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Research Article

Prime Ideals and Strongly Prime Ideals of Skew Laurent Polynomial Rings

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We first study connections between α-compatible ideals of R and related ideals of the skew Laurent polynomials ring $R[x,x^{-1};\alpha]$, where α is an automorphism of R. Also we investigate the relationship of $P(R)$ and $N_r(R)$ of R with the prime radical and the upper nil radical of the skew Laurent polynomial rings. Then by using Jordan's ring, we extend above results to the case where α is not surjective.

1. Introduction

Throughout the paper, R always denotes an associative ring with unity. We use $P(R)$, $N_r(R)$, and $N(R)$ to denote the prime radical, the upper nil radical, and the set of all nilpotent elements of R, respectively.

Recall that for a ring R with an injective ring endomorphism $\alpha : R \to R$, $R[x;\alpha]$ is the Ore extension of R. The set $\{x^j\}_{j\geq 0}$ is easily seen to be a left Ore subset of $R[x;\alpha]$, so that one can localize $R[x;\alpha]$ and form the skew Laurent polynomials ring $R[x,x^{-1};\alpha]$. Elements of $R[x,x^{-1};\alpha]$ are finite sum of elements of the form $x^{-j}rx^i$, where $r \in R$ and i,j are nonnegative integers. Multiplication is subject to $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$.

Now we consider Jordan's construction of the ring $A(R,\alpha)$ (see [1], for more details). Let $A(R,\alpha)$ or A be the subset $\{x^{-i}rx^i \mid r \in R,\ i \geq 0\}$ of the skew Laurent polynomial ring $R[x,x^{-1};\alpha]$. For each $j \geq 0$, $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that the set of all such elements forms a subring of $R[x,x^{-1};\alpha]$ with $x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{(i+j)}$ for $r,s \in R$ and $i,j \geq 0$. Note that α is actually an automorphism of $A(R,\alpha)$, given by $x^{-i}rx^i$ to $x^{-i}\alpha(r)x^i$, for each $r \in R$ and $i \geq 0$. We have $R[x,x^{-1};\alpha] \cong A[x,x^{-1};\alpha]$, by way of an isomorphism which maps $x^{-i}rx^j$ to $\alpha^{-i}(r)x^{j-i}$. For an α-ideal I of R, put $\Delta(I) = \cup_{i\geq 0}x^{-i}Ix^i$. Hence $\Delta(I)$ is α-ideal of A. The constructions $I \to \Delta(I)$, $J \to J \cap R$ are inverses, so there is an order-preserving bijection between the sets of α-invariant ideals of R and α-invariant ideals of A.

According to Krempa [2], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. R is called an α-rigid ring [3] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are *reduced* (i.e., R has no nonzero nilpotent element) by Hong et al. [3]. Properties of α-rigid rings have been studied in Krempa [2], Hirano [4], and Hong et al. [3, 5].

On the other hand, a ring R is called *2-primal* if $P(R) = N(R)$ (see [6]). Every reduced ring is obviously a 2-primal ring. Moreover, 2 primal rings have been extended to the class of rings which satisfy $N_r(R) = N(R)$, but the converse does not hold [7, Example 3.3]. Observe that R is a 2-primal ring if and only if $P(R) = N_r(R) = N(R)$, if and only if $P(R)$ is a *completely semiprime ideal* (i.e., $a^2 \in P(R)$ implies that $a \in P(R)$ for $a \in R$) of R. We refer to [6–12] for more detail on 2 primal rings.

Recall that a ring R is called *strongly prime* if R is prime with no nonzero nil ideals. An ideal P of R is *strongly prime* if R/P is a strongly prime ring. All (strongly) prime ideals are taken to be proper. We say an ideal P of a ring R is *minimal (strongly) prime* if P is minimal among (strongly) prime ideals of R. Note that (see [13]) $N_r(R) = \cap\{P \mid P$ is a minimal strongly prime ideal of $R\}$.

Recall that an ideal P of R is *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. Every completely prime ideal of R is strongly prime and every strongly prime ideal is prime.

