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Research Article
Semi-Hausdorff Fuzzy Filters

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The notion of fuzzy filters was studied by Vicente and Aranguren (1988), Lowen (1979), and Ramakrishnan and Nayagam (2002). The notion of fuzzily compactness was introduced and studied by Ramakrishnan and Nayagam (2002). In this paper, an equivalent condition of fuzzily compactness is studied and a new notion of semi-Hausdorffness on fuzzy filters, which cannot be defined in crisp theory of filters, is introduced and studied.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [1]. The theory of fuzzy filters was studied in [2, 3]. The notion of fuzzy topological spaces is introduced in [4] and studied in [5, 6]. The notion of quasicoincident was introduced in [7], and the notion of disjointness was studied in [8] for defining separation axioms. The notion of fuzzily compact sets was introduced and studied in [9]. In this paper, an equivalent condition of fuzzy compactness through fuzzy filter convergence is studied in Section 3. The notion of Hausdorff interval-valued fuzzy filter was introduced and studied in [3]. In this paper, a new notion of semi-Hausdorffness, which cannot be defined in the usual theory of filters, is introduced and studied to some extend in Section 2.

Here we give a brief review of preliminaries.

Definition 1.1 [1]. A function $\mu : X \to [0,1]$ is called a fuzzy subset of X.

Definition 1.2 [7]. A fuzzy set μ is said to be quasicoincident with γ at $x \in X$ if $\mu(x) + \gamma(x) > 1$.

Definition 1.3 [8]. Two fuzzy sets μ and γ are said to be disjoint if $\mu \leq \gamma^c$.

Note 1.4. By Definition 1.2, two fuzzy sets μ and γ are said to be not quasicoincident if $\mu(x)+\gamma(x)\leq 1$ for all $x\in X$, and hence disjointness in Definition 1.3 and not quasicoincidence are equivalent.

Definition 1.5 [10]. A fuzzy topological space (X,δ) is said to be a nearly fuzzy Hausdorff space (n.f. T_2 space) if for every pair of elements $x\neq y$ of X, there exist no quasicoincident (disjoint) fuzzy open sets $\mu,\nu\in\delta$ such that $\mu(x)>1/2$ and $\nu(y)>1/2$.

Definition 1.6 [9]. Let (X,δ) be a fuzzy topological space. A collection δ_0 of fuzzy open sets is called a fuzzily open cover for $A\subseteq X$ if for every $z\in A$, there exists $\gamma\in\delta_0$ such that $\gamma(z)\geq 1/2$.

Definition 1.7 [9]. A subset $A\subseteq X$ of a fuzzy topological space (X,δ) is said to be fuzzily compact if for every fuzzily open cover σ for A, there exists a finite subcollection δ_0 of σ such that for every $z\in A$, there exists $\gamma\in\delta_0$ with $\gamma(z)\geq 1/2$.

Definition 1.8 [2]. A collection $\Im$ of fuzzy sets is said to be a fuzzy filter if

(1) $0\notin\Im$;
(2) if $\mu,\nu\in\Im$, then $\mu\wedge\nu\in\Im$;
(3) if $\mu\in\Im$ and $\nu\geq\mu$, then $\nu\in\Im$.

Definition 1.9 [3]. Let $(X,\Im)$ be a fuzzy filter. Let $Y\subseteq X$. Then $(Y,\Im\mid Y)$ is called the subfilter if no element of $\Im$ vanishes on Y.

Definition 1.10 [3]. Let $(X,\Im_1)$ and $(Y,\Im_2)$ be fuzzy filters.

(1) A function $f:X\to Y$ is said to be a fuzzy filter continuous if $f^{-1}(\gamma)\in\Im_1$ for every $\gamma\in\Im_2$.
(2) A function $f:X\to Y$ is said to be a fuzzy filter open if $f(\mu)\in\Im_2$ for every $\mu\in\Im_1$.
(3) An injective function $f:X\to Y$ is said to a be fuzzy filter homeomorphism if f is both fuzzy filter continuous and fuzzy filter open.

Definition 1.11 [3]. Let $(X_\alpha,\Im_\alpha)$ be an indexed family of fuzzy filters. Let $X=\prod X_\alpha$. Now, the product fuzzy filter $\Im=\prod\Im_\alpha$ is the smallest fuzzy filter for which the projection maps $p_\alpha:X\to X_\alpha$ defined by $p_\alpha((x_\alpha))=x_\alpha$ are fuzzy filter continuous.

Definition 1.12 [3]. Let $f:(X_1,\Im_1)\to X_2$ be a surjective map. Then $\Im=\{\mu\in I^{X_2}\mid f^{-1}(\mu)\in\Im_1\}$ is called a quotient fuzzy filter on X_2.

Definition 1.13 [3]. A sequence $\{x_n\}$ of $(X,\Im)$ is said to converge fuzzy filterly to x if for every $\mu\in\Im$ such that $\mu(x)>1/2$, there exist N such that $\mu(x_n)>1/2$ for all $n\geq N$, equivalently, $\mu^c(x_n)<1/2$ for all $n\geq N$.

