i^\star \subseteq A$ (see [1, Section 7.17]).

Definition 2.2. Let (X,τ) be a topological space. An *open pair family* $(\mathcal{A},\mathcal{A}^\star)=\{(A_i,A_i^\star)\mid i\in I\}$ is said to be

(1) τ-*open cocushioned* (see [1, page 163]) if for each $I'\subseteq I$, it satisfies

$$\bigcap_{i\in I'}\{A_i\mid A_i\in\mathcal{A}\}\subseteq \operatorname{int}_\tau\bigcap_{i\in I'}\{A_i^\star\mid A_i^\star\in\mathcal{A}^\star\}, \tag{2.1}$$

(2) τ-*open cushioned* if for each $I'\subseteq I$, it satisfies

$$\operatorname{cl}_\tau\left(\bigcup_{i\in I'}\{A_i\mid A_i\in\mathcal{A}\}\right)\subseteq\bigcup_{i\in I'}\{A_i^\star\mid A_i^\star\in\mathcal{A}^\star\}. \tag{2.2}$$

(3) τ-*open weakly cushioned* if for each $I'\subseteq I$, it satisfies

$$\operatorname{cl}_\tau\left(\bigcup_{i\in I'}\{A_i\mid A_i\in\mathcal{A}\}\right)\subseteq\bigcup_{i\in I'}\{\operatorname{cl}_\tau A_i^\star\mid A_i^\star\in\mathcal{A}^\star\}. \tag{2.3}$$

Definition 2.3 (see [2]). A bitopological space (X,τ_0,τ_1) is $(c,1-c)$-*regular*, $c\in\{0,1\}$, if for each $x\in X$ and each τ_c-open set U containing x, there exists a τ_c-open set V such that $x\in V\subseteq\operatorname{cl}_{\tau_{1-c}}V\subseteq U$. (X,τ_0,τ_1) is said to be *pairwise regular* if it is $(0,1)$-regular and $(1,0)$-regular.

Definition 2.4 (see [2]). A bitopological space (X,τ_0,τ_1) is said to be *pairwise normal* if given a τ_0-closed set A and a τ_1-closed set B with $A\cap B=\varnothing$, there exist a τ_1-open set U and a τ_0-open set V such that $A\subset U$, $B\subset V$, and $U\cap V=\varnothing$.

Remark 2.5. (1) It is worth noting that the notion of pairbase of Definition 2.1 differs from that of Fletcher and Lindgren in [1, page 163], Kopperman in [20, Definition 1.4], and Salbany in [5, Definition 2.3] (see Remark 2.10).

(2) If, in Definition 2.2(1), (3), we put $\mathcal{A}=\mathcal{A}^\star$, then for each $I'\subseteq I$ we have $\bigcap_{i\in I'}\{A_i\mid A_i\in\mathcal{A}\}=\operatorname{int}\bigcap_{i\in I'}\{A_i\mid A_i\in\mathcal{A}\}$ and $\operatorname{cl}_\tau(\bigcup_{i\in I'}\{A_i\mid A_i\in\mathcal{A}\})=\bigcup_{i\in I'}\{\operatorname{cl}_\tau A_i\mid A_i\in\mathcal{A}\}$. Thus the notions of τ-open cocushioned and τ-open weakly cushioned pair families of Definition 2.2 extend the notions of τ-interior preserving family and τ-closure preserving family, respectively.

Definition 2.6. A bitopological space (X,τ_0,τ_1) has a κ-τ_c-*open cocushioned*, κ-τ_{1-c}-*open weakly cushioned pairbase for* τ_c, $c\in\{0,1\}$, if and only if there are nested collections of τ_c-open families, $\mathcal{A}_{ca}=\{A_{cai}\mid a\in\kappa,\ i\in I_a\}$ and $\mathcal{A}_{ca}^\star=\{A_{cai}^\star\mid a\in\kappa,\ i\in I_a\}$ such that

(1) for each $a\in\kappa$, $(\mathcal{A}_{ca},\mathcal{A}_{ca}^\star)$ is a τ_c-open cocushioned, τ_{1-c}-open weakly cushioned enclosing pair family,

(2) $\cup\{(\mathcal{A}_{ca},\mathcal{A}_{ca}^\star)\mid a\in\kappa\}$ is pairbase for τ_c.

Theorem 2.7. *Let (X,τ_0,τ_1) be a quasi-metrizable bitopological space. Then for each $c\in\{0,1\}$, τ_c has a σ-τ_c-open cocushioned, σ-τ_{1-c}-open weakly cushioned pairbase.*

Proof. Let (X,τ_0,τ_1) be the above-mentioned space and d its quasi-metric ($\tau_d=\tau_0$, $\tau_{d^{-1}}=\tau_1$). For any $m\in\omega$, we put

$$\mathcal{Q}^m=\left\{B^\star\left(x,\frac{1}{m}\right)\mid x\in X\right\}=\left\{B^{-1}\left(x,\frac{1}{m}\right)\cap B\left(x,\frac{1}{m}\right)\mid x\in X\right\} \tag{2.4}$$

and we consider a well-ordering $<$ of $\{B^\star(x,1/m) \mid x \in X\}$ for a fixed m.

For a fixed n and x running X we put

$$\begin{aligned} S_n\left(B^\star\left(x,\frac{1}{m}\right)\right) &= \left\{y \mid B^\star\left(y,\frac{1}{n}\right) \subset B^\star\left(x,\frac{1}{m}\right)\right\}, \\ S'_n\left(B^\star\left(x,\frac{1}{m}\right)\right) &= S_n\left(B^\star\left(x,\frac{1}{m}\right)\right) \setminus \cup\left\{B^\star\left(z,\frac{1}{m}\right) \mid B^\star\left(z,\frac{1}{m}\right) < B^\star\left(x,\frac{1}{m}\right)\right\}. \end{aligned} \tag{2.5}$$

We first note that the different $S'_n(B^\star(x,1/m))$—as x runs through X—have $d^\star$-distance larger than $1/n$ (as in [27, page 252]). Next, put

$$\begin{aligned} E_{0n}\left(B^\star\left(x,\frac{1}{m}\right)\right) &= \cup\left\{B\left(y,\frac{1}{3n}\right) \mid y \in S'_n\left(B^\star\left(x,\frac{1}{m}\right)\right)\right\}, \\ E^\star_{0n}\left(B^\star\left(x,\frac{1}{m}\right)\right) &= \cup\left\{B\left(y,\frac{2}{3n}\right) \mid y \in S'_n\left(B^\star\left(x,\frac{1}{m}\right)\right)\right\}, \\ E_{1n}\left(B^\star\left(x,\frac{1}{m}\right)\right) &= \cup\left\{B^{-1}\left(y,\frac{1}{3n}\right) \mid y \in S'_n\left(B^\star\left(x,\frac{1}{m}\right)\right)\right\}, \\ E^\star_{1n}\left(B^\star\left(x,\frac{1}{m}\right)\right) &= \cup\left\{B^{-1}\left(y,\frac{2}{3n}\right) \mid y \in S'_n\left(B^\star\left(x,\frac{1}{m}\right)\right)\right\}. \end{aligned} \tag{2.6}$$

We put $E_{cn}(B^\star(x,1/m)) = \mathscr{E}_{cnm}(x)$, $E^\star_{cn}(B^\star(x,1/m)) = \mathscr{E}^\star_{cnm}(x)$, $c \in \{0,1\}$, and we conclude the following.

(1) *For each $c \in \{0,1\}$, $\{(\mathscr{E}_{cnm}(x),\mathscr{E}^\star_{cnm}(x)) \mid x \in X\}$ is a τ_c-open cocushioned, τ_{1-c}-open weakly cushioned, τ_c-open enclosing pair family.*

We prove it for the case $c = 0$. The case $c = 1$ is similar.

