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Research Article

Characterization for the Convergence of Krasnoselskij Iteration for Non-Lipschitzian Operators

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We establish the convergence of Krasnoselskij iteration for various classes of non-Lipschitzian operators.

1. Introduction

Let X be a real Banach space; B a nonempty, convex subset of X; and $T : B \to B$ an operator. Let $x_0 \in B$. The following iteration is known as Krasnoselskij iteration (see [1]):

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n. \tag{1.1}$$

The map $J : X \to 2^{X^*}$ given by $Jx := \{f \in X^* : \langle x, f\rangle = \|x\|^2,\ \|f\| = \|x\|\}$, for all $x \in X$, is called *the normalized duality mapping*. It is easy to see that we have

$$\langle y, j(x)\rangle \le \|x\|\|y\|, \quad \forall x, y \in X,\ \forall j(x) \in J(x). \tag{1.2}$$

Denote

$$\Psi := \{\psi \mid \psi : [0,+\infty) \longrightarrow [0,+\infty) \text{ is astrictly increasing map with } \psi(0) = 0\}. \tag{1.3}$$

Definition 1.1. Let X be a real Banach space, and let B be a nonempty subset of X. A map $T : B \to B$ is called uniformly pseudocontractive if there exists a map $\psi \in \Psi$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \psi(\|x-y\|), \quad \forall x, y \in B. \tag{1.4}$$

A map $S : X \to X$ is called uniformly accretive if there exists a map $\psi \in \Psi$ and $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy, j(x-y)\rangle \geq \psi(\|x-y\|), \quad \forall x, y \in X. \tag{1.5}$$

Taking $\psi(a) := \psi(a) \cdot a$, for all $a \in [0,+\infty)$, $(\psi \in \Psi)$, reduces to the usual definitions of ψ-strongly pseudocontractive and ψ-strongly accretive. Taking $\psi(a) := \gamma \cdot a^2$, $\gamma \in (0,1)$, for all $a \in [0,+\infty)$, $(\psi \in \Psi)$, we get the usual definitions of strongly pseudocontractive and strongly accretive. Therefore, the class of strongly pseudocontractive maps is included stricly in the class of ψ-strongly pseudocontractive maps. The *example* from [2] shows that this inclusion is proper. Remark, further, that the class of ψ-strongly pseudocontractive maps is also included strictly in the class of uniformly pseudocontractive maps (see also [3]).

We will give a characterization for the convergence of (1.1) when applied to uniformly pseudocontractive operators. For this purpose, we need the following lemma similar to [4, Lemma 1]. Next, $\mathbb{N}$ denotes the set of all natural numbers.

Lemma 1.2. *Let $\{a_n\}$ be a positive bounded sequence and assume that there exists $n_0 \in \mathbb{N}$ such that*

$$a_{n+1} \leq (1-\lambda)a_n + \lambda a_{n+1} - \lambda \frac{\psi(a_{n+1})}{a_{n+1}} + \lambda \varepsilon_n, \quad \forall n \geq n_0, \tag{1.6}$$

where $\lambda \in (0,1)$, $\varepsilon_n \geq 0$, for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\varepsilon_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

Proof. There exists an $M > 0$ such that $a_n \leq M$, for all $n \in \mathbb{N}$. Denote $a := \liminf a_n$. We will prove that $a = 0$. Suppose on the contrary that $a > 0$. Then there exists an $N_1 \in \mathbb{N}$ such that

$$a_n \geq \frac{a}{2}, \quad \forall n \geq N_1. \tag{1.7}$$

From $\lim_{n\to\infty}\varepsilon_n = 0$, we know that there exists an $N_2 \in \mathbb{N}$ such that

$$\varepsilon_n \leq \frac{\psi(a/2)}{2M}, \quad \forall n \geq N_2. \tag{1.8}$$

Set $N_0 := \max\{N_1, N_2\}$. Using the fact that $-(1/M) \geq -(1/a_{n+1})$,we get the following:

$$\begin{aligned} a_{n+1} &\leq (1-\lambda)a_n + \lambda a_{n+1} - \lambda \frac{\psi(a_{n+1})}{a_{n+1}} + \lambda \varepsilon_n \\ &\leq (1-\lambda)a_n + \lambda a_{n+1} - \lambda \frac{\psi(a/2)}{M} + \lambda \frac{\psi(a/2)}{2M} \\ &\leq (1-\lambda)a_n + \lambda a_{n+1} - \lambda \frac{\psi(a/2)}{2M}, \end{aligned} \tag{1.9}$$

which implies that $(1-\lambda)a_{n+1} \leq (1-\lambda)a_n - \lambda((\psi(a/2))/2M)$, or

$$a_{n+1} \leq a_n - \frac{\lambda}{1-\lambda}\frac{\psi(a/2)}{2M} \leq a_n - \lambda \frac{\psi(a/2)}{2M}, \tag{1.10}$$