According to Hong et al. [5], for an endomorphism α of a ring R, an α-ideal I is called to be *α-rigid ideal* if $a\alpha(a) \in I$ implies that $a \in I$ for $a \in R$. Hong et al. [5] studied connections between α-rigid ideals of R and related ideals of some ring extensions. Also they studied relationship of $P(R)$ and $N_r(R)$ of R with the prime radical and the upper nil radical of the Ore extension $R[x;\alpha,\delta]$ of R in the cases when either $P(R)$ or $N_r(R)$ is an α-rigid ideal of R and obtaining the following result. Let $P(R)$ (resp., $N_r(R)$) be an α-rigid δ-ideal of R. Then $P(R[x;\alpha,\delta]) \subseteq P(R)[x;\alpha,\delta]$ (resp., $N_r(R[x;\alpha,\delta]) \subseteq N_r(R)[x;\alpha,\delta]$).

In [14], the authors defined α-compatible rings, which are a generalization of α-rigid rings. A ring R is called *α-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. In this case, clearly the endomorphism α is injective. In [14, Lemma 2.2], the authors showed that R is α-rigid if and only if R is α-compatible and reduced. Thus, the α-compatible ring is a generalization of α-rigid ring to the more general case where R is not assumed to be reduced.

Motivated by the above facts, for an endomorphism α of a ring R, we define *α-compatible ideals* in R which are a generalization of α-rigid ideals. For an ideal I, we say that I is an α-compatible ideal of R if for each $a, b \in R$, $ab \in I \Leftrightarrow a\alpha(b) \in I$. The definition is quite natural, in the light of its similarity with the notion of α-rigid ideals, where in Proposition 2.3, we will show that I is an α-rigid ideal if and only if I is an α-compatible ideal and completely semiprime.

In this paper, we first study connections between α-compatible ideals of R and related ideals of the skew Laurent polynomial ring $R[x, x^{-1};\alpha]$, where α is an automorphism of R. Also we investigate the relationship of $P(R)$ and $N_r(R)$ of R with the prime radical and the upper nil radical of the skew Laurent polynomials. Then by using Jordan's ring, we extend above results to the case where α is not surjective.

2. Prime ideals and strongly prime ideals of skew Laurent polynomial rings

Recall that an ideal I of R is called an *α-ideal* if $\alpha(I) \subseteq I$; I is called *α-invariant* if $\alpha^{-1}(I) = I$. If I is an α-ideal, then $\overline{\alpha} : R/I \to R/I$ defined by $\overline{\alpha}(a+I) = \alpha(a) + I$ is an endomorphism. Then we have the following proposition.

Proposition 2.1. *Let I be an ideal of a ring R. Then the following statements are equivalent:*

(1) *I is an α-compatible ideal;*

(2) *R/I is $\overline{\alpha}$-compatible.*

Proof. It is clear. □

Proposition 2.2. *Let I be an α-compatible ideal of a ring R. Then*

(1) *I is α-invariant;*

(2) *if $ab \in I$, then $a\alpha^n(b) \in I, \alpha^n(a)b \in I$ for every positive integer n; conversely, if $a\alpha^k(b)$ or $\alpha^k(a)b \in I$ for some positive integer k, then $ab \in I$.*

Proof. This follows from [15, Lemma 2.2 and Proposition 2.3]. □

Recall from [16] that a one-sided ideal I of a ring R has the *insertion of factors property* (or simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$ (Bell in 1970 introduced this notion for $I = 0$).

Proposition 2.3 (see [15, Proposition 2.4]). *Let R be a ring, I an ideal of R, and $\alpha : R \to R$ an endomorphism of R. Then the following conditions are equivalent:*

(1) *I is α-rigid ideal of R;*

(2) *I is α-compatible, semiprime and has the IFP;*

(3) *I is α-compatible and completely semiprime.*

For an α-ideal I of R, put $\Delta(I) = \cup_{i\geq 0} x^{-i} I x^i$.