Definition 1.14 [9]. Let (X,δ) and (Y,σ) be fuzzy topological spaces. A point $x\in X$ is said to be a fuzzily limit point of A if for every fuzzy open set $\mu\in\delta$ such that $\mu(x)\geq 1/2$, $\mu(z)\geq 1/2$ for some $z\in A-\{x\}$. A subset C of X is said to be a fuzzily closed set if it contains all its fuzzily limit points.

Definition 1.15 [9]. A function $f:X\to Y$ is said to be nearly fuzzy continuous if $f^{-1}(A)$ is fuzzily closed in (X,δ) for every fuzzily closed set A in (Y,σ).

2. Semi-Hausdorff fuzzy filters

Remark 2.1. From [3, Theorem 3.2], in a Hausdorff fuzzy filter, any sequence of points of X converges filterly uniquely if it converges.

But the converse need not be true, which is seen from the following example.

Example 2.2. Let X be an uncountable set and $\mathfrak{I} = \{\mu \in I^X \mid \mu^c \text{ has a countable support}\}$. Clearly, $(X,\mathfrak{I})$ is a fuzzy filter. Let $\{x_n\}$ be any sequence of points of X. Let $x \in X$ be an arbitrary point. Consider $\mu \in \mathfrak{I}$ such that

$$\mu(z) = \begin{cases} 1 & (z \neq x_n,\ x \text{ or } z = y), \\ \dfrac{1}{4} & (z = x_n), \\ \dfrac{3}{4} & (z = x). \end{cases} \tag{2.1}$$

Clearly, $\mu(x) > 1/2$. But $\mu(x_n) < 1/2$ for all $n \in Z_+$ and hence x_n does not converge to x fuzzily. So no sequence converges and hence every sequence converges filterly uniquely if it converges. But $(X,\mathfrak{I})$ is not a Hausdorff fuzzy filter. Let $x, y \in X$ such that $x \neq y$. Suppose there exists $\mu, \gamma \in \mathfrak{I}$ such that $\mu(x) > 1/2$, $\gamma(y) > 1/2$ and $\mu(z) + \gamma(z) \leq 1$, for all $z \in X$. Since $\mu, \gamma \in \mathfrak{I}$, μ^c, and γ^c have countable supports, say, $\{x_n\}_{n\in Z_+}$ and $\{y_m\}_{m\in Z_+}$, respectively. Hence μ and γ have value 1 on $X - \{x_n, y_m\}$ $n,m \in Z_+$. Since X is uncountable, there exists $z \in X - \{x_n, y_m\}$ $n,m \in Z_+$ such that $\mu(z) = 1$ and $\gamma(z) = 1$, a contradiction to the fact that $\mu(z) + \gamma(z) \leq 1$, for all $z \in X$.

Definition 2.3. A fuzzy filter $(X,\mathfrak{I})$ is said to be a semi-Hausdorff fuzzy filter (s.H.F filter) if every sequence of points converges fuzzy filterly to at most one point.

Definition 2.4 [3]. Let $(X,\mathfrak{I})$ be a fuzzy filter. Then $(X,\mathfrak{I})$ is said to be a nearly fuzzy T_1 filter (n.f. T_1) if for every $x, y \in X$, $x \neq y$, there exists $\mu, \gamma \in \mathfrak{I}$ such that $\mu(x) > 1/2$, $\gamma(y) > 1/2$ and $\mu(y) \leq 1/2$, $\gamma(x) \leq 1/2$.

Theorem 2.5. *Every s.H.F filter is an n.f. T_1 filter.*

The proof of Theorem 2.5 is immediate from definitions.

The converse need not be true, which is seen from the following example.

Example 2.6. Let X be an infinite set and $\mathfrak{I} = \{\mu \in I^X \mid \mu^c \text{ has finite support}\}$. Clearly, $(X,\mathfrak{I})$ is a fuzzy filter. To prove that $(X,\mathfrak{I})$ is an n.f. T_1, let $x, y \in X$, $x \neq y$. Let $\mu, \gamma \in \mathfrak{I}$ such that

$$\begin{aligned} \mu(z) &= \begin{cases} \dfrac{1}{4} & (z = x), \\ 1 & (z \neq x), \end{cases} \\ \gamma(z) &= \begin{cases} \dfrac{1}{4} & (z = y), \\ 1 & (z \neq y). \end{cases} \end{aligned} \tag{2.2}$$

Then supports of μ^c and γ^c are $\{x\}$ and $\{y\}$, respectively. Therefore, $\gamma(x) > 1/2$, $\gamma(y) \leq 1/2$ and $\mu(y) > 1/2$, $\mu(x) \leq 1/2$. To prove that $(X,\Im)$ is not an s.H.F filter, consider x_n of X such that $x_i \neq x_j$ for $i \neq j$. Now, x_n converges fuzzy filterly to each point of X. Let $x \in X$ and $\mu \in \Im$ such that $\mu(x) > 1/2$. Since μ^c has finite support and x_n is an infinite sequence of distinct points, $\mu(x_n) = 1$ for all but finite number of points of x_n. Therefore, x_n converges fuzzy filterly to x and hence to all points of X. Hence$(X,\Im)$ is not an s.H.F filter.

The proof of the following theorem is immediate.