It is evident that for each $c \in \{0,1\}$, the pair family $\{(\mathscr{E}_{cnm}(x),\mathscr{E}^\star_{cnm}(x)) \mid x \in X\}$ is enclosing. For each $A \subseteq X$, let $y \in \cap\{\mathscr{E}_{onm}(t) \mid t \in A\}$, then there exists $\kappa_t \in S'_n(B^\star(t,1/m))$ such that $d(\kappa_t, y) < 1/3n$. If $a \in B(y,1/6n)$, then $d(\kappa_t,a) < 1/2n < 2/3n$, hence

$$\cap\{\mathscr{E}_{onm}(t) \mid t \in A\} \subseteq \operatorname{int}_{\tau_0} \cap\{\mathscr{E}^\star_{onm}(t) \mid t \in A\}. \tag{2.7}$$

Let $y \in \cap\{X \setminus \operatorname{cl}_{\tau_1}\mathscr{E}^\star_{onm}(t) \mid t \in A\}$, then there is $\kappa_t \in S'_n(B^\star(t,1/m))$ such that $d(\kappa_t,y) \geq 2/3n$. Suppose that $a \in B^{-1}(y,1/6n)$, then $d(\kappa_t,a) > 1/2n$. Thus $a \in \cap\{X \setminus \operatorname{cl}_{\tau_1}\mathscr{E}_{onm}(t) \mid t \in A\}$, consequently

$$\cap\{X \setminus \operatorname{cl}_{\tau_1}\mathscr{E}^\star_{onm}(t) \mid t \in A\} \subseteq \operatorname{int}_{\tau_1} \cap\{X \setminus \operatorname{cl}_{\tau_1}\mathscr{E}_{onm}(t) \mid t \in A\}. \tag{2.8}$$

(2) *For each $c \in \{0,1\}$, $\cup\{(\mathscr{E}_{cnm}(x),\mathscr{E}^\star_{cnm}(x)) \mid m,\ n \in \omega,\ x \in X\}$ is a pairbase for τ_c.*

We make use of the following three statements:

(i) for each $m \in \omega$, $\cup\{S'_n(B^\star(x,1/m)) \mid n \in \omega,\ x \in X\}$ is a covering of X (see [27, page 253]),

(ii) for each $m \in \omega$, each element of $\{S'_n(B^\star(x,1/m)) \mid n \in \omega,\ x \in X\}$ has $d^\star$-diameter at most $2/m$ (see [27, page 253]),

(iii) $S_n(B^\star(x,1/m)) \neq \varnothing$ for $n > m$.

We prove (2) for the case $c = 0$.

Given $x \in X$ and $\epsilon > 0$, we choose $m \in \omega$ such that $3/m < \epsilon$. From (i) there is a $y \in X$ and $n \in \omega$, such that $x \in S'_n(B^\star(y,1/m)) \subseteq E_{0n}(B^\star(y,1/m)) \subseteq E^\star_{0n}(B^\star(y,1/m))$.

We prove that $E^{\star}_{0n}(B^{\star}(y,1/m)) \subseteq B(x,\epsilon)$. In fact; if $a \in E^{\star}_{0n}(B^{\star}(y,1/m))$, then there is a $\beta \in S'_n(B^{\star}(y,1/m))$ such that $d(\beta,a) < 2/3n$ and, since β and x belong to $S'_n(B^{\star}(y,1/m))$, (ii) implies that $d(x,\beta) < 2/m$. Because of (iii) we have that $d(x,a) < 2/m + 2/3m < 3/m < \epsilon$. Similarly, it is proved for the case $c = 1$.

If $\{(\mathcal{A}_{cp},\mathcal{A}^{\star}_{cp}) \mid p \in \omega\} = \bigcup_{(n,m)\in\omega\times\omega}\{(\mathcal{E}_{cnm}(x),\mathcal{E}^{\star}_{cnm}(x)),\ m < n < p+2,\ p \in \omega,\ x \in X\}$, then for each $c \in \{0,1\}$, $\{(\mathcal{A}_{cp},\mathcal{A}^{\star}_{cp}) \mid p \in \omega\}$ is nested collection of τ_c-open cocushioned, τ_{1-c}-open weakly cushioned enclosing τ_c-open pair families and $\cup\{(\mathcal{A}_{cp},\mathcal{A}^{\star}_{cp}) \mid p \in \omega\}$ is a pairbase for τ_c. □

Fox in [22] (see [1, Theorem 7.15]) and Künzi in [28, Theorem 5] prove the conjecture of Lindgren and Fletcher in [24] that quasi-metrizability is equivalent to the availability of a local quasi-uniformity with a countable base and with a local quasi-uniformity for an inverse. In our program, we construct local quasi-uniformities $\mathcal{U}$ and $\mathcal{U}^{-1}$, in a more general form: they have decreasing bases.

Theorem 2.8. *If in a pairwise regular bitolopogical space (X,τ_0,τ_1) for each $c \in \{0,1\}$, τ_c has a κ-τ_c-open cocushioned, κ-τ_{1-c}-open weakly cushioned pairbase for τ_c, then there is a local quasi-uniformity $\mathcal{U}$ which has a decreasing base with cofinality κ, $\tau(\mathcal{U}) = \tau_0$ such that $\mathcal{U}^{-1}$ is a local quasi-uniformity with $\tau(\mathcal{U}^{-1}) = \tau_1$.*

Proof. Let for $c \in \{0,1\}$, $\{(\mathcal{A}_{ca},\mathcal{A}^{\star}_{ca}) \mid a \in \kappa\}$ be nested collection of τ_c-open cocushioned, τ_{1-c}-open weakly cushioned enclosing τ_c-open pair families, $\mathcal{A}_{ca} = \{A_{cai} \mid i \in I_a\}$ and $\mathcal{A}^{\star}_{ca} = \{A^{\star}_{cai} \mid i \in I_a\}$. Suppose that every pair family $(\mathcal{A}_{ca},\mathcal{A}^{\star}_{ca})$ contains (X,X) and $(\emptyset,\emptyset)$.

For each $a \in \kappa$ and each $x \in X$, we put

$$\begin{aligned}
\mathcal{K}^a_x &= \operatorname{int}_{\tau_0}\bigcap_{i\in I_a}\{A^{\star}_{oai} \mid x \in A_{oai}\},\\
\Lambda^a_x &= \operatorname{int}_{\tau_1}\bigcap_{i\in I_a}\{X\setminus \operatorname{cl}_{\tau_1}A_{oai} \mid x \in X\setminus \operatorname{cl}_{\tau_1}A^{\star}_{oai}\},\\
\mathcal{M}^a_x &= \operatorname{int}_{\tau_1}\bigcap_{i\in I_a}\{A^{\star}_{1ai} \mid x \in A_{1ai}\},\\
N^a_x &= \operatorname{int}_{\tau_0}\bigcap_{i\in I_a}\{X\setminus \operatorname{cl}_{\tau_0}A_{1ai} \mid x \in X\setminus \operatorname{cl}_{\tau_0}A^{\star}_{1ai}\}.
\end{aligned} \tag{2.9}$$

We form

$$V_a = \cup\{\Lambda^a_x \times \mathcal{K}^a_x \mid x \in X\}, \qquad W_a = \cup\{N^a_x \times \mathcal{M}^a_x \mid x \in X\} \tag{2.10}$$

and show that each of the families $\{V_a \mid a \in \kappa\}$ and $\{W_a \mid a \in \kappa\}$ forms a decreasing base for a local quasi-uniformity compatible with τ_0 and τ_1, respectively. We prove it for the first family.

The family $\{V_a \mid a \in \kappa\}$ is decreasing. In fact, if $\beta \geq a$, then $\mathcal{K}^{\beta}_x \subseteq \mathcal{K}^a_x$ and $\Lambda^{\beta}_x \subseteq \Lambda^a_x$, hence $V_{\beta} \subseteq V_a$.

Next, if $x \in X$, $A \in \tau_0$ and $x \in A$, since the space is pairwise regular and $\cup\{(\mathscr{A}_{oa}, \mathscr{A}^{\star}_{oa}) \mid a \in \kappa\}$ is pairbase for τ_0, we can choose $\beta, \gamma \in \kappa$ such that $(C, C^{\star}) \in (\mathscr{A}_{o\gamma}, \mathscr{A}^{\star}_{o\gamma})$, $(B, B^{\star}) \in (\mathscr{A}_{o\beta}, \mathscr{A}^{\star}_{o\beta})$ and $x \in C \subseteq \mathrm{cl}_{\tau_1} C^{\star} \subseteq B \subseteq \mathrm{cl}_{\tau_1} B^{\star} \subseteq A$. Since the collection $\{(\mathscr{A}_{oa}, \mathscr{A}^{\star}_{oa}) \mid a \in \kappa\}$ is nested, there is a $\delta \in \kappa$ such that $(C, C^{\star})$ and $(B, B^{\star})$ belong to $(\mathscr{A}_{o\delta}, \mathscr{A}^{\star}_{o\delta})$.

We have that $V_{\delta}[C] = \cup\{\mathscr{K}^{\delta}_{y} \mid \Lambda^{\delta}_{y} \cap C \neq \varnothing,\ y \in X\}$ and we prove that $V_{\delta}[C] \subseteq A$. In fact; if $y \in B$, then $\mathscr{K}^{\delta}_{y} \subseteq B^{\star} \subseteq A$. If $y \notin B$, then $y \in X \setminus \mathrm{cl}_{\tau_1} C^{\star}$ and $\Lambda^{\delta}_{y} \subseteq X \setminus \mathrm{cl}_{\tau_1} C$ or $\Lambda^{\delta}_{y} \cap C = \varnothing$. Hence $V_{\delta}[C] \subseteq A$ and from Lemma 1.9 the family $\Gamma = \{V_a \mid a \in \kappa\}$ is a decreasing base for a local quasi-uniformity such that $\tau(\Gamma) = \tau_0$.

There also holds that $V^{-1}_{\delta}[x] = \bigcup_y \{\Lambda^{\delta}_{y} : x \in \mathscr{K}^{\delta}_{y}\}$, hence it is τ_1-open.