Proposition 2.4. (1) *If I is an α-compatible ideal of R, then $\Delta(I)$ is an α-compatible ideal of A.*

(2) *If J is an α-compatible ideal of A, then $J = \Delta(J_0)$ and J_0 is an α-compatible ideal of R.*

(3) *If P is a completely (semi)prime α-compatible ideal of R, then $\Delta(P)$ is a completely (semi)prime α-compatible ideal of A.*

(4) *If Q is a completely (semi)prime α-compatible ideal of A, then $Q = \Delta(Q_0)$ and Q_0 is a completely (semi)prime α-compatible ideal of R.*

(5) *If P is a prime α-compatible ideal of R, then $\Delta(P)$ is a prime α-compatible ideal of A.*

Proof. (1) Since I is an α-ideal of R, $\Delta(I)$ is an ideal of A. Now, let $(x^{-i}rx^i)(x^{-j}sx^j) \in \Delta(I)$. Hence $x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j} \in \Delta(I)$ and that $\alpha^j(r)\alpha^i(s) \in I$. Thus $\alpha^j(r)\alpha^{i+1}(s) \in I$, since I is α-compatible. Consequently $(x^{-i}rx^i)\alpha(x^{-j}sx^j) \in \Delta(I)$. Therefore $\Delta(I)$ is α-compatible.

(2) Let $J_0 = J \cap J$ and $r \in J_0$. Then $\alpha^n(r) \in J_0$ for each $n \geq 0$. Hence $\alpha^n(x^{-n}rx^n) = r \in J$ for each $n \geq 0$. Thus $x^{-n}rx^n \in J$, since J is α-compatible. Therefore $\Delta(J_0) \subseteq J_0$. Now, let $x^{-m}rx^m \in J$. Then $\alpha^m(x^{-m}rx^m) \in J$ and that $r \in J$, since J is α-compatible. Thus $J \subseteq \Delta(J_0)$. Consequently, $\Delta(J_0) = J$.

(3) Let $(x^{-i}rx^i)(x^{-j}sx^j) \in \Delta(P)$. Then $x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j} \in \Delta(P)$ and that $\alpha^j(r)\alpha^i(s) \in P$. Hence $rs \in P$, by Proposition 2.2. Thus $r \in P$ or $s \in P$, since P is completely prime. Consequently, $x^{-i}rx^i \in \Delta(P)$ or $x^{-j}sx^j \in \Delta(P)$.

(4) By (2), $Q = \Delta(Q_0)$ and Q_0 is a α-compatible ideal of R. Since Q is completely (semi)prime and $Q = \Delta(Q_0)$, hence Q_0 is completely (semi)prime. Let $(x^{-i}rx^i)A(x^{-j}sx^j) \subseteq \Delta(P)$. Then $rRs \subseteq P$, by Proposition 2.2. Hence $r \in P$ or $s \in P$, since P is prime. Consequently $x^{-i}rx^i \in \Delta(P)$ or $x^{-j}sx^j \in \Delta(P)$. Therefore $\Delta(P)$ is a prime ideal of A. □

Theorem 2.5. *Let P be a strongly prime α-compatible ideal of R. Then $\Delta(P)$ is a strongly prime ideal of A.*

Proof. Since P is a prime α-compatible ideal of R, hence $\Delta(P)$ is a prime ideal of A, by Proposition 2.4. We show that $\Delta(P)$ is a strongly prime ideal of A. Assume $J = I/\Delta(P)$ is a nil ideal of $A/\Delta(P)$. Let $a \in I_i$. Then $x^{-i}ax^i \in I$. Since $I/\Delta(P)$ is a nil ideal, hence $(x^{-i}ax^i)^n \in \Delta(P)$ for some $n \geq 0$. Hence $x^{-i}a^nx^i = x^{-j}px^j$ for some $p \in P$ and $j \geq 0$. Thus $\alpha^j(a^n) = \alpha^i(p) \in P$. Hence $a^n \in P$, since P is α-compatible. Then $(I_i + P)/P$ is a nil ideal of R/P for each $i \geq 0$. Hence $I_i \subseteq P$, for each $i \geq 0$. Therefore $I \subseteq \Delta(P)$. Consequently, $\Delta(P)$ is a strongly prime ideal of A. □

Note that if I is an α-ideal of R, then $I[x, x^{-1}; \alpha]$ is an ideal of the skew Laurent polynomials ring $R[x, x^{-1}; \alpha]$.