Theorem 2.7. *Let $(X,\Im)$ be an s.H.F filter. Let $Y \subseteq X$. Then $(Y,\Im \mid Y)$ is also an s.H.F filter if no element of $\Im$ vanishes on Y.*

Theorem 2.8. *An s.H.F filter is invariant under every bijective fuzzy filter open map.*

Proof. Let $f : (X,\Im_1)\to(Y,\Im_2)$ be a fuzzy filter open map and $(X,\Im_1)$ be an s.H.F filter. Suppose $(Y,\Im_2)$ is not an s.H.F filter, then there exists $(y_n) \in Y$ such that (y_n) converges fuzzy filterly to y and y'. Since f is bijective, $f^{-1}(y_n)$ is a sequence of points of X. Let $f^{-1}(y_n) = x_n$. Let $f^{-1}(y) = a$ and $f^{-1}(y') = b$. Since f is bijective, $a \neq b$. To prove that $f^{-1}(y_n) = x_n \in X$ converges to a and b, let $\mu \in \Im_1$ be a fuzzy filter open set such that $\mu(a) > 1/2$. Now, $f(\mu) \in \Im_2$ and $f(\mu)(y) = \mu(a) > 1/2$. Therefore, $f(\mu)(y_n) > 1/2$ for all but finite number of n's. Since f is 1-1, f is invariant and hence $\mu(x_n) = f^{-1}(f(\mu))(x_n) = f(\mu)(y_n) > 1/2$ for all but finite number of n's. Hence $x_n\to a$ fuzzy filterly. Similarly, $x_n\to b$ fuzzy filterly, which is a contradiction to the fact that $(X,\Im_1)$ is an s.H.F filter. □

Lemma 2.9. *Let $f : (X,\Im_1)\to(Y,\Im_2)$ be fuzzy filter continuous and let x_n converge to x fuzzy filterly. Then $f(x_n)$ converges to $f(x)$ fuzzy filterly.*

The proof is immediate.

Definition 2.10 [4]. Let $\mathbf{B}$ be a base for a fuzzy filter. Now, the collection $\Im_{\mathbf{B}} = \{\mu \mid$ there exists some $\gamma \in \mathbf{B}$ such that $\gamma \leq \mu\}$ is the fuzzy filter generated by $\mathbf{B}$.

Definition 2.11 [4]. A collection S of fuzzy sets is said to be a subbase for a fuzzy filter $\Im$ in finite intersections of members of S that form a base for $\Im$.

The proof of the following lemma is immediate.

Lemma 2.12. *A function is fuzzy filter continuous if and only if the inverse image of subbasic fuzzy filter open set is fuzzy filter open.*

Lemma 2.13. *Let $(X_\alpha,\Im_\alpha)$ be any indexed family of fuzzy filters. Then $(X_\alpha,\Im_\alpha)$ is fuzzy filter homeomorphic to a subspace of the product fuzzy filter $(\prod X_\alpha,\prod\Im_\alpha)$ (each X_α is a nonempty set).*

Proof. Since X_α is nonempty, we can fix $x_\beta \in X_\beta$ for all $\beta \neq \alpha$. Define $f : X_\alpha\to\prod X_\alpha$ such that $f(x_\alpha) = (x_j)$, where $x_j = x_\alpha, j = \alpha$ and $x_j = x_\beta, j = \beta \neq \alpha$. Then f is well defined and 1-1. To prove that f is fuzzy filter continuous, consider a subbasic fuzzy filter open set $p_\alpha^{-1}(\mu_\alpha) \in \prod\Im_\alpha$, where μ_α is fuzzy filter open in X_α, and let $p_\alpha : \prod X_\alpha\to X_\alpha$ be the projection map. Also, $f^{-1}(p_\alpha^{-1}(\mu_\alpha))(x_\alpha) = (p_\alpha^{-1}(\mu_\alpha))f(x_\alpha) = \mu_\alpha(x_\alpha)$, which is filter open. By Lemma 2.12 f is fuzzy filter continuous.

Consider a fuzzy filter open set μ_α in $(X_\alpha,\Im_\alpha)$. Let $S = \{(x_j) \mid x_j = x_\alpha,\ j = \alpha$ and $x_j = x_\beta$ for all $j = \beta \neq \alpha\}$.

Now, by definition of f,

$$f^{-1}(x) = \begin{cases} x_\alpha & x \in S, \\ 0 & x \notin S. \end{cases} \tag{2.3}$$

Clearly, $p_\alpha^1(\mu_\alpha)/S = f(\mu_\alpha)$. Hence $f(\mu_\alpha)$ is fuzzy filter open in S as a subspace of $(\prod X_\alpha, \prod \Im_\alpha)$. Therefore, $f^{-1} : S \to (X_\alpha, \Im_\alpha)$ is fuzzy filter continuous and $(X_\alpha, \Im_\alpha)$ is fuzzy filter homeomorphic to a subspace of $(\prod X_\alpha, \prod \Im_\alpha)$. □

Theorem 2.14. *Let $(X_\alpha, \Im_\alpha)$ be an indexed family of fuzzy filters. Then $(\prod X_\alpha, \prod \Im_\alpha)$ is an s.H.F filter if and only if each $(X_\alpha, \Im_\alpha)$ is an s.H.F filter.*