Similarly we conclude that $E = \{W_a \mid a \in \kappa\}$ is a decreasing base for a local quasi-uniformity such that $\tau(E) = \tau_1$, and $W^{-1}_{a}[x] = \bigcup_y \{N^{a}_{y} \mid x \in \mathscr{M}^{a}_{y}\}$ τ_0-open set.

Let $F = \Gamma \bigvee E^{-1}$. Then $\tau(F) = \tau_0$ and $\tau(F^{-1}) = \tau_1$. Hence, by Lemma 1.10, F (resp., F^{-1}) is a base for a local quasi-uniformity. We now pick up a decreasing family $\mathscr{B}$ of entourages of the form $V_a \cap W^{-1}_{a}$, where V_a and W_a belong to Γ and E, respectively. This family is a decreasing base for a local quasi-uniformity $\mathscr{U}$, which induces the topology τ_0 as well, thus $\tau(\mathscr{U}) = \tau_0$. There also hold $\tau(\mathscr{U}^{-1}) = \tau_1$ and the proof is completed. □

Theorem 2.9. *A pairwise regular bitolopogical space (X, τ_0, τ_1) is quasi-metrizable if and only if for each $c \in \{0,1\}$, τ_c has a σ-τ_c-open cocushioned, σ-τ_{1-c}-open weakly cushioned pairbase for τ_c.*

Proof. Sufficiency: we conclude it from Theorem 2.8 for $\kappa = \aleph_0$. *Necessity*: it results from Theorem 2.7. □

Remark 2.10. A natural extension of the notion of a locally finite family is the notion of interior preserving family. Köfner in [17] proves that, in a T_1 space the existence of a σ-interior preserving base is equivalent to the admission of a non-Archimedean quasimetric. Unfortunately, this result cannot be extended to the class of all quasimetric spaces since the Köfner plane is a quasi-metric space whose topology does not have a σ-interior preserving base. Hence, a generalization of the Bing-Nagata-Smirnov's metrization theorem in asymmetric spaces cannot be based on the existence of a σ-interior preserving base or σ-locally finite base. Hence, to succeed a Bing-Nagata-Smirnov's type quasi-metrization theorem, we have to find a common generalization of the notions of σ-interior preserving (resp., σ-closure preserving) base and of metrizability. This generalization exists if we use the notion of an open pair family. By Remark 2.5 and Theorems 2.7 and 2.8, it is clear that the notions of σ-open cocushioned pairbase and σ-weakly open cushioned pairbase of Definition 2.2 satisfy this natural and basic requirement. On the other hand, in the Fox-Kopperman's approach, the members of the second family $\mathscr{A}^{\star}$ (of the pair family $(\mathscr{A}, \mathscr{A}^{\star})$, which they use), are: arbitrary (in the Fox case) or closed (in the Kopperman case). Thus, the equality $\mathscr{A} = \mathscr{A}^{\star}$ may be applied only if the space is zero dimensional (see [20, pages 103-104]) in the Kopperman case, and only if the space is discrete in the Fox case. Moreover, Fox and Kopperman prove the sufficient part of the quasi-metrization problem by using, for each $n \in \mathbb{N}$, as pair bases the families $\{(B_n(x, r_1), B_n(x, r_2)) \mid x \in X\}$, $\{(B^{-1}_{n}(x, r'_1), B^{-1}(x, r'_2)) \mid x \in X\}$, $r_2 > r_1$, $r'_2 > r'_1$. This proof ensures a simple and immediate result, but following this procedure in the case of metric spaces (where $r_1 = r_2 = r_1' = r_2' = r$ and $B(x,r) = B^{-1}(x,r)$), we get families of the form $\{B(x,r) \mid x \in X\}$. But these families cannot give the necessary part of the metrization problem, because they are not in general locally finite or interior preserving. In contrast, if we make use of Theorem 2.7 for metric spaces, then the σ-pairbases coincide with the usual σ-locally finite base which the Bing-Nagata-Smirnov's theorem gives. This allows us to approach the quasi-metrization problem more naturally from that of Fox and Kopperman.

We present an alternative form of Theorem 2.9.

Definition 2.11 (see ([29])). A subset of a bitopological space (X,τ_0,τ_1) is *$(c,1-c)$-regular*, $c \in \{0,1\}$, if and only if it is equal to the τ_c-interior of its τ_{1-c}-closure.

If a pairwise regular bitopological space (X,τ_0,τ_1) has as pairbase for τ_c, the pair family $(\mathcal{A},\mathcal{A}^\star)$, $c \in \{0,1\}$, then the pair family $(\operatorname{int}_{\tau_c} \operatorname{cl}_{\tau_{1-c}} \mathcal{A}, \operatorname{int}_{\tau_c} \operatorname{cl}_{\tau_{1-c}} \mathcal{A}^\star)$ is a pairbase for τ_c as well. Hence, from now on, we may consider that the members of pairbases of a pairwise regular bitopological space (X,τ_0,τ_1) are $(c,1-c)$-regular (if (X,τ_0,τ_1) is a bitopological space and $A \subset X$, then for each $c \in \{0,1\}$, we have $\operatorname{int}_{\tau_c} \operatorname{cl}_{\tau_{1-c}} A = \operatorname{int}_{\tau_c} \operatorname{cl}_{\tau_{1-c}} \operatorname{int}_{\tau_c} \operatorname{cl}_{\tau_{1-c}} A$.) and τ_c-open subsets of X.

The pair families we use in Theorems 2.7 and 2.8 have the following form:

$(\mathcal{P},\mathcal{P}^\star) = (\mathcal{A},\mathcal{A}^\star) \cup (X \setminus \operatorname{cl}_{\tau_0} \mathcal{B}^\star, X \setminus \operatorname{cl}_{\tau_0} \mathcal{B})$ which is τ_0-open cocushioned pairbase for τ_0 and

$(\mathcal{Q},\mathcal{Q}^\star) = (\mathcal{B},\mathcal{B}^\star) \cup (X \setminus \operatorname{cl}_{\tau_1} \mathcal{A}^\star, X \setminus \operatorname{cl}_{\tau_1} \mathcal{A})$ which is τ_1-open cocushioned pairbase for τ_1.

If we suppose that the members of $\mathcal{A},\mathcal{A}^\star$ and $\mathcal{B},\mathcal{B}^\star$ are $(0,1)$-regular sets and $(1,0)$-regular sets, respectively, then from $X \setminus \operatorname{cl}_{\tau_0}(X \setminus \operatorname{cl}_{\tau_1} A) = \operatorname{int}_{\tau_0} \operatorname{cl}_{\tau_1} A = A$ for each $A \in \mathcal{A} \cup \mathcal{A}^\star$ and $X \setminus \operatorname{cl}_{\tau_1}(X \setminus \operatorname{cl}_{\tau_0} B) = \operatorname{int}_{\tau_1} \operatorname{cl}_{\tau_0} B = B$ for each $B \in \mathcal{B} \cup \mathcal{B}^\star$, we conclude that

$$(X \setminus \operatorname{cl}_{\tau_1} \mathcal{P}^\star, X \setminus \operatorname{cl}_{\tau_1} \mathcal{P}) = (\mathcal{Q},\mathcal{Q}^\star), \qquad (X \setminus \operatorname{cl}_{\tau_0} \mathcal{Q}^\star, X \setminus \operatorname{cl}_{\tau_0} \mathcal{Q}) = (\mathcal{P},\mathcal{P}^\star). \tag{2.11}$$

Definition 2.12. Let $(\mathcal{A},\mathcal{A}^\star)$ be an open pair family of a topological space (X,τ) and let $\hat{\tau}$ be an arbitrary topology on X. We call *$\hat{\tau}$-conjugate pair family of* $(\mathcal{A},\mathcal{A}^\star)$, the pair family $(X \setminus \operatorname{cl}_{\hat{\tau}} \mathcal{A}^\star, X \setminus \operatorname{cl}_{\hat{\tau}} \mathcal{A})$.

According to the previous statement, Theorem 2.9 is equivalent to the following.

Theorem 2.13. *A pairwise regular bitopological space (X,τ_0,τ_1) is quasi-metrizable if and only if τ_0 has a σ-τ_0-open cocushioned pairbase whose τ_1-conjugate pair family is σ-τ_1-open cocushioned pairbase for τ_1.*

An alternation of the τ_0 and τ_1 yields a dual statement.

An immediate consequence of Theorem 2.9 is the following theorem.