Theorem 2.6. *Let α be an automorphism of R. Let I be a semiprime α-compatible ideal of R. Assume $f(x) = \sum_{i=r}^{n} a_i x^i$ and $g(x) = \sum_{j=s}^{m} b_j x^j \in R[x, x^{-1}; \alpha]$. Then the following statements are equivalent:*

(1) $f(x)R[x, x^{-1}; \alpha]g(x) \subseteq I[x, x^{-1}; \alpha]$;

(2) $a_i R b_j \subseteq I$ *for each* i, j.

Proof. (1)⇒(2). Assume $f(x)R[x, x^{-1}; \alpha]g(x) \subseteq I[x, x^{-1}; \alpha]$. Then

$$(a_r x^r + \cdots + a_n x^n)c(b_s x^s + \cdots + b_m x^m) \in I[x, x^{-1}; \alpha] \quad \text{for each } c \in R. \tag{†}$$

Hence $a_n\alpha^n(cb_m) \in I$. Thus $a_ncb_m \in I$, since I is α-compatible. Next, replacing c by $cb_{m-1}da_ne$, where $c, d, e \in R$. Then $(a_r x^r + \cdots + a_n x^n)cb_{m-1}da_ne(b_s x^s + \cdots + b_{m-1}x^{m-1}) \in I[x, x^{-1}; \alpha]$. Hence $a_n\alpha^n(cb_{m-1}da_neb_{m-1}) \in I$ and that $a_ncb_{m-1}da_neb_{m-1} \in I$, since I is α-compatible. Thus $(Ra_nRb_{m-1})^2 \subseteq I$. Hence $Ra_nRb_{m-1} \subseteq I$, since I is semiprime. Continuing this process, we obtain $a_nRb_k \subseteq I$, for $k = s, \ldots, m$. Hence from α-compatibility of I, we get $(a_r x^r + \cdots + a_n x^n)R[x, x^{-1}; \alpha](b_s x^s + \cdots + b_{m-1}x^{m-1}) \subseteq I[x, x^{-1}; \alpha]$. Using induction on $n + m$, we obtain $a_iRb_j \subseteq I$ for each i, j.

(2)⇒(1). It follows from Proposition 2.2. □

Corollary 2.7. *Let α be an automorphism on R. If I is a (semi)prime α-compatible ideal of R, then $I[x, x^{-1}; \alpha]$ is a (semi)prime ideal of $R[x, x^{-1}; \alpha]$.*

Proof. Assume that I is a prime α-compatible ideal of R. Let $f(x) = \sum_{i=r}^{n} a_i x^i$ and $g(x) = \sum_{j=s}^{m} b_j x^j \in R[x, x^{-1}; \alpha]$ such that $f(x)R[x, x^{-1}; \alpha]g(x) \subseteq I[x, x^{-1}; \alpha]$. Then $a_iRb_j \subseteq I$ for each i, j, by Theorem 2.6. Assume $g(x) \notin I[x, x^{-1}; \alpha]$. Hence $b_j \notin I$ for some j. Thus $a_i \in I$ for each $i = r, \ldots, n$, since I is prime. Therefore $f(x) \in I[x, x^{-1}; \alpha]$. Consequently, $I[x, x^{-1}; \alpha]$ is a prime ideal of $R[x, x^{-1}; \alpha]$. □

Theorem 2.8. *If each minimal prime ideal of R is α-compatible, then* $P(R[x,x^{-1};\alpha]) \subseteq \Delta(P(R))[x, x^{-1};\alpha]$.