Proof. Let $(X_\alpha, \Im_\alpha)$ be a family of s.H.F filters. Suppose $(\prod X_\alpha, \prod \Im_\alpha)$ is not an s.H.F filter, there exists a sequence $\{x_n\}$ of points of $\prod X_\alpha$, which converges fuzzy filterly to distinct points x and y. Since $x \neq y$, there exists an index β such that $x_\beta \neq y_\beta$. Since the projection map $p_\beta : \prod X_\alpha \to X_\beta$ is fuzzy filter continuous in the product fuzzy filter, by Lemma 2.9, $p_\beta(x_n)$ converges to x_β and y_β fuzzy filterly with $x_\beta \neq y_\alpha$, a contradiction to s.H.F filterness of $(X_\beta, \Im_\beta)$. Hence $(\prod X_\alpha, \prod \Im_\alpha)$ is an s.H.F filter.

Now, we prove the converse part.

Let $(\prod X_\alpha, \prod \Im_\alpha)$ be an s.H.F filter. By Lemma 2.13, each X_α is fuzzy filter homeomorphic to a subspace of $(\prod X_\alpha, \prod \Im_\alpha)$. By Theorems 2.7 and 2.8, $(X_\alpha, \Im_\alpha)$ is an s.H.F filter. □

Note 2.15. Let $(Y, \Im)$ be a fuzzy filter. For defining the pointwise convergence filter on Y^X, if we take $S(x,\mu) = \{f \in Y^X \mid f(x) \in \mu\} = \{f \in Y^X \mid \mu(f(x)) > 1/2\}$ or $S(p,\mu) = \{f \in Y^X \mid f(p) \in \mu\}$, where $x \in X$, μ is a fuzzy filter open set in Y (i.e., $\mu \in \Im$) and p is a fuzzy point, analogous to the crisp theory, the collection $S(x,\mu)$, and $S(p,\mu)$ need not be a subbasis for a filter as seen from the following example.

Example 2.16. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. Let $\mathbf{B} = \{\mu^{l,m,n}_{y_1,y_2,y_3}, \nu^{l_0,m_0,n_0}_{y_1,y_2,y_3}, \eta^{l_0,m,n}_{y_1,y_2,y_3}\}$, where $m, n \leq 1/2 < l$ and $l + l_0 \leq 1$, $m + m_0 \leq 1$, $n + n_0 \leq 1$. Now, $(Y, \Im)$ is a filter generated by $\mathbf{B}$. Let $S(x,\mu) = \{f \in Y^X \mid f(x) \in \mu\} = \{f \in Y^X \mid \mu(f(x)) > 1/2\}$. Now, since $S(x_1,\mu) \cap S(x_1,\nu) = \phi$, the collection $S(x,\mu)$ is not a subbasis for a filter. Similarly, the collection $S(p,\mu) = \{f \in Y^X \mid f(p) \in \mu\}$ fails to be a subbasis for a filter.

Note 2.17. If $S(x,\mu) = \{f \in Y^X \mid \mu(f(x)) > 0\}$, this collection is a subbasis for the filter. Consider some $S(x_1,\mu)$ and $S(x_2,\gamma)$. Now, we prove that $\mu(f(x_1)) > 0$ and $\gamma(f(x_2)) > 0$ for some function $f \in Y^X$. Since $(\mu \wedge \gamma) \neq 0$, there exists $y \in Y$ such that $(\mu \wedge \gamma)(y) > 0$. Let $f : X \to Y$ be defined by $f(x_1) = f(x_2) = y$. Therefore, $\mu(f(x_1)) = \mu(y) > 0$. Similarly, $\gamma(f(x_2)) > 0$. Hence $f \in S(x_1,\mu) \cap S(x_2,\gamma)$.

Also, since $\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n \neq 0$, min $(\mu_1(y), \mu_2(y), \ldots, \mu_n(y)) \neq 0$ for some y. Consider the function $f : X \to Y$ such that $f(x_i) = y$ for all i. Now, clearly, $\mu_1 f(x_1) = \mu_1(y) > 0$. Similarly, $\mu_i f(x_i) > 0$ for all i. Therefore, $f \in S(x_1,\mu_1) \cap S(x_2,\mu_2) \cap \cdots \cap S(x_n,\mu_n)$.

Note 2.18. Here, we do not arrive at some basic results.

So to generalize fuzzy pointwise convergence filter, we need the following definition.

Definition 2.19. A fuzzy filter $(X,\Im)$ is said to be a strong fuzzy filter if $\mu,\nu \in \Im$, then $(\mu \wedge \nu)(x) > 1/2$ for some $x \in X$.

Note 2.20. Let $(Y,\Im)$ be a strong fuzzy filter. Then $S(x,\mu) = \{f \in Y^X \mid f(x) \in \mu\} = \{f \in Y^X \mid \mu(f(x)) > 1/2\}$. If $\mu,\gamma \in \Im$, $\mu \wedge \gamma(y) > 1/2$ for some y. Now, there exists a function f such that $f(x_1) = y$ and $f(x_2) = y$. Therefore, $\mu(f(x_1)) > 1/2$ and $\gamma(f(x_2)) > 1/2$. Hence $f \in S(x_1,\mu) \cap S(x_2,\gamma)$ and $\{S(x,\mu)\}$ is a subbasis for a filter.