THEOREM 2.14 (see Fox [22]). *Let (X,τ_0,τ_1) be a pairwise regular bitopological space. The following statements are equivalent.*

(i) (X,τ_0,τ_1) *is quasimetrizable.*

(ii) *For each $c \in \{0,1\}$, τ_c has a σ-τ_c-cocushioned, σ-τ_{1-c}-cushioned pairbase.*

(iii) *For each $c \in \{0,1\}$, τ_c has a σ-τ_c-open cocushioned, σ-τ_{1-c}-open weakly cushioned pairbase.*

Proof. (ii)⇒(iii). Let for $c \in \{0,1\}$, $(\mathcal{A}_c,\mathcal{A}_c{}^\star) = \{(\mathcal{A}_{cn},\mathcal{A}_{cn}^\star) \mid n \in \omega\}$ be a σ-τ_c-cocushioned, σ-τ_{1-c}-cushioned pairbase for τ_c topology. Let also $\mathcal{A}_{cn} = \{A_{cni} \mid i \in I_n\}$ and $\mathcal{A}_{cn}^\star = \{A_{cni}^\star \mid i \in I_n\}$. We put

$$\begin{gathered} V_{cn}(x) = \operatorname{int}_{\tau_c} \cap \{A_{cni}^\star \mid x \in A_{cni},\ (A_{cni},A_{cni}^\star) \in (\mathcal{A}_{cn},\mathcal{A}_{cn}^\star),\ i \in I_n\}, \\ V(A_{cni}^\star) = \cup \{V_{cn}(x) \mid x \in A_{cni}^\star\}. \end{gathered} \tag{2.12}$$

Let now $B_{cni}(x) = \operatorname{int}_{\tau_c} A_{cni}$, $B_{cni}^\star = V(A_{cni}^\star)$, $\mathcal{B}_{cn} = \{B_{cni} \mid i \in I_n\}$ and $\mathcal{B}_{cn}^\star = \{B_{cni}^\star \mid i \in I_n\}$. We put $(\mathcal{B}_c,\mathcal{B}_c^\star) = \{(\mathcal{B}_{cn},\mathcal{B}_{cn}^\star) \mid n \in \omega\}$. Then

(1) $(\mathcal{B}_c,\mathcal{B}_c^\star)$ *is a σ-pairbase for τ_c-topology which its members are open sets*. Indeed, since $(\mathcal{A}_c,\mathcal{A}_c^\star)$ is a σ-pairbase for τ_c-topology, for $x \in O \in \tau$ we can find m such that $(A_{cmi},A_{cmi}^\star),(A_{cmj},A_{cmj}^\star) \in \mathcal{A}_{cm}$ and $x \in \operatorname{int}_{\tau_c} A_{cmi} \subseteq A_{cmi} \subseteq A_{cmi}^\star \subseteq A_{cmj} \subseteq A_{cmj}^\star \subseteq O$. Thus $x \in \operatorname{int}_{\tau_c} A_{cmi} \subseteq A_{cmi} \subseteq V_m(A_{cmi}^\star) \subseteq A_{cmj}^\star \subseteq O$. Hence, $x \in B_{cmi} \subseteq B_{cmi}^\star \subseteq O$.

(2) *For each $n \in \omega$, $(\mathcal{B}_{cn},\mathcal{B}_{cn}^\star)$ is a τ_c-open cocushioned τ_{1-c}-open weakly cushioned pair family*. In fact; if $I \subseteq I_n$, then

$$\begin{aligned} \cap\{B_{cni} \mid i \in I\} &= \cap\{\operatorname{int}_{\tau_c} A_{cni} \mid i \in I\} \subseteq \cap\{A_{cni} \mid i \in I\} \subseteq \operatorname{int}_{\tau_c} \cap\{A_{cni}^\star \mid i \in I\} \\ &\subseteq \operatorname{int}_{\tau_c} \cap\{V(A_{cni}^\star) \mid i \in I\} = \operatorname{int}_{\tau_c} \cap\{B_{cni}^\star \mid i \in I\}, \\ \operatorname{cl}_{\tau_{1-c}} \cup\{B_{cni} \mid i \in I\} &= \operatorname{cl}_{\tau_{1-c}} \cup\{\operatorname{int}_{\tau_c} A_{cni} \mid i \in I\} \subseteq \operatorname{cl}_{\tau_{1-c}} \cup\{A_{cni} \mid i \in I\} \subseteq \cup\{A_{cni}^\star \mid i \in I\} \\ &\subseteq \cup\{V(A_{cni}^\star) \mid i \in I\} \subseteq \cup\{\operatorname{cl}_{\tau_{1-c}} V(A_{cni}^\star) \mid i \in I\} \\ &= \cup\{\operatorname{cl}_{\tau_{1-c}} B_{cni}^\star \mid i \in I\}. \end{aligned} \tag{2.13}$$

(iii)⇒(ii). Let for $c \in \{0,1\}$, $(\mathcal{A}_c,\mathcal{A}_c{}^\star) = \{(\mathcal{A}_{cn},\mathcal{A}_{cn}^\star) \mid n \in \omega\}$ be a σ-τ_c-open cocushioned, σ-τ_{1-c}-open weakly cushioned pairbase for τ_c topology. Then, the pair family $(\mathcal{A}_c,\operatorname{cl}_{\tau_{1-c}} \mathcal{A}_c{}^\star) = \{(\mathcal{A}_{cn},\operatorname{cl}_{\tau_{1-c}} \mathcal{A}_{cn}^\star) \mid n \in \omega\}$ is a σ-τ_c-cocushioned, σ-τ_{1-c}-cushioned pairbase for τ_c.

The equivalence of (i) and (ii) is given by the quasi-metrization theorem of Fox [22]. □

THEOREM 2.15 (see Kelly [2]). *If (X,τ_0,τ_1) is a pairwise regular space and τ_0 and τ_1 have countable bases, then (X,τ_0,τ_1) is quasimetrizable.*

Proof. Let $\mathcal{P} = \{P_n \mid n \in \omega\}$ and $\mathcal{Q} = \{Q_n \mid n \in \omega\}$ be countable bases of τ_0 and τ_1, respectively. If $\mathcal{A}_n = \cup\{P_m \mid m \le n\}$ and $\mathcal{B}_n = \cup\{Q_m \mid m \le n\}$, then the pair families $\{(\mathcal{A}_n, \mathcal{A}_n) \mid n \in \omega\}$ and $\{(\mathcal{B}_n,\mathcal{B}_n) \mid n \in \omega\}$ satisfy the suppositions of Theorem 2.9. □

Theorem 2.16 (see Lane [3]). *If (X,τ_0,τ_1) is a pairwise regular space, τ_0 has a σ-τ_0- and τ_1-locally finite base $\mathcal{P} = \{P_n \mid n \in \omega\}$ and τ_1 has a σ-τ_0- and τ_1-locally finite base $\mathcal{Q} = \{Q_n \mid n \in \omega\}$, then (X,τ_0,τ_1) is quasimetrizable.*

Proof. If $\mathcal{P} = \{P_n \mid n \in \omega\}$ is a σ-τ_0- and τ_1-locally finite base for τ_0, then the pair-family $\mathcal{A} = \{(\mathcal{A}_n,\mathcal{A}_n) \mid n \in \omega\}$ where $\mathcal{A}_n = \cup\{P_m \mid m \leq n\}$ is a σ-τ_0-open cocushioned, σ-τ_1-open weakly cushioned pairbase for τ_0. Similarly, we can find a pair family $\mathcal{B} = \{(\mathcal{B}_n,\mathcal{B}_n) \mid n \in \omega\}$ ($\mathcal{B}_n = \cup\{Q_n \mid m \leq n\}$) which is σ-τ_1-cocushioned, σ-τ_0-weakly cushioned pairbase for τ_1. By Theorem 2.9, (X,τ_0,τ_1) is quasimetrizable. □

Theorem 2.17 (see Raghavan-Reilly [8]). *A pairwise Hausdorff bitopological space (X,τ_0,τ_1) is quasi-metrizable if and only if for each point $x \in X$, one can assign τ_c neighborhood bases $\{S(n,c;x) \mid n \in \omega\}$, $c \in \{0,1\}$ such that*

(i) *$y \notin S(n-1,c;x)$ implies $S(n,c;x) \cap S(n,1-c;y) = \varnothing$;*
(ii) *$y \in S(n,c;x)$ implies $S(n,c;y) \subset S(n-1,c;x)$, $c \in \{0,1\}$.*

Proof. For each $c \in \{0,1\}$, $(\mathcal{A}_c,\mathcal{A}_c^\star) = \{(\mathcal{A}_{cn},\mathcal{A}_{cn}^\star) \mid n \in \omega\}$, $(\mathcal{A}_{cn},\mathcal{A}_{cn}^\star) = \{(S(n,c,x), S(n-1,c,x)) \mid x \in X\}$ is a σ-τ_c-open cocushioned, σ-τ_{1-c}-open weakly cushioned pairbase for τ_c. The space is quasimetrizable by Theorem 2.9. □

Theorem 2.18 (see Parrek [6]). *A bitopological space (X,τ_0,τ_1) is quasi-metrizable if and only if there are functions $g : \omega \times X \to \tau_0$ and $h : \omega \times X \to \tau_1$ such that*

(i) *for every $x \in X$, $\{g(n,x) \mid n \in \omega\}$ is a τ_0-neighborhood base of x, and $y \in g(n,x)$ implies $g(n,y) \subset g(n-1,x)$,*
(ii) *for every $x \in X$, $\{h(n,x) \mid n \in \omega\}$ is a τ_1-neighborhood base of x, and $y \in h(n,x)$ implies $h(n,y) \subset h(n-1,x)$,*
(iii) *$y \in g(n,x)$ if and only if $x \in h(n,y)$.*