Proof. Let P be a minimal prime ideal of R. By Proposition 2.4, $\Delta(P)$ is a α-compatible ideal of A. Assume $(a^{-i}rx^i)A(x^{-j}sx^j) \subseteq \Delta(P)$. Then $rRs \subseteq P$, since $\Delta(P)$ is α-compatible. Hence $r \in P$ or $s \in P$. Thus $a^{-i}rx^i \in \Delta(P)$ or $x^{-j}sx^j \in \Delta(P)$. Therefore $\Delta(P)$ is a prime ideal of A. Thus $\Delta(P)[x,x^{-1};\alpha]$ is a prime ideal of $A[x,x^{-1};\alpha]$, by Corollary 2.7. Consequently, $P(R[x,x^{-1};\alpha]) \subseteq \Delta(P(R))[x,x^{-1};\alpha]$. □

In [14], the authors give some examples of α-compatible rings however they are not α-rigid. Note that there exists a ring R for which every nonzero proper ideal is α-compatible but R is not α-compatible. For example, let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field, and the endomorphism α of R is defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ for $a,b,c \in F$.

The following examples show that there exists α-compatible ideals which are not α-rigid.

Example 2.9 (see [15], Example 2.5). Let F be a field. Let $R = \{\begin{pmatrix} f & f_1 \\ 0 & f \end{pmatrix} \mid f, f_1 \in F[x]\}$, where $F[x]$ is the ring of polynomials over F. Then R is a subring of the 2×2 matrix ring over the ring $F[x]$. Let $\alpha : R \to R$ be an automorphism defined by $\alpha\left(\begin{pmatrix} f & f_1 \\ 0 & f \end{pmatrix}\right) = \begin{pmatrix} f & uf_1 \\ 0 & f \end{pmatrix}$, where u is a fixed nonzero element of F. Let $p(x)$ be an irreducible polynomial in $F[x]$. Let $I = \{\begin{pmatrix} 0 & f_1 \\ 0 & 0 \end{pmatrix} \mid f_1 \in \langle p(x)\rangle\}$, where $\langle p(x)\rangle$ is the principal ideal of $F[x]$ generated by $p(x)$. Then I is an α-compatible ideal of R but is not α-rigid. Indeed, since $\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix}\alpha\left(\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in I$, but $\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} \notin I$ for $g(x) \notin \langle p(x)\rangle$. Thus I is not α-rigid.

Example 2.10 (see [17], Example 2). Let $\mathbb{Z}_2$ be the field of integers modulo 2 and $A = \mathbb{Z}_2[a_0,a_1,a_2,b_0,b_1,b_2,c]$ be the free algebra of polynomials with zero constant term in noncommuting indeterminates $a_0,a_1,a_2,b_0,b_1,b_2,c$ over $\mathbb{Z}_2$. Note that A is a ring without unity. Consider an ideal of $\mathbb{Z}_2 + A$, say I, generated by a_0b_0, $a_1b_2 + a_2b_1$, $a_0b_1 + a_1b_0$, $a_0b_2 + a_1b_1 + a_2b_0$, a_2b_2, a_0rb_0, a_2rb_2, $(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$ with $r \in A$ and $r_1r_2r_3r_4$ with $r_1,r_2,r_3,r_4 \in A$. Then I has the IFP. Let $\alpha : R \to R$ be an inner automorphism (i.e., there exists an invertible element $u \in R$ such that $\alpha(r) = u^{-1}ru$ for each $r \in R$). Then I is α-compatible, since I has the IFP. But I is not α-rigid, since I is not completely semiprime.

Theorem 2.11. *Let α be an automorphism of R. If each minimal prime ideal of R is α-compatible, then* $P(R[x,x^{-1};\alpha]) \subseteq P(R)[x,x^{-1};\alpha]$.

Proof. It follows from Corollary 2.7. □

The following example shows that there exists a ring R such that all minimal prime ideals are α-compatible, but are not α-rigid.

Example 2.12 (see [15], Example 2.11). Let $R = \mathrm{Mat}_2(\mathbb{Z}_4)$ be the 2×2 matrix ring over the ring $\mathbb{Z}_4$. Then $P(R) = \{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{ij} \in \overline{2\mathbb{Z}}\}$ is the only prime ideal of R. Let $\alpha : R \to R$ be the endomorphism defined by $\alpha\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix}$. Then α is an automorphism of R and $P(R)$ is α-compatible. However, $P(R)$ is not α-rigid, since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \in P(R)$, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin P(R)$.