Definition 2.21. The filter generated by the subbasis $\{S(x,\mu)\}_{x\in X,\mu\in\Im}$ is called the point wise convergence filter on Y^X with respect to the strong fuzzy filter $(Y,\Im)$.

Remark 2.22. In the above pointwise convergence filter on Y^X with respect to a strong fuzzy filter $(Y,\Im)$, $f_n \to f$ if and only if $f_n(x) \to f(x)$ fuzzy filterly for every $x \in X$.

Proof. Assume $f_n \to f$ in the above filter. To prove that $f_n(x) \to f(x)$ fuzzy filterly for every $x \in X$, consider $x \in X$ and a fuzzy filter open set $\mu \in \Im$ such that $\mu(f(x)) > 1/2$. Hence $f \in S(x,\mu)$. Since $f_n \to f$ and $f \in S(x,\mu)$, there exists N such that $f_n \in S(x,\mu)$ for all $n \geq N$. Hence $\mu(f_n(x)) > 1/2$ for all $n \geq N$. Hence $f_n(x) \to f(x)$ fuzzy filterly for every $x \in X$.

Conversely, suppose $f_n(x) \to f(x)$ fuzzy filterly for every $x \in X$. To prove that $f_n \to f$ in the strong filter, let $S(x,\mu)$ be a subbasic open set containing f. Then $\mu(f(x)) > 1/2$. Since $f_n(x) \to f(x)$ fuzzy filterly, $\mu(f_n(x)) > 1/2$ for all $n \geq N$ for some N. Hence $f_n \in S(x,\mu)$ for all $n \geq N$. Hence $f_n \to f$. □

Definition 2.23. A filter (X,τ) is said to be semi-Hausdorff filter (semi-T_2 filter) if and only if every sequence in X has at most one limit.

Corollary 2.24. *The pointwise convergence filter on Y^X is a semi-T_2 filter if $(Y,\Im)$ is an s.H.F filter.*

Proof. Let $(Y,\Im)$ be an s.H.F filter. Suppose the above filter on Y^X is not a semi-T_2 filter, then there exists $f_n \in Y^X$ such that $f_n \to f$ and $f_n \to g$ with $f \neq g$. By the above remark $f_n(x) \xrightarrow{f} f(x)$ for all $x \in X$, and $f_n(x) \xrightarrow{f} g(x)$ for all $x \in X$. Since $(Y,\Im)$ is an s.H.F filter, $f(x) = g(x)$ for all $x \in X$, which contradicts the fact that $f \neq g$. Hence the above filter on Y^X is a semi-T_2 filter. □

Theorem 2.25. *Let $(Y,\Im)$ be a strong fuzzy filter such that Y^X is semi-T_2 filter for every indexing set X in the pointwise convergence filter with respect to $(Y,\Im)$. Then $(Y,\Im)$ is an s.H.F filter.*

Proof. Suppose that $(Y,\Im)$ is not an s.H.F filter, then there exists $y_n \in Y$ such that $y_n \xrightarrow{f} x$, and $y_n \xrightarrow{f} y$ such that $x \neq y$. Define $f_n, f_x, f_y : X \to Y$ by $f_n(z) = y_n$, $f_x(z) = x$ and $f_y(z) = y$, for all $z \in X$. Then clearly, f_n, f_x, and f_y are elements of Y^X. Now, we claim that $f_n \to f_x$ and $f_n \to f_y$. Consider a subbasic filter open set $S(t,\mu)$ containing f_x, where $t \in X$ and $\mu \in \Im$. Hence $\mu(f_x(t)) > 1/2$. So $\mu(x) > 1/2$. Since $y_n \xrightarrow{f} x$ and $\mu(x) > 1/2$, we have $\mu(y_n) > 1/2$ for every $n \geq N$ for some N. Therefore, $\mu(f_n(t)) = \mu(y_n) > 1/2$ for every $n \geq N$. Hence $f_n \in S(t,\mu)$ for every $n \geq N$. So $f_n \to f_x$. Similarly, $f_n \to f_y$. So we get a contradiction to semi-T_2 ness of Y^X. Hence $(Y,\Im)$ is an s.H.F filter. □

Definition 2.26. Let $(X,\Im)$ be a fuzzy filter. A subset S of X is sequentially fuzzy filterly compact if every sequence in S has subsequence converging fuzzy filterly to a point in S.

Definition 2.27. Let $(X,\Im)$ be a fuzzy filter. A subset S of X is sequentially fuzzy filterly closed if no sequence in S converges fuzzy filterly to a point in the complement of S (S^c).