Proof. It is obvious that the pair family $(\mathcal{A},\mathcal{A}^\star) = \{(\mathcal{A}_n,\mathcal{A}_n^\star) \mid n \in \omega\}$ where $(\mathcal{A}_n,\mathcal{A}_n^\star) = \{(g(n,x),\ g(n-1,x)) \mid x \in X\}$ is a σ-τ_0-open cocushioned, σ-τ_1-open weakly cushioned pairbase for τ_0 and the pair family $(\mathcal{B},\mathcal{B}^\star) = \{(\mathcal{B}_n,\mathcal{B}_n^\star) \mid n \in \omega\}$ where $(\mathcal{B}_n,\mathcal{B}_n^\star) = \{(h(n,x),h\ (n-1,x)) \mid x \in X\}$ is a σ-τ_1-open cocushioned, σ-τ_0-open weakly cushioned pairbase for τ_1. By Theorem 2.9, (X,τ_0,τ_1) is quasimetrizable. □

Salbany [5] gives the next definitions: let $\mathcal{Q} = \{(A_a,B_a) \mid a \in A\}$ be a set of pairs of subsets of a bitopological space (X,τ_0,τ_1). Then $\mathcal{Q}$ is (τ_1,τ_0)-*locally finite* if each $x \in X$ has a τ_0-neighborhood $V_{\tau_0}(x)$ and a τ_1-neighborhood $V_{\tau_1}(x)$ such that $V_{\tau_0}(x) \cap B_a \neq \varnothing$ and $V_{\tau_1}(x) \cap A_a \neq \varnothing$ for only finitely many pairs (A_a,B_a). For each $c \in \{0,1\}$, the topology τ_c has an *open* S-*pairbase* $\{(A_a,B_a) \mid a \in A\}$ if A_a is τ_c-open, B_a is τ_{1-c}-open, $A_a \cup B_a = X$, and for $x \in U \in \tau_c$ there is a such that $x \in X \setminus B_a \subset A_a \subset U$.

It is evident that Salbany's definition of S-pairbase differs from that of Definition 2.1.

Theorem 2.19 (see Salbany [5]). *Let (X,τ_0,τ_1) be a pairwise regular bitopological space such that τ_0 has a σ-(τ_1,τ_0)-locally finite S-pairbase and τ_1 has a σ-(τ_0,τ_1)-locally finite S-pairbase. Then (X,τ_0,τ_1) is quasimetrizable.*

Proof. Let (X,τ_0,τ_1) be a bitopological space and τ_c has a σ-(τ_{1-c},τ_c)-locally finite S-pairbase $\mathcal{Q}_c = \{(\mathcal{A}_{nc},\mathcal{B}_{nc}) \mid n \in \omega\}$, where $(\mathcal{A}_{nc},\mathcal{B}_{nc}) = \{(A_{nci},B_{nci}) \mid i \in I_n\}$, $c \in \{0,1\}$.

Then $\mathcal{P}_c = \{(\Gamma_{nc}, \Delta_{nc}) \mid n \in \omega\}$, where $(\Gamma_{nc}, \Delta_{nc}) = \{(X \setminus \mathrm{cl}_{\tau_c} B_{nci}, A_{nci}) \mid i \in I_n\}$ is a σ-τ_c-open cocushioned, τ_{1-c}-open weakly cushioned pairbase (in the meaning of Definition 2.2) for τ_c. We prove it for $c = 0$.

It is clear that $\mathcal{P}_c$ is a pairbase (in the meaning of Definition 2.2) for τ_c. Let $I \subseteq I_n$ and let for a fixed $n \in \omega$, $x \in \cap\{X \setminus \mathrm{cl}_{\tau_0} B_{n0i} \mid i \in I\} \subset \cap\{A_{n0i} \mid i \in I\}$. Without loss of generality we can assume that for each $i \in I$, the sets A_{n0i} and B_{n0i} are (0,1)-regular and (1,0)-regular, respectively. If $x \in A_{n0i}$ only for a finite number of values i, then there is a τ_0-open neighborhood $V_{\tau_0}(x)$ of x such that $V_{\tau_0}(x) \subset \cap\{A_{n0i} \mid i \in I\}$. Otherwise, for each τ_1-open neighborhood $V_{\tau_1}(x)$ of x, $V_{\tau_1}(x) \cap A_{n0i} \neq \emptyset$, except for finitely many (A_{n0i}, B_{n0i}). By (τ_1, τ_0)-locally finiteness of the space, there is τ_0-open neighborhood $V_{\tau_0}(x)$ of x such that $V_{\tau_0}(x) \cap B_{n0i} \neq \emptyset$ for only finitely many B_{n0i}. Hence, there is a τ_0-open neighborhood $V_{\tau_0}(x)$ of x such that $V_{\tau_0}(x) \subset \cap\{X \setminus \mathrm{cl}_{\tau_0} B_{n0i} \mid i \in I\} \subset \cap\{A_{n0i} \mid i \in I\}$. It follows that $x \in \mathrm{int}_{\tau_0} \cap\{A_{n0i} \mid i \in I\}$. Thus, $\cap(X \setminus \mathrm{cl}_{\tau_1} B_{n0i}) \subseteq \mathrm{int}_{\tau_0} \cap A_{n0i}$ which implies that $\{(X \setminus \mathrm{cl}_{\tau_0} B_{n0i}, A_{n0i}) \mid i \in I_n\}$ is a τ_0-open cocushioned pair family. Similarly, $\{(X \setminus \mathrm{cl}_{\tau_1} A_{n0i}, X \setminus \mathrm{cl}_{\tau_1}(X \setminus \mathrm{cl}_{\tau_0} B_{n0i})) \mid i \in I_n\} = \{(X \setminus \mathrm{cl}_{\tau_1} A_{n0i}, B_{n0i}) \mid i \in I_n\}$ is τ_1-open cocushioned pair family. Hence $\mathcal{P}_0 = \{(\Gamma_{n0}, \Delta_{n0}) \mid n \in \omega\}$ is σ-τ_0-open cocushioned, σ-τ_1-open weakly cushioned pairbase for τ_0. Similarly, we can prove that $\mathcal{P}_1 = \{(\Gamma_{n1}, \Delta_{n1}) \mid n \in \omega\}$ is σ-τ_1-open cocushioned, σ-τ_0-open weakly cushioned pairbase for τ_1. By Theorem 2.9, the space is quasimetrizable. □

Theorem 2.20 (Salbany's conjecture [5, page 504]). *Every pairwise regular bitopological space (X, τ_0, τ_1) such that τ_0 has a σ-(τ_1, τ_0) locally finite S-pairbase and τ_1 has a σ-(τ_0, τ_1) locally finite S-pairbase is pairwise normal.*

The proof is an immediate consequence of [2, Theorem 2.19 and Proposition 4.2].

Theorem 2.21 (Patty's conjecture [4], Salbany [5]). *If (X, τ_0, τ_1) is pairwise regular and τ_0 has a σ-τ_1-locally finite base and τ_1 has a σ-τ_0-locally finite base, then X is quasimetrizable.*

Proof. Let τ_0 has a σ-τ_1 locally finite base $\mathcal{P} = \{\mathcal{P}_n \mid n \in \omega\}$, $\mathcal{P}_n = \{P_{ni} \mid i \in I_n\}$ and τ_1 has a σ-τ_0 locally finite base $\mathcal{Q} = \{\mathcal{Q}_n \mid n \in \omega\}$, $\mathcal{Q}_n = \{Q_{nj} \mid j \in J_n\}$. Then for each $n \in \omega$, $\mathcal{P}_n$ and $\mathcal{Q}_n$ are point finite. Thus $\mathcal{P}^\star_{nm} = \{(\cup P_{mi'}, P_{ni}) \mid \mathrm{cl}_{\tau_1} P_{mi'} \subset P_{ni}\}$ is a σ-τ_0-open cocushioned, σ-τ_1-open weakly cushioned pairbase for τ_0 and $\mathcal{Q}^\star_{nm} = \{(\cup Q_{mj'}, Q_{nj}) \mid \mathrm{cl}_{\tau_0} Q_{mj'} \subset Q_{nj}\}$ is a σ-τ_1-open cocushioned, σ-τ_0-open weakly cushioned pairbase for τ_1. By Theorem 2.9, (X, τ_0, τ_1) is quasimetrizable. □

Romaguera in [9, page 329] gives the next definitions.

Definition 2.22. Let (X, τ_0, τ_1) be a bitopological space and $\mathcal{F} = \{f_i \mid i \in I\}$ a family of τ_0-l.s.c. and τ_1-u.s.c. real-valued functions on X. Then the family $\mathcal{F}$ is *pairwise relative complete* if for every $J \subset I$, $f(x) = \inf\{f_j(x) \mid j \in J\}$ is τ_0-l.s.c. and $F(x) = \sup\{f_j(x) \mid j \in J\}$ is τ_1-u.s.c.