Theorem 2.13. *Let α be an automorphism of R. If P is a completely (semi)prime α-compatible ideal of R, then* $P[x,x^{-1};\alpha]$ *is a completely (semi)prime ideal of* $R[x,x^{-1};\alpha]$.

Proof. Let P be a completely prime ideal of R. R/P is domain, hence it is a reduced ring. R/P is an $\overline{\alpha}$-compatible ring, hence R/P is $\overline{\alpha}$-rigid, by [14, Lemma 2.2]. Let $\overline{f(x)}$, $\overline{g(x)} \in R/P[x,x^{-1};\overline{\alpha}]$ such that $\overline{f(x)}\,\overline{g(x)} = 0$. Then $\overline{f(x)} = 0$ or $\overline{g(x)} = 0$, by a same way as used in [3, Proposition 6]. Thus $R[x,x^{-1};\alpha]/P[x,x^{-1};\alpha] \cong R/P[x,x^{-1};\overline{\alpha}]$ is domain and $P[x,x^{-1};\alpha]$ is a completely prime ideal of $R[x,x^{-1};\alpha]$. □

Corollary 2.14. *Let α be an automorphism on R. If $P(R)$ is an α-rigid ideal of R, then $P(R[x,x^{-1};\alpha]) \subseteq P(R)[x,x^{-1};\alpha]$.*

Proof. $P(R)$ is α-rigid, hence $P(R)$ is a completely semiprime α-compatible ideal of R, by Proposition 2.3. Therefore $P(R[x,x^{-1};\alpha]) \subseteq P(R)[x,x^{-1};\alpha]$, by Theorem 2.13. □

Theorem 2.15. *Let α be an automorphism of R. If P is a strongly (semi)prime α-compatible ideal of R, then $P[x,x^{-1};\alpha]$ is a strongly (semi)prime ideal of $R[x,x^{-1};\alpha]$.*

Proof. By Corollary 2.7, $P[x,x^{-1};\alpha]$ is a prime ideal of $R[x,x^{-1};\alpha]$. Hence $R[x,x^{-1};\alpha]/P[x,x^{-1};\alpha] \simeq R/P[x,x^{-1};\overline{\alpha}]$ is a prime ring. We claim that zero is the only nil ideal of $R/P[x,x^{-1};\overline{\alpha}]$. Let J be a nil ideal of $R/P[x,x^{-1};\overline{\alpha}]$. Assume I be the set of all leading coefficients of elements of J. First we show that I is an ideal of R/P. Clearly, I is a left ideal of R/P. Let $\overline{a} \in I$ and $\overline{r} \in R/P$. Then there exists $\overline{f(x)} = \overline{a_0} + \cdots + \overline{a_{n-1}}x^{n-1} + \overline{a}x^n \in J$. Hence $(\overline{f(x)}\overline{r})^m = 0$, for some nonnegative integers m. Thus $\overline{a}\,\overline{\alpha}^n(\overline{ra})\cdots\overline{\alpha}^{(m-1)n}(\overline{ra})\overline{\alpha}^{mn}(\overline{r}) = 0$, since it is the leading coefficient of $(\overline{f(x)}\overline{r})^m$. Therefore $(\overline{ar})^m = 0$, since R/P is $\overline{\alpha}$-compatible. Consequently, I is an ideal of R/P. Clearly I is a nil ideal of R/P. Hence $I = 0$ and so $J = 0$. Therefore $P[x,x^{-1};\alpha]$ is a strongly prime ideal of $R[x,x^{-1};\alpha]$. □

Theorem 2.16. *Let α be an automorphism of R. If each minimal strongly prime ideal of R is α-compatible, then $N_r(R[x,x^{-1};\alpha]) \subseteq N_r(R)[x,x^{-1};\alpha]$.*