Theorem 2.28. *In an s.H.F filter $(X,\Im)$, every set, which is sequentially fuzzy filterly compact, is sequentially fuzzy filterly closed.*

Proof. Let S be a sequentially fuzzy filterly compact subset of $(X,\Im)$. Suppose S is not sequentially fuzzy filterly closed, then there is a sequence x_n in S such that $x_n \to x$ fuzzy filterly and $x \notin S$. Since S is sequentially fuzzy filterly compact, there is a subsequence x_{n_k} converging fuzzy filterly to a point $y \in S$. But as a subsequence of a fuzzy filterly convergent sequence converging to $x, x_{n_k} \to x$ fuzzy filterly. So we have $x_{n_k} \xrightarrow{f} x$ and $x_{n_k} \xrightarrow{f} y$ with $x \neq y$, a contradiction to the fact that S is an s.H.F filter, being a subspace of an s.H.F filter. Hence S is a sequentially fuzzy filterly closed. □

Theorem 2.29. *A fuzzy filter $(X,\Im)$ is an s.H.F filter if and only if the diagonal set $\Delta = \{(x,x) \mid x \in X\}$ is sequentially fuzzy filterly closed.*

Proof. If X is an s.H.F filter, and suppose that there is a sequence $(x_n,x_n) \in \Delta$ converging fuzzy filterly to $(x,y) \notin \Delta$, then $x_n \xrightarrow{f} x$ and $x_n \xrightarrow{f} y$. Take $\mu \in \Im$ such that $\mu(x) > 1/2$. Let $p_i : X \times X \to X$, $i = 1,2$ be the projection maps on the ith coordinate. We have $p_1^{-1}(\mu)(x,y) = \mu(p_1(x,y)) = \mu(x) > 1/2$. Since p_1 is filter continuous in the product filter, $p_1^{-1}(\mu)$ is a fuzzy filter open set such that $p_1^{-1}(\mu)(x,y) = \mu(x) > 1/2$ and hence $p_1^{-1}(\mu)(x_n,x_n) > 1/2$ for all but finite number of n's. Therefore, we get $\mu(x_n) = \mu(p_1(x_n,x_n)) = p_1^{-1}(\mu)(x_n,x_n) > 1/2$ for all but finite number of n's. So $x_n \to x$ fuzzy filterly. Similarly, $x_n \to y$ fuzzy filterly. Hence we have $x_n \to x$ and $x_n \to y$ with $x \neq y$ which contradicts s.H.F filterness of X. Therefore, Δ is sequentially closed.

Conversely, let Δ be a sequentially fuzzy filterly closed, and suppose X is not an s.H.F filter, we have a sequence x_n of X such that $x_n \to x$ and $x_n \to y$ fuzzy filterly with $x \neq y$. Now, we claim that $(x_n,x_n) \to (x,y)$ fuzzy filterly. Let a filter open set μ in the product fuzzy filter such that $\mu(x,y) > 1/2$. Hence we have a filter open set $\mu_1 \times \mu_2$ such that $\mu_1 \times \mu_2(x,y) > 1/2$ and $\mu_1 \times \mu_2 \subseteq \mu$. Since $x_n \to x$ and $x_n \to y$ fuzzy filterly and $\mu_1(x) > 1/2$, we get $\mu_1(x_n) > 1/2$ for all but finite number of n's. Similarly, $\mu_2(x_n) > 1/2$ for all but finite number of n's. Hence $\mu(x_n,x_n) \geq \mu_1 \times \mu_2(x_n,x_n) > 1/2$ for all but finite number of n's, and hence $(x_n,x_n) \to (x,y)$ fuzzy filterly with $x \neq y$ which is a contradiction to the fact that Δ is sequentially fuzzy filterly closed. □

Definition 2.30. A function $f : (X,\Im_X) \to (Y,\Im_Y)$ is said to be a sequentially fuzzy filterly continuous function if and only if $x_n \xrightarrow{f} x \Rightarrow f(x_n) \xrightarrow{f} f(x)$.

Theorem 2.31. *Let $f : (X,\Im_X) \to (Y,\Im_Y)$ and $g : (X,\Im_X) \to (Y,\Im_Y)$ be sequentially fuzzy filterly continuous functions. If Y is an s.H.F filter, then the set $A = \{x \in X \mid f(x) = g(x)\}$ is sequentially fuzzy filterly closed. Conversely, if A is sequentially fuzzy filterly closed for all X and for all sequentially fuzzy filterly continuous functions f, g, then Y is an s.H.F filter.*

Proof. Let Y be an s.H.F filter. Suppose A is not sequentially fuzzy filterly closed, then there exists $x_n \in A$ such that $x_n \xrightarrow{f} x$ with $x \notin A$. Hence $f(x_n) = g(x_n)$ and $f(x) \neq g(x)$. Since $x_n \xrightarrow{f} x$ and f, g are sequentially fuzzy filterly continuous functions, we have $f(x_n) \xrightarrow{f} f(x)$ and $g(x_n) \xrightarrow{f} g(x)$. Since $f(x_n) = g(x_n)$, $f(x_n) \xrightarrow{f} f(x)$ and $f(x_n) \xrightarrow{f} g(x)$ with $f(x) \neq g(x)$, we have a contradiction to the fact that Y is an s.H.F filter.

Now, the converse follows from the previous theorem by taking $X = Y \times Y$, and f, g are projections. □

Remark 2.32. The authors acknowledged the referees for pointing out that if F is a fuzzy filter, then $F^* = F \cup \{\phi\}$ is a fuzzy topology, and hence results in [3] and in this paper can be proved easily by this remark and more general theorems for fuzzy topological spaces.

3. Fuzzy filter convergence

In this section, new notions of fuzzy filter convergence and fuzzily cluster points are introduced and some fuzzy topological properties are studied through those notions.