Definition 2.23. (X, τ_0, τ_1) is the *pairwise initial bitopological space induced by family $\mathcal{F} = \{f_i : X \to \mathbb{R} \mid i \in I\}$*, if τ_0 is the initial topology (a topology τ on X is said to be *initial topology* with respect to the family $\mathcal{F} = \{f_i : X \to R \mid i \in I\}$, if τ is the coarsest topology on X which makes all f_is continuous) of $\mathcal{F}$ with respect to $\tau_0^\star = \{\emptyset, \mathbb{R}, (a, +\infty),\ a \in \mathbb{R}\}$

and τ_1 is the initial topology of $\mathcal{F}$ with respect to $\tau_1^\star = \{\varnothing,\mathbb{R},(-\infty,a),\ a\in\mathbb{R}\}$. Denote $\tau_0 = \tau_0^\star(\mathcal{F})$ and $\tau_1 = \tau_1^\star(\mathcal{F})$.

Theorem 2.24 (Romaguera [9]). *The space (X,τ_0,τ_1) is quasimetrizable if and only if it is the pairwise initial space induced by a σ-pairwise relatively complete family $\mathcal{F}$.*

Proof. Let $\mathcal{F} = \bigcup_{n\in\omega}\mathcal{F}_n$ with $\tau_0^\star(\mathcal{F}) = \tau_0$, $\tau_1^\star(\mathcal{F}) = \tau_1$ and every $\mathcal{F}_n = \{f_i \mid i\in I_n\}$ pairwise relative complete. Suppose that $\mathcal{P} = \{\mathcal{P}_n \mid n\in\omega\}$, $\mathcal{P}_n = \{(f_i^{-1}(m/2^n,+\infty), f_i^{-1}(m/2^n,+\infty)) \mid i\in I_n, m\in\omega\}$ and $\mathcal{Q} = \{\mathcal{Q}_n \mid n\in\omega\}$, $\mathcal{Q}_n = \{(f_i^{-1}(-\infty,m/2^n), f_i^{-1}(-\infty, m/2^n)) \mid i\in I_n,\ m\in\omega\}$.

Let $x\in\cap\{f_j^{-1}(m/2^n,+\infty) \mid j\in J\subset I_n,\ m\in\omega\}$. For each $m\in Z$, let $J_{mn} = \{j\in J \mid m/2^n \le f_j(x)\le (m+1)/2^n\}$. Since for each $J\subset I_n$, $\inf\{f_j(x)\mid j\in J\}\le f_j(x)\le\sup\{f_j(x)\mid j\in J\}$, $J_{mn}\ne\varnothing$ only for a finite m. By hypothesis, $f_{mn} = \inf\{f_j \mid j\in J_{mn}\}$ is τ_0-l.s.c. and $F_{mn} = \sup\{f_j \mid j\in J_{mn}\}$ is τ_1-u.s.c. Thus $U_n(x) = \bigcap_m f_{mn}^{-1}(m/2^n,+\infty)$ is a τ_0-open neighborhood of x such that $U_n(x)\subset\cap\{f_j^{-1}(m/2^n,+\infty)\mid j\in J,\ m\in\omega\}$. Similarly, if $x\in\cap\{f_j^{-1}(-\infty,m/2^n)\mid j\in J\subset I_n,\ m\in\omega\}$, then $V_n(x) = \bigcap_m F_{mn}^{-1}(-\infty,m/2^n)$ is a τ_1-open neighborhood of x such that $V_n(x)\subset\cap\{f_j^{-1}(-\infty,m/2^n)\mid j\in J,\ m\in\omega\}$. Hence $\mathcal{P}$ is σ-τ_0-open cocushioned and $\mathcal{Q}$ is σ-τ_1-open cocushioned.

Clearly, $X\setminus \mathrm{cl}_{\tau_1} f_i^{-1}(m/2^n,+\infty) = f_i^{-1}(-\infty,m/2^n)$ and $X\setminus\mathrm{cl}_{\tau_0} f_i^{-1}(-\infty,m/2^n) = f_i^{-1}(m/2^n,+\infty)$, hence $\mathcal{P}$ is σ-τ_1-open weakly cushioned and $\mathcal{Q}$ is σ-τ_0-open weakly cushioned. By Theorem 2.9 the space is quasimetrizable. □

3. From bitopologies to topologies

We come now to the case of topological spaces.

Definition 3.1. A σ-τ-open cocushioned pairbase (resp., σ-τ-interior-preserving base) $(\mathcal{A},\mathcal{A}^\star)$ (resp., $\mathcal{A}$) of a topological space (X,τ) is *complementary*, if there is a topology $\hat{\tau}$ on X such that $(X\setminus\mathrm{cl}_{\hat{\tau}}\mathcal{A}^\star, X\setminus\mathrm{cl}_{\hat{\tau}}\mathcal{A})$ (resp., $X\setminus\mathrm{cl}_{\hat{\tau}}\mathcal{A}$) is a σ-$\hat{\tau}$-open cocushioned pairbase (resp., σ-$\hat{\tau}$-interior-preserving base) for $\hat{\tau}$.

Theorem 3.2. *A topological space (X,τ) is quasimetrizable if and only if it has a complementary σ-τ-open cocushioned pairbase.*

The proof is an immediate consequence of Definition 3.1 and Theorem 2.13.

Theorem 3.3 (Bing-Nagata-Smirnov theorem). *The following conditions on a regular space (X,τ) are equivalent:*

(i) *the space (X,τ) is metrizable,*

(ii) *the space (X,τ) has a σ-discrete base,*

(iii) *the space (X,τ) has a σ-locally finite base.*

Proof. (iii)⇒(i). Let $\mathcal{E} = \{\mathcal{E}_n \mid n\in\omega\}$ be a σ-locally finite base for τ. Then, $\mathcal{A} = \{\mathcal{A}_n \mid n\in\omega\}$, where $\mathcal{A}_n = \cup\{\mathcal{E}_m \mid m\le n\}$ is a nested collection of locally finite families. Thus, the pair family $\mathcal{P} = \{(\mathcal{K}_n,\mathcal{K}_n)\mid n\in\omega\}$ where $(\mathcal{K}_n,\mathcal{K}_n) = \{(A,A)\cup(X\setminus\mathrm{cl}_\tau A, X\setminus\mathrm{cl}_\tau A)\mid A\in\mathcal{A}_n\}$ is σ-open cocushioned pairbase for τ and its τ-conjugate $X\setminus\ \mathrm{cl}_\tau\ \mathcal{P} = \{(X\setminus \mathrm{cl}_\tau\mathcal{K}_n, X\setminus\mathrm{cl}_\tau\mathcal{K}_n)\mid n\in\omega\} = \{(\mathcal{K}_n,\mathcal{K}_n)\mid n\in\omega\}$ is σ-open cocushioned pairbase for τ as well. By Theorem 2.13 the space (X,τ,τ) is quasimetrizable which implies that (X,τ) is metrizable. The rest is obvious. $\square$

Corollary 3.4. *A topological space (X,τ) is metrizable if τ has a complementary σ-τ-interior preserving base.*

Theorem 3.5 (see Köfner [17]). *Let (X,τ) be a topological space. The following statements are equivalent:*

(i) *(X,τ) is non-Archimedean quasi-metrizable;*
(ii) *(X,τ) has a complementary σ-τ-interior preserving base;*
(iii) *(X,τ) has a σ-τ-interior preserving base.*

Proof. (iii)⇒(ii). Let $\mathcal{K} = \{\mathcal{K}_n \mid n \in \omega\}$ be a σ-τ-interior preserving base for τ. Denote $V_n(x) = \operatorname{int}_\tau \cap\{K \mid x \in K,\ K \in \mathcal{K}_n\}$, $\widehat{V}_n(x) = \{t \mid x \in V_n(t)\}$ and $\widehat{\tau}$ the topology which for every $x \in X$, it has as neighborhood system of x the family $\mathcal{N}_x = \{\widehat{V}_n(x) \mid n \in \omega\}$. Let $\mathcal{A} = \{\mathcal{A}_n \mid n \in \omega\}$ where $\mathcal{A}_n = \{\mathcal{K}_n \cup (X \setminus \operatorname{cl}_\tau \widehat{V}_n(x)) \mid x \in X\}$. It is clear that $\operatorname{cl}_{\widehat{\tau}} K = K$ and $\operatorname{cl}_\tau \widehat{V}_n(x) = \widehat{V}_n(x)$. We show that $\mathcal{A}$ is a complementary σ-interior preserving base for τ. It is clear that $\mathcal{A}$ is a base for τ and $X \setminus \operatorname{cl}_{\widehat{\tau}}\mathcal{A}$ is a base for $\widehat{\tau}(X \setminus \operatorname{cl}_{\widehat{\tau}}(X \setminus \operatorname{cl}_\tau \widehat{V}_n(x)) = \widehat{V}_n(x))$.