since $-(\lambda/(1-\lambda)) \leq -\lambda$. Thus $\lambda(\psi(a/2))/2M \leq a_n - a_{n+1}$, which implies that $\sum \lambda < \infty$, in contradiction to $\sum \lambda = \infty$. Therefore, $\liminf a_n = 0$. Hence there exists a subsequence $\{a_{n_j}\} \subset \{a_n\}$ such that $\lim_{j\to\infty} a_{n_j} = 0$. Fix $\varepsilon > 0$. Then there exists an $n_3 \in \mathbb{N}$ such that

$$a_{n_j} < \frac{\varepsilon}{4}, \quad \forall j \geq n_3. \tag{1.11}$$

Also there exists an $n_4 \in \mathbb{N}$ such that

$$\varepsilon_n < \frac{\psi(\varepsilon/4)}{2M}, \quad \forall n \geq n_4. \tag{1.12}$$

Define $n_0 := \max\{n_3, n_4, N_0\}$. We claim that $a_{n_j+k} < \varepsilon/4$ for each $j > n_0$ and each $k > 0$. Suppose not. Then there exists an n_0 and a $k > 0$ such that

$$a_{n_j+k} \geq \frac{\varepsilon}{4}. \tag{1.13}$$

For this n_j, let k denote the smallest positive integer for which (1.13) is true. Then $a_{n_j+k-1} \leq \varepsilon/4$.

From (1.6),

$$\begin{aligned} a_{n_j+k} &\leq (1-\lambda)a_{n_j+k-1} + \lambda a_{n_j+k} - \lambda\frac{\psi(a_{n_j+k})}{a_{n_j+k}} + \lambda\varepsilon_{n_j+k-1} \\ &\leq (1-\lambda)a_{n_j+k-1} + \lambda a_{n_j+k} - \frac{\lambda\psi(\varepsilon/4)}{a_{n_j+k}} + \lambda\frac{\psi(\varepsilon/4)}{2M} \\ &\leq (1-\lambda)a_{n_j+k-1} + \lambda a_{n_j+k} - \lambda\frac{\psi(\varepsilon/4)}{2M}, \end{aligned} \tag{1.14}$$

which implies that $a_{n_j+k} \leq (\varepsilon/4) - (\lambda/(1-\lambda))(\psi(\varepsilon/4)/2M)$. This leads to the contradiction:

$$\frac{\varepsilon}{4} \leq a_{n_j+k} \leq \frac{\varepsilon}{4} - \frac{\lambda}{1-\lambda}\frac{\psi(\varepsilon/4)}{2M} < \frac{\varepsilon}{4}. \tag{1.15}$$

Therefore, $a_{n_j+k} < \varepsilon/4$, for all $k \in \mathbb{N}$, and each $j > n_0$, hence $\lim_{n\to\infty} a_n = 0$. □

2. Main result

Theorem 2.1. *Let X be a real Banach space, B a nonempty, closed, convex, bounded subset of X. Let $T : B \to B$ be a uniformly pseudocontractive and uniformly continuous operator with $F(T) \neq \emptyset$. Then for $x_0 \in B$, the Krasnoselskij iteration* (1.1) *converges to the fixed point of T if and only if* $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Proof. Since T is a self-map of B, which is bounded and convex, then, from (1.1), each $x_n \in B$, so $\{x_n\}$ is bounded for each $n \in \mathbb{N}$. Uniqueness of the fixed point follows from (1.4). If $\{x_n\}$ converges to the fixed point of T, that is, $\lim_{n\to\infty} x_n = x^*$, then, obviously, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Conversely, we will prove that if $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, then $\lim_{n\to\infty} x_n = x^*$. Suppose that

$x_n = x^*$ for some $n \in \mathbb{N}$. Then from (1.1), it follows that $x_m = x^*$ for each $m > n$, and the theorem is proved. Now suppose that $x_n \neq x^*$ for each $n \in \mathbb{N}$. Using (1.1) and (1.2),

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \langle x_{n+1} - x^*, j(x_{n+1} - x^*)\rangle \\
&\quad = \langle (1-\lambda)(x_n - x^*) + \lambda(Tx_n - Tx^*), j(x_{n+1} - x^*)\rangle \\
&\quad = (1-\lambda)\langle (x_n - x^*), j(x_{n+1} - x^*)\rangle + \lambda\langle Tx_n - Tx^*, j(x_{n+1} - x^*)\rangle \\
&\quad \leq (1-\lambda)\|x_n - x^*\|\|x_{n+1} - x^*\| + \lambda\langle Tx_{n+1} - Tx^*, j(x_{n+1} - x^*)\rangle + \lambda\langle Tx_n - Tx_{n+1}, j(x_{n+1} - x^*)\rangle \\
&\quad \leq (1-\lambda)\|x_n - x^*\|\|x_{n+1} - x^*\| + \lambda\|x_{n+1} - x^*\|^2 - \lambda\psi(\|x_{n+1} - x^*\|) + \lambda\|Tx_n - Tx_{n+1}\|\|x_{n+1} - x^*\| \\
&\quad \leq \|x_{n+1} - x^*\|\left((1-\lambda)\|x_n - x^*\| + \lambda\|x_{n+1} - x^*\| - \lambda\frac{\psi(\|x_{n+1} - x^*\|)}{\|x_{n+1} - x^*\|} + \lambda\|Tx_n - Tx_{n+1}\|\right).
\end{aligned} \tag{2.1}$$