Corollary 2.17. *Let α be an automorphism of R. If $N_r(R)$ is an α-rigid ideal of R, then $N_r(R[x,x^{-1};\alpha]) \subseteq N_r(R)[x,x^{-1};\alpha]$.*

Proof. $N_r(R)$ is α-rigid, hence $N_r(R)$ is a completely semiprime α-compatible ideal of R, by Proposition 2.3, and that $N_r(R)$ is a strongly semiprime ideal of R. Therefore $N_r(R[x,x^{-1};\alpha]) \subseteq N_r(R)[x,x^{-1};\alpha]$, by Theorem 2.15. □

Example 2.12 also shows that there exists a ring R such that all minimal strongly prime ideals are α-compatible, but are not α-rigid.

Theorem 2.18. *Assume each minimal prime ideal of R is α-compatible. Then the following are equivalent:*

(1) $P(R[x,x^{-1};\alpha])$ *is completely semiprime;*

(2) $P(R[x,x^{-1};\alpha]) = \Delta(P(R))[x,x^{-1};\alpha]$ *and* $P(R)$ *is completely semiprime.*

Proof. (1)⇒(2). Suppose that $P(R[x,x^{-1};\alpha])$ is a completely semiprime ideal of $R[x,x^{-1};\alpha]$. It is enough to show that $\Delta(P(R))[x,x^{-1};\alpha] \subseteq P(R[x,x^{-1};\alpha])$, by Theorem 2.8. Let Q be a minimal prime ideal of $R[x,x^{-1};\alpha]$ and $P = A \cap Q$. Since $P(R[x,x^{-1};\alpha])$ is a completely semiprime ideal of $R[x,x^{-1};\alpha]$, P is a completely semiprime ideal of A. Clearly P is an α-invariant ideal of A. Hence $P = \Delta(P_0)$. We claim that P_0 is a minimal prime ideal of

R. Since P is completely prime, P_0 is a completely prime ideal of R. Let I be a minimal prime ideal of R such that $I \subseteq P_0$. By assumption, I is α-compatible. Hence $\Delta(I)$ is a prime α-compatible ideal of A. Thus $\Delta(I)[x, x^{-1}; \alpha]$ is a prime ideal of $R[x, x^{-1}; \alpha]$ and $\Delta(I)[x, x^{-1}; \alpha] \subseteq \Delta(P_0)[x, x^{-1}; \alpha] \subseteq Q$. Since Q is a minimal prime ideal of $R[x, x^{-1}; \alpha]$, hence $\Delta(I)[x, x^{-1}; \alpha] = \Delta(P_0)[x, x^{-1}; \alpha] = Q$. Therefore $\Delta(I) = \Delta(P_0)$ and that $I = P_0$. Consequently $\Delta(P(R))[x, x^{-1}; \alpha] \subseteq P(R[x, x^{-1}; \alpha])$ and that $P(R[x, x^{-1}; \alpha]) = \Delta(P(R))[x, x^{-1}; \alpha]$. Since $P(R[x, x^{-1}; \alpha]) = \Delta(P(R))[x, x^{-1}; \alpha]$ and $P(R[x, x^{-1}; \alpha])$ is completely semiprime, hence $P(R)$ is completely semiprime.

(2)⇒(1). Since $P(R)$ is α-compatible and completely semiprime, $\Delta(P(R))$ is an α-compatible completely semiprime ideal of A. Hence $\Delta(P(R))[x, x^{-1}; \alpha]$ is a completely semiprime ideal of $A[x, x^{-1}; \alpha] = R[x, x^{-1}; \alpha]$. Thus $P(R[x, x^{-1}; \alpha]) = \Delta(P(R))[x, x^{-1}; \alpha]$ is a completely semiprime ideal of $R[x, x^{-1}; \alpha]$. □

Corollary 2.19. *Let α be an automorphism of R. Let each minimal prime ideal of R be α-compatible. Then the following are equivalent:*

(1) $P(R[x, x^{-1}; \alpha])$ *is completely semiprime;*

(2) $P(R[x, x^{-1}; \alpha]) = P(R)[x, x^{-1}; \alpha]$ *and $P(R)$ is completely semiprime.*

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