It remains to prove the following:

(1) *for each $n \in \omega$, $\mathcal{A}_n$ is τ-interior preserving.* Let A be any subset of X and $t \in \cap\{X \setminus \operatorname{cl}_\tau \widehat{V}_n(x) \mid x \in A\} = \cap\{X \setminus \widehat{V}_n(x) \mid x \in A\}$. Suppose that $\lambda \in V_n(t)$. Then from $x \notin V_n(t)$ and $V_n(\lambda) \subseteq V_n(t)$ we conclude that $x \notin V_n(\lambda)$ which implies that $\lambda \in X \setminus \widehat{V}_n(x)$. Hence, $\cap\{X \setminus \operatorname{cl}_\tau \widehat{V}_n(x) \mid x \in A\}$ is τ-open. The rest is obvious;

(2) *for each $n \in \omega$, $X \setminus \operatorname{cl}_{\widehat{\tau}}\mathcal{A}_n$ is $\widehat{\tau}$-interior preserving.* Let $t \in \cap\{X \setminus \operatorname{cl}_{\widehat{\tau}}(X \setminus \widehat{V}_n(x)) \mid x \in A\} = \cap\{\widehat{V}_n(x) \mid x \in A\}$ and $\lambda \in \widehat{V}_n(t)$. Then from $V_n(x) \subseteq V_n(t) \subseteq V_n(\lambda)$ we conclude that $\lambda \in \widehat{V}_n(x)$. Let now $t \in \cap\{X \setminus \operatorname{cl}_{\widehat{\tau}} K \mid K \in \mathcal{K}_n\} = \cap\{X \setminus K \mid K \in \mathcal{K}_n\}$ and $\lambda \in \widehat{V}_n(t)$. Suppose that $\lambda \in K$ for some $K \in \mathcal{K}_n$. Then $V_n(t) \subseteq V_n(\lambda) \subseteq K$, an absurdity. Hence, $\cap\{X \setminus \operatorname{cl}_{\widehat{\tau}}(X \setminus \operatorname{cl}_\tau \widehat{V}_n(x)) \mid x \in A\}$ and $\cap\{X \setminus \operatorname{cl}_{\widehat{\tau}} K \mid K \in \mathcal{K}_n\}$ are $\widehat{\tau}$-open. The implication (ii)⇒(iii) is evident.

The equivalence of (i) and (iii) is taken from Köfner [17, Proposition 1]. $\square$

Theorem 3.6 (see Ribeiro [7]). *A T_1 topological space is quasimetrizable if and only if for each $x \in X$, there is a base $\mathcal{A}_x = \{A(x,n) \mid n \in \omega\}$ for the neighborhood system of x such that $A(x,0) = X$ and such that if $y \in A(x,n)$, then $A(y,n) \subset A(x,n-1)$.*

Proof. Let $V_n(x) = \operatorname{int}_\tau \cap\{A(y,n-1) \mid x \in A(y,n),\ y \in X\}$, $\widehat{V}_n(x) = \{t \mid x \in V_n(t)\}$ and $\widehat{\tau}$ the topology which generates the collection $\{\widehat{V}_n(x) \mid n \in \omega, x \in X\}$. Let $(\mathcal{A},\mathcal{A}^\star) = \{(\mathcal{A}_n,\mathcal{A}_n^\star) \mid n \in \omega\}$ where $(\mathcal{A}_n,\mathcal{A}_n^\star) = \{(A(x,n),A(x,n-1)) \mid x \in X\}$. We prove that $(\mathcal{A},\mathcal{A}^\star)$ is a complementary σ-τ-open cocushioned pairbase for τ.

By hypothesis, for each $B \subseteq X$, $\cap\{A(x,n) \mid x \in B\} \subseteq \operatorname{int}_\tau \cap \{A(x,n-1) \mid x \in B\}$. It remains to prove that $(X \setminus \operatorname{cl}_{\widehat{\tau}}\mathcal{A}^\star, X \setminus \operatorname{cl}_{\widehat{\tau}}\mathcal{A}) = \{(X \setminus \operatorname{cl}_{\widehat{\tau}}\mathcal{A}_n^\star, X \setminus \operatorname{cl}_{\widehat{\tau}}\mathcal{A}_n) \mid n \in \omega\}$ where $(X \setminus \operatorname{cl}_{\widehat{\tau}}\mathcal{A}_n^\star, X \setminus \operatorname{cl}_{\widehat{\tau}}\mathcal{A}_n) = \{X \setminus \operatorname{cl}_{\widehat{\tau}} A(x,n-1), X \setminus \operatorname{cl}_{\widehat{\tau}} A(x,n) \mid x \in X\}$ is σ-τ-open cocushioned pairbase for $\widehat{\tau}$. Let B be any subset of X and $t \in \cap\{X \setminus \operatorname{cl}_{\widehat{\tau}} A(x,n-1) \mid x \in B\}$. We prove that $\widehat{V}_{n+2}(t) \subset \cap\{X \setminus \operatorname{cl}_{\widehat{\tau}} A(x,n) \mid x \in B\}$. Indeed, let $\lambda \in \widehat{V}_{n+2}(t)$. Suppose that there exists $x \in B$ such that $\lambda \in \operatorname{cl}_{\widehat{\tau}} A(x,n)$. Then there exists $\kappa \in \widehat{V}_{n+2}(\lambda) \cap A(x,n)$. Since

$t \in V_{n+2}(\lambda)$ and $\lambda \in V_{n+2}(\kappa)$, respectively, we have $t \in A(\lambda, n+1)$ and $\lambda \in A(\kappa, n+1)$ ($\lambda \in A(\lambda, n+2)$ and $\kappa \in A(\kappa, n+2)$). Thus $t \in A(\lambda, n+1) \subseteq A(\kappa, n) \subseteq A(x, n-1)$, an absurdity. It follows that $\cap\{X \setminus \mathrm{cl}_{\hat{\tau}} A(x, n-1) \mid x \in B\} \subseteq \mathrm{int}_{\hat{\tau}} \cap \{X \setminus \mathrm{cl}_{\hat{\tau}} A(x, n) \mid x \in B\}$. By Theorem 3.2 the space is quasimetrizable. □

Theorem 3.7 (Sion-Zelmer [18], Norman [19]). *Let (X, τ) be a topological space. If τ has a σ-point finite base for τ, then there exists a quasimetric d which generates τ.*

Proof. Every σ-τ-point-finite base for τ is a σ-τ-interior preserving base for τ. The rest is obvious. □

Kopperman in [20] gives the following definitions.

Definition 3.8. Given an $R \subseteq X \times 2^X$, $\tau(R)$ is the topology $\{P : \text{if } x \in P \text{ then for some finite } \mathcal{F} \subseteq R(x), \cap\mathcal{F} \subseteq P\}$, and R is *basic* if whenever $x \in P \in \tau(R)$, there is an $A \in R(x)$ such that $A \subseteq P$.

Definition 3.9. A set relation on X is a relation on the power set of X, and such a relation G is *enclosing* if $(A,B) \in G \Rightarrow A \subseteq B$. For an enclosing G and a topology τ on $X : dG = \{(x, A) : x \in A \in \mathrm{Dom}(G)\}$, $rG = \{(x, B) : \exists (A,B) \in G, x \in A\}$, the topology arising from G is $\tau(dG)$. G is

(i) a *pairgenerator* (for τ) if $\tau(dG) = \tau(rG)(= \tau)$,
(ii) a *pairbase* (for τ) if it is a pairgenerator (for τ) and dG, rG are basic,
(iii) *τ-cushioned* if $\mathrm{cl}(\cup \mathrm{Dom}(H)) \subseteq \cup Rg(H)$ whenever $H \subseteq G$,
(iv) *τ-cocushioned* if $\cap \mathrm{Dom}(H) \subseteq \mathrm{int}(\cap Rg(H))$ whenever $H \subseteq G$,
(v) *σ-τ-(co)cushioned* if it is a countable union of τ-(co)cushioned sets,
(vi) *σ-self-(co)cushioned* if it is σ-$\tau(dG)$-(co)cushioned.

Definition 3.10. The *conjugate* of an enclosing set relation G on X is $G^\star = \{(X \setminus B, X \setminus A) : (A,B) \in G\}$.

Theorem 3.11 (see Kopperman [20]). *Let (X, τ) be a topological space. The following are equivalent:*

(i) *(X, τ) is quasimetrizable,*
(ii) *there is an enclosing set relation G on X such that τ arises from G and both G and $G^\star$ are σ-self-cocushioned pairbases.*

Proof. Let G be an enclosing set relation on X such that both G and $G^\star$ are σ-self-cocushioned pairbases (in the sense of Definition 3.9). Then $G = \{G(n) \mid n \in \omega\}$, $G^\star = \{G^\star(n) \mid n \in \omega\}$ and for each $n \in \omega$, $G(n)$ is $\tau(dG)$-cocushioned and $G^\star(n)$ is $\tau(dG^\star)$-cocushioned. Let $V_n(x) = \mathrm{int}_\tau \cap \{A^\star_{ni} \mid x \in A_{ni}, (A_{ni}, A^\star_{ni}) \in G(n),\ i \in I_n\}$, $V(A^\star_{ni}) = \cup \{V_n(x) \mid x \in A^\star_{ni}\}$, $\mathcal{K}_{ni} = \{\mathrm{int}_\tau A_{ni} \mid (A_{ni}, A^\star_{ni}) \in G(n)\}$ and $\mathcal{K}^\star_{ni} = \{V(A^\star_{ni}) \mid (A_{ni}, A^\star_{ni}) \in G(n)\}$. Then similar to Theorem 2.14, it is proved that $(\mathcal{K}, \mathcal{K}^\star) = \{(\mathcal{K}_n, \mathcal{K}^\star_n) \mid n \in \omega\}$, where $(\mathcal{K}_n, \mathcal{K}^\star_n) = \{(K_{ni}, K^\star_{ni}) \mid i \in I_n\}$ is a σ-τ-open cocushioned pairbase for τ.