Hence

$$\|x_{n+1} - x^*\| \leq (1-\lambda)\|x_n - x^*\| + \lambda\|x_{n+1} - x^*\| - \lambda\frac{\psi(\|x_{n+1} - x^*\|)}{\|x_{n+1} - x^*\|} + \lambda\|Tx_n - Tx_{n+1}\|. \tag{2.2}$$

Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and T is uniformly continuous, it follows that

$$\lim_{n\to\infty}\|Tx_n - Tx_{n+1}\| = 0. \tag{2.3}$$

Set $a_n = \|x_n - x^*\|$, $\varepsilon_n = \|Tx_n - Tx_{n+1}\|$ and use Lemma 1.2 to obtain the conlcusion. □

Remark 2.2. (1) If B is not bounded, then Theorem 2.1 holds under the assumption that $\{x_n\}$ is bounded.

(2) If $T(B)$ is bounded, then $\{x_n\}$ is bounded.

(3) If T is strongly pseudocontractive, then automatically $F(T) \neq \emptyset$.

3. Further results

Let I denote the identity map. A map $T : B \to B$ is called pseudocontractive if there exists $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$.

Remark 3.1. The operator T is a (uniformly, strongly) pseudocontractive map if and only if $(I-T)$ is a (uniformly, strongly) accretive map.

Remark 3.2. (1) Let $T, S : X \to X$, and let $f \in X$ be given. A fixed point for the map $Tx = f + (I-S)x$, for all $x \in X$, is a solution for $Sx = f$.

(2) Let $f \in X$ be a given point. If S is an accretive map, then $T = f - S$ is a strongly pseudocontractive map.

Consider Krasnoselskij iteration with $Tx = f + (I - S)x$,

$$x_{n+1} = (1-\lambda)x_n + \lambda\big(f + (I-S)x_n\big). \quad (3.1)$$

Remarks 3.1 and 3.2 and Theorem 2.1 lead to the following result.

Corollary 3.3. *Let X be a real Banach space and let $S : X \to X$ be a uniformly accretive and uniformly continuous operator, with $(I - S)(X)$ bounded. Suppose that $Sx = f$ has a solution. Then for any $x_0 \in X$, the Krasnoselskij iteration* (3.1) *converges to the solution of $Sx = f$ if and only if* $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Let S be an accretive operator. The operator $Tx = f - Sx$ is strongly pseudocontractive for a given $f \in X$. A solution for $Tx = x$ becomes a solution for $x + Sx = f$. Consider Krasnoselskij iteration with $Tx := f - Sx$,

$$x_{n+1} = (1-\lambda)x_n + \lambda\big(f - Sx_n\big). \quad (3.2)$$

Again, using Remarks 3.1 and 3.2 and Theorem 2.1, we obtain the following result.

Corollary 3.4. *Let X be a real Banach space and let $S : X \to X$ be an accretive and uniformly continuous operator, with $(I - S)(X)$ bounded. Suppose that $x + Sx = f$ has a solution. Then for $x_0 \in X$, the Krasnoselskij iteration* (3.2) *converges to the solution of $x+Sx = f$ if and only if* $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Remark 3.5. If (1.4) holds for all $x \in B$ and $y := x^* \in F(T)$, then such a map is called *uniformly hemicontractive*. It is trivial to see that our results hold for the uniformly hemicontractive maps.

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References

[1] M. A. Krasnosel'skii, "Two remarks on the method of successive approximations," *Uspekhi Matematicheskikh Nauk*, vol. 10, no. 1, pp. 123–127, 1955.

[2] M. O. Osilike, "Iterative solution of nonlinear equations of the ϕ-strongly accretive type," *Journal of Mathematical Analysis and Applications*, vol. 200, no. 2, pp. 259–271, 1996.

[3] C. E. Chidume and C. O. Chidume, "Convergence theorems for fixed points of uniformly continuous generalized ϕ-hemi-contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 303, no. 2, pp. 545–554, 2005.

[4] Ş. M. Şoltuz, "New technique for proving the equivalence of Mann and Ishikawa iterations," *Revue d'Analyse Numérique et de Théorie de l'Approximation*, vol. 34, no. 1, pp. 103–108, 2005.