Definition 3.1. Let (X,δ) be a fuzzy topological space and let $(X,\mathfrak{I})$ be a fuzzy filter. A point $x \in X$ is said to be a fuzzily cluster point of $(X,\mathfrak{I})$ ($(X,\mathfrak{I})$ accumulates x) if for every $\mu \in \delta$ with $\mu(x) \geq 1/2$, there exists $z \in X$ such that $\mu(z) + \nu(z) > 1$, for all $\nu \in \mathfrak{I}$.

Definition 3.2. Let (X,δ) be a fuzzy topological space. A fuzzy filter $(X,\mathfrak{I})$ is said to converge to a point $x \in X$ if for every $\mu \in \delta$ with $\mu(x) \geq 1/2$, there exists $\nu \in \mathfrak{I}$ such that $\nu \leq \mu$.

The proof of the following note is immediate from definitions.

Note 3.3. If a strong fuzzy filter $(X,\mathfrak{I})$ converges to a point $x \in X$, then $\mathfrak{I}$ accumulates x.

Theorem 3.4. *Let (X,δ) be a fuzzy topological space and let $(X,\mathfrak{I})$ be a fuzzy filter. A point $x \in X$ is a fuzzily cluster point of $(X,\mathfrak{I})$ if and only if $\overline{\mu}(x) > 1/2$, for all $\mu \in \mathfrak{I}$.*

Proof. Let $x \in X$ be a fuzzily cluster point. Suppose there exists $\mu \in \mathfrak{I}$ such that $\overline{\mu}(x) \leq 1/2$, $\overline{\mu}^c(x) \geq 1/2$, and $\overline{\mu}^c \in \delta$. Clearly, $\overline{\mu}^c(z) + \overline{\mu}(z) \leq 1$, for all $z \in X$, and hence $\overline{\mu}^c(z) + \mu(z) \leq 1$, for all $z \in X$. So there exists $\mu \in \mathfrak{I}$, $\overline{\mu}^c \in \delta$ such that $\overline{\mu}^c(z) + \mu(z) \leq 1$, for all $z \in X$. This contradicts the fact that $x \in X$ is a fuzzily cluster point of $(X,\mathfrak{I})$. Now, we prove the converse part. Suppose $\overline{\mu}(x) > 1/2$, for all $\mu \in \mathfrak{I}$, we have to prove that x is a fuzzily cluster point of $(X,\mathfrak{I})$. By assuming the contrary, we have $\mu \in \mathfrak{I}$ and $\nu \in \delta$ such that $\nu(x) \geq 1/2$ and $\mu(z) + \nu(z) \leq 1$, for all $z \in X$. So $\mu(z) \leq \nu^c(z)$, for all $z \in X$ and hence ν^c is a fuzzy closed set containing μ. Hence $\overline{\mu}(z) \leq \nu^c(z) = 1 - \nu(z)$, for all $z \in X$. So we have $\overline{\mu}(x) \leq 1/2$, a contradiction to our hypothesis. □

Theorem 3.5. *A fuzzy topological space (X,δ) is n.f. $T_2 \Rightarrow$ every convergent strong fuzzy filter in X converges uniquely.*

Proof. Let (X,δ) be n.f. T_2, and let $\mathfrak{I}$ be any strong fuzzy filter on X. Suppose $\mathfrak{I}$ converges to two distinct points x and y, by n.f. T_2 ness of (X,δ), there exist $\mu, \nu \in \delta$ with $\mu(x) > 1/2$, $\nu(y) > 1/2$, and $\mu(z) + \nu(z) \leq 1$, for all $z \in X$. Since $\mathfrak{I}$ converges to x, $\mu \in \mathfrak{I}$. Similarly,

$\nu \in \Im$. So by strong fuzzy filterness of $\Im$, $\mu \wedge \nu(z) > 1/2$ for some $z \in X$. So $\mu(z) + \nu(z) > 1$, a contradiction. □

Theorem 3.6. *Let (X, δ) and (Y, σ) be fuzzy topological spaces. Let $f : X \to Y$ be any map as follows.*

(a) *If f is fuzzy continuous, then $\Im \to x$ implies $f(\Im) \to f(x)$.*

(b) *If $\Im \to x$ implies $f(\Im) \to f(x)$ for every fuzzy filter $\Im$ on X, then f is nearly fuzzy continuous.*

Proof. (a) Assume that f is fuzzy continuous and $\Im \to x$. To prove that $f(\Im) \to f(x)$, let $\mu \in \sigma$ such that $\mu(f(x)) \geq 1/2$. Since f is fuzzy continuous, $f^{-1}(\mu) \in \delta$, and clearly, $f^{-1}(\mu)(x) = \mu(f(x)) \geq 1/2$. Since $\Im \to x$, $f^{-1}(\mu) \in \Im$. Hence $f(f^{-1}(\mu)) \in f(\Im)$. Since $f(f^{-1}(\mu)) \leq \mu$, $\mu \in f(\Im)$. Hence $f(\Im) \to f(x)$.