Let $\hat{\tau} = \tau(dG^\star) = \tau(rG^\star)$. We prove that $(X \setminus \mathrm{cl}_{\hat{\tau}} \mathcal{K}^\star, X \setminus \mathrm{cl}_{\hat{\tau}} \mathcal{K}) = \{(X \setminus \mathrm{cl}_{\hat{\tau}} K^\star_{ni}, X \setminus \mathrm{cl}_{\hat{\tau}} K_{ni}) \mid n \in \omega, i \in I_n\}$ is a σ-τ-open cocushioned pairbase for $\hat{\tau}$, and thus $(\mathcal{K}, \mathcal{K}^\star)$ is a complementary σ-τ-open cocushioned pairbase for τ.

Let $x \in O \in \hat{\tau}$. By hypothesis, there is $m \in \omega$, $(A_{mi}, A^{\star}_{mi}), (A_{mj}, A^{\star}_{mj}), (A_{mk}, A^{\star}_{mk}) \in G(m)$ such that

$$x \in \mathrm{int}_{\hat{\tau}}(X \setminus A^{\star}_{mk}) = X \setminus A^{\star}_{mk} \subseteq X \setminus A_{mk} \subseteq X \setminus A^{\star}_{mj} \subseteq X \setminus A_{mj} \subseteq X \setminus A^{\star}_{mi} \subseteq X \setminus A_{mi} \subseteq O. \tag{3.1}$$

From $\mathrm{cl}_{\hat{\tau}} A_{mk} = A_{mk}$, $\mathrm{cl}_{\hat{\tau}} A_{mi} = A_{mi}$ and $\mathrm{cl}_{\hat{\tau}} V(A^{\star}_{mi}) \subseteq \mathrm{cl}_{\hat{\tau}} A_{mk}$ we conclude that

$$x \in X \setminus A^{\star}_{mk} \subseteq X \setminus A_{mk} \subseteq X \setminus \mathrm{cl}_{\hat{\tau}} V(A^{\star}_{mi}) \subseteq X \setminus \mathrm{cl}_{\hat{\tau}} A_{mi} \subseteq X \setminus \subseteq X \setminus \mathrm{cl}_{\hat{\tau}}(\mathrm{int}_{\tau} A_{mi}) \subseteq O \tag{3.2}$$

and finally

$$x \in X \setminus \mathrm{cl}_{\hat{\tau}} K^{\star}_{mi} \subseteq X \setminus \mathrm{cl}_{\hat{\tau}} K_{mi} \subseteq O. \tag{3.3}$$

Hence, $(X \setminus \mathrm{cl}_{\hat{\tau}} \mathcal{K}^{\star}, X \setminus \mathrm{cl}_{\hat{\tau}} \mathcal{K})$ is a pairbase for $\hat{\tau}$.

It remains to prove that for each $n \in \omega$, $\{(X \setminus \mathrm{cl}_{\hat{\tau}} K^{\star}_{ni}, X \setminus \mathrm{cl}_{\hat{\tau}} K_{ni}) \mid i \in I_n\}$ is a $\hat{\tau}$-open cocushioned pair family. Indeed, for any $I \subseteq I_n$ we have

$$\begin{aligned}
\cap\{X \setminus \mathrm{cl}_{\hat{\tau}} K^{\star}_{mi} \mid i \in I\} &= \cap\{X \setminus \mathrm{cl}_{\hat{\tau}} V(A^{\star}_{mi}) \mid i \in I\} \subseteq \cap\{X \setminus A^{\star}_{mi} \mid i \in I\} \\
&\subseteq \mathrm{int}_{\hat{\tau}} \cap\{X \setminus A_{mi} \mid i \in I\} = \mathrm{int}_{\hat{\tau}} \cap\{X \setminus \mathrm{cl}_{\hat{\tau}} A_{mi} \mid i \in I\} \\
&\subseteq \mathrm{int}_{\hat{\tau}} \cap\{X \setminus \mathrm{cl}_{\hat{\tau}}(\mathrm{int}_{\tau} A_{mi}) \mid i \in I\} \\
&= \mathrm{int}_{\hat{\tau}} \cap\{X \setminus \mathrm{cl}_{\hat{\tau}} K_{mi} \mid i \in I\}.
\end{aligned} \tag{3.4}$$

The rest is obvious. □

Theorem 3.12 (Fox [22], Künzi [28]). *A topological space (X, τ) is quasimetrizable if and only if it admits a local quasiuniformity $\mathcal{U}$ with a countable base such that $\mathcal{U}^{-1}$ is a local quasiuniformity.*

Proof. Let (X, τ) be a topological space and let $\mathcal{V} = \{V_n \mid n \in \omega\}$ be a decreasing base for a local quasiuniformity compatible with τ such that $\mathcal{V}^{-1} = \{V_n^{-1} \mid n \in \omega\}$ is a base for a local quasiuniformity. Then $(\mathcal{A}, \mathcal{A}^{\star}) = \cup\{(\mathcal{A}_n, \mathcal{A}^{\star}_n) \mid n \in \omega\}$, where $(\mathcal{A}_n, \mathcal{A}^{\star}_n) = \{(V_n(x), V_n^3(x)) \cup (X \setminus \mathrm{cl}_{\tau} V_n^{-3}(x), X \setminus \mathrm{cl}_{\tau} V_n^{-1}(x)) \mid V_n \in \mathcal{V},\ x \in X\}$ is a σ-$\tau_{\mathcal{V}}$-open cocushioned pairbase for τ and $(X \setminus \mathrm{cl}_{\tau_{\mathcal{V}^{-1}}} \mathcal{A}^{\star}, X \setminus \mathrm{cl}_{\tau_{\mathcal{V}^{-1}}} \mathcal{A})$ is a σ-$\tau_{\mathcal{V}^{-1}}$-interior preserving pairbase for $\tau_{\mathcal{V}^{-1}}$. Thus $(\mathcal{A}, \mathcal{A}^{\star})$ is a complementary σ-τ-open cocushioned pairbase for τ. □

In [21], Hung characterizes quasimetrizable topologies in terms of neighborhood properties, drawing inspiration from a characterization of γ-spaces.

We recall from Hung [21, pages 40-41] the following definitions.

An *assignment* is a map $g : X \times \omega \to 2^X$ such that, for any $x \in X$, $n \in \omega$, $x \in g(x, n)$. For each $A \subset X$, $g(A, n)$ denotes $\cup\{g(x, n) \mid x \in A\}$. An assignment g induces a topology

$$\{T \subseteq X \mid \text{for each } x \in T, \text{ there is an } n \in \omega \text{ such that } g(x, n) \subseteq T\}. \tag{3.5}$$

An assignment is *decreasing* if for all $x \in X$, $n \in \omega$, $g(x, n+1) \subseteq g(x, n)$. For a class $\mathcal{A}$ of subsets of X, an assignment *cushions* members in $\mathcal{A}$ if, whenever $A \in \mathcal{A}$ and $n \in \omega$, there is an $m \in \omega$ such that $g(g(A,m),m) \subseteq g(A,n)$. The dual assignment is defined by $g^\star(x,n) = \{y \mid x \in g(y,n)\}$. Clearly, $g^\star(x,n)$ is also an assignment and it is decreasing if $g(x,n)$ is also decreasing. A topological space is *locally finite* if each element has a finite neighborhood.

Theorem 3.13 (see [21, page 41]). *If a decreasing assignment cushions countable relatively locally finite sets, then it cushions all sets.*

Theorem 3.14 (see Hung [21]). *A topological space (X,τ) is quasimetrizable if and only if τ is induced by a decreasing assignment that cushions all countable relatively locally finite sets (and hence all sets).*

Proof. It is clear that $(\mathscr{A},\mathscr{A}^\star) = \{(\mathscr{A}_n,\mathscr{A}_n^\star) \mid n \in \omega\}$ where $(\mathscr{A}_n,\mathscr{A}_n^\star) = \{(g(x,n), g(g(g(x,n),n),n)) \cup (X \setminus \mathrm{cl}_\tau(g^\star(g^\star(g^\star(x,n),n),n)), X \setminus \mathrm{cl}_\tau g^\star(x,n)) \mid x \in X\}$ is a complementary σ-τ-open cocushioned pairbase for τ. $\square$

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