b) If $\Im \to x$ implies $f(\Im) \to f(x)$ for every fuzzy filter $\Im$ on X, we have to prove that $f : X \to Y$ is nearly fuzzy continuous. Let A be a fuzzily closed set in Y. Now, we prove that $f^{-1}(A)$ is fuzzily closed in X. If $f^{-1}(A) = X$, then it is fuzzily closed. Suppose $f^{-1}(A) \neq X$, let $x \notin f^{-1}(A)$. Clearly, $f(x) \notin A$. Since A is fuzzily closed, there exists $\mu \in \sigma$ such that $\mu(f(x)) \geq 1/2$ and $\mu^{-1}[1/2, 1] \cap A = \phi$. Now, let $\Im$ be a fuzzy filter generated by $\mathbf{B} = \{\nu \in \delta \mid \nu(x) \geq 1/2\}$. Clearly, $\Im \to x$. So by hypothesis, $f(\Im) \to f(x)$. Hence $\mu \in f(\Im)$. Clearly, $f^{-1}(\mu) \in \Im$ and $f^{-1}(\mu)(x) \geq 1/2$. So we have $\nu \in \mathbf{B} \leq f^{-1}(\mu)$. Hence we have $\nu \in \delta$ and $\nu(x) \geq 1/2$. Now, we claim that $\nu^{-1}[1/2, 1] \wedge f^{-1}(A) = \phi$. If not, $z \in \nu^{-1}[1/2, 1] \cap f^{-1}(A)$, $f(z) \in A$, and $\nu(z) \geq 1/2$. Hence we have $f^{-1}(\mu)(z) \geq 1/2$ and $f(z) \in A$. So we have $f(z) \in \mu^{-1}[1/2, 1] \wedge A$, which is a contradiction. So x is not a fuzzily limit point of $f^{-1}(A)$ and hence $f^{-1}(A)$ is fuzzily closed. □

Theorem 3.7. *A fuzzy topological space (X, δ) is fuzzily compact if and only if each strong fuzzy filter in X has at least one fuzzily cluster point.*

Proof. By [7, Theorem 8], it is enough to prove that every collection of fuzzily closed sets with finite intersection property has nonempty intersection if and only if every strong fuzzy filter in X has at least one fuzzily cluster point.

"If" part. Assume the hypothesis, let $\Im$ be a strong fuzzy filter in X. It is enough to prove that $\Im$ has at least one fuzzily cluster point. Now, for all $\mu \in \Im$, we have that $(\overline{\mu})^{-1}(1/2, 1]$ is fuzzily closed. Since $\Im$ is a strong fuzzy filter, we have $\mu \wedge \nu(x) > 1/2$ for every pair of $\mu, \nu \in \Im$ for some $x \in X$. So $\Omega = \{(\overline{\mu})^{-1}(1/2, 1] \mid \mu \in \Im\}$ is clearly a collection of fuzzily closed sets with finite intersection property, and hence by hypothesis, we have $\bigcap_{\mu \in \Im} (\overline{\mu})^{-1}(1/2, 1] \neq \phi$. Therefore, we have $z \in X$ such that $\overline{\mu}(z) > 1/2$, for all $\mu \in \Im$. By Theorem 3.4, z is a fuzzily cluster point of $\Im$.

"Only if" part. Now, we assume the hypothesis. Let Ω be a collection of fuzzily closed sets that satisfies finite intersection property. Let $A \in \Omega$ and if $z \notin A$, by definition of fuzzily closed set, then z is not a fuzzily limit point of A, and hence there exists $\nu \in \delta$ such that $\nu(z) \geq 1/2$ and $\nu(y) < 1/2$, for all $y \in A$. So we have a fuzzy closed set ν^c with $\nu^c(z) \leq 1/2$ and $\nu^c(y) > 1/2$, for all $y \in A$. So for every $A \in \Omega$ and for each $z \notin A$, there exists a fuzzy closed set $\gamma_{z,A}$ with $\gamma_{z,A}(y) > 1/2$, for all $y \in A$ and $\gamma_{z,A}(z) \leq 1/2$.

Now, consider $S = \{\gamma_{z,A} \mid A \in \Omega \text{ and } z \notin A\}$. By finite intersection property of Ω, for any finite subcollection S_0 of S, we have $\wedge_{\mu \in S_0}(t) > 1/2$ for some $t \in X$. So the fuzzy filter

$\mathfrak{I}$ generated by S is clearly a strong fuzzy filter. Now, by hypothesis, this strong fuzzy filter $\mathfrak{I}$ has at least one fuzzily cluster point. Let it be x.

By Theorem 3.4, $\overline{\mu}(x) > 1/2$, for all $\mu \in \mathfrak{I}$. Hence $\overline{\gamma}_{z,A}(x) > 1/2$, for all $\gamma_{z,A} \in S$. Since $\gamma_{z,A}$ is fuzzily closed, $\gamma_{z,A}(x) > 1/2$, for all $z \notin A$ and $A \in \Omega$. Clearly, $x \in A$ for all $A \in \Omega$. If $x \notin B$, for some $B \in \Omega$, we have $\gamma_{x,B} \in S$ such that $\gamma_{x,B}(y) > 1/2$, for all $y \in B$ and $\gamma_{x,B}(x) \leq 1/2$, which is a contradiction. Hence $\bigcap_{A\in\Omega} A \neq \Phi$. Hence the theorem is proved. □

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