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Research Article

Convergence Rates for Probabilities of Moderate Deviations for Multidimensionally Indexed Random Variables

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Let $\{X, X_{\overline{n}}; \overline{n} \in Z_+^d\}$ be a sequence of i.i.d. real-valued random variables, and $S_{\overline{n}} = \sum_{\overline{k}\leq\overline{n}} X_{\overline{k}}$, $\overline{n} \in Z_+^d$. Convergence rates of moderate deviations are derived; that is, the rates of convergence to zero of certain tail probabilities of the partial sums are determined. For example, we obtain equivalent conditions for the convergence of the series $\sum_{\overline{n}} b(\overline{n})\psi^2(a(\overline{n}))P\{|S_{\overline{n}}| \geq a(\overline{n})\phi(a(\overline{n}))\}$, where $a(\overline{n}) = n_1^{1/\alpha_1}\cdots n_d^{1/\alpha_d}$, $b(\overline{n}) = n_1^{\beta_1}\cdots n_d^{\beta_d}$, ϕ and ψ are taken from a broad class of functions. These results generalize and improve some results of Li et al. (1992) and some previous work of Gut (1980).

1. Introduction

Let Z_+^d be the set of all positive integer d–dimensional lattice points with coordinate-wise partial ordering, $\leq$; that is, for every $\overline{m} = (m_1, \ldots, m_d)$, $\overline{n} = (n_1, \ldots, n_d) \in Z_+^d$, $\overline{m} \leq \overline{n}$ if and only if $m_i \leq n_i$, $i = 1, 2, \ldots, d$, where $d \geq 1$ is a fixed integer. $|\overline{n}|$ denote $\prod_{i=1}^d n_i$ and $\overline{n} \to \infty$ means $n_i \to \infty$, $i = 1, 2, \ldots, d$. Throughout the paper, $\{X, X_n, X_{\overline{n}}; n \in Z_+, \overline{n} \in Z_+^d\}$ are i.i.d. random variables with $EX = 0$ and $EX^2 = \sigma^2$. Let $S_n = \sum_{k=1}^n X_k$, and $S_{\overline{n}} = \sum_{\overline{k}\leq\overline{n}} X_{\overline{k}}$. we define

$$a(\overline{n}) = n_1^{1/\alpha_1}\cdots n_d^{1/\alpha_d}, \qquad b(\overline{n}) = n_1^{\beta_1}\cdots n_d^{\beta_d}, \tag{1.1}$$

where $\alpha_1, \ldots, \alpha_d$; $\beta_1, \ldots, \beta_d$ are real numbers with $0 < \alpha_i \leq 2$, $\beta_i \geq -1$, $i = 1, 2, \ldots, d$. Further, set $\alpha = \max\{\alpha_i; 1 \leq i \leq d\}$, $s = \max\{\alpha_i(\beta_i + 1); 1 \leq i \leq d\}$, $r = \max\{\alpha_i(\beta_i + 2); 1 \leq i \leq d\}$, $q = \operatorname{card}\{i : \alpha_i(\beta_i + 2) = r\}$, and $p = \operatorname{card}\{i : \alpha_i(\beta_i + 2) = 1\}$.

Hartman and Wintner [1] studied the fundamental strong laws of classic Probability Theory for i.i.d random variables $\{X, X_n; n \in Z_+\}$ and obtained the following Hartman-Wintner law of the iterated logarithm (LIL).

Theorem 1.1. *$EX = 0$ and $EX^2 = \sigma^2 < \infty$ if and only if*

$$\limsup_{n\to\infty} \frac{S_n}{(2n \log\log n)^{1/2}} = -\liminf_{n\to\infty} \frac{S_n}{(2n \log\log n)^{1/2}} = \sigma \ a.s. \tag{1.2}$$

Afterward, the study of the estimate of the convergence rate in the above relation (1.2) has been attracting the attention of various researchers over the last few decades. Darling and Robbins [2], Davis [3], Gafurov [4], and Li [5] have obtained some good results on the estimate of convergence rate in (1.2). The best result is probably the one given by Li et al. [6]. For easy reference, we restate their result in Theorem 1.2.

Theorem 1.2. *Let $\phi(t)$ and $\psi(t)$ be two positive real-valued functions on $[1, \infty)$ such that $\phi(t)$ is nondecreasing, $\lim_{t\to\infty}\phi(t) = \infty$, and $\psi(t) = O(\phi(t))$ as $t \to \infty$. For $t \geq 0$, let $\sigma^2(t) = EX^2 I\{|X| < \sqrt{t}\} - (EXI\{|X| < \sqrt{t}\})^2$, and $\sigma_n^2 \triangleq \sigma^2(n\phi^2(n))$. Then the following results are equivalent:*

$$\sum_{n\geq 1} \frac{\psi^2(n)}{n} P\left\{|S_n| \geq n^{1/2}\phi(n)\right\} < \infty, \tag{1.3}$$

$$\sum_{n\geq 1} \frac{\psi^2(n)}{n\phi(n)} \exp\left(-\frac{\phi^2(n)}{2\sigma_n^2}\right) < \infty. \tag{1.4}$$

If, in addition, $EX^2 I\{|X| > t\} = O(1/\log\log t)$ as $t \to \infty$, then (1.3) is equivalent to the following:

$$\sum_{n\geq 1} \frac{\psi^2(n)}{n\phi(n)} \exp\left(-\frac{1}{2}\phi^2(n)\right) < \infty. \tag{1.5}$$

Note that the above result is the best possible for n.

Further Strassen [7] introduced an almost sure invariance principle for the Brownian motion and obtained Strassen law of the iterated logarithm. In [8], Wichura generalized Strassen laws of the iterated logarithm for the stochastic processes with multidimensional indices and derived the following version of LIL for multidimensionally indexed i.i.d. random variables $\{X, X_{\overline{n}}; \overline{n} \in Z_+^d\}$, which is called the Wichura LIL.

Theorem 1.3. *$EX = 0, EX^2 = \sigma^2$, and $EX^2(\log^+|X|)^{d-1} < +\infty$ if and only if*

$$\limsup_{|\overline{n}|\to\infty} \frac{S_{\overline{n}}}{(2\sigma^2|\overline{n}| \log\log|\overline{n}|)^{1/2}} = \sqrt{d} \ a.s. \tag{1.6}$$

However the analogue of Theorem 1.2 is not yet available in the situation, where the i.i.d. random variables are multidimensionally indexed. This motivates us to consider the estimate of convergence rate in the relation (1.6). Towards this end, we consider the equivalence of the following statements:

$$\sum_{\overline{n}} \frac{\psi^2(|\overline{n}|)}{|\overline{n}|} P\{|S_{\overline{n}}| \geq |\overline{n}|\phi(|\overline{n}|)\} < \infty, \tag{1.7}$$

$$\sum_{\overline{n}} \frac{\psi^2(|\overline{n}|)}{|\overline{n}|\phi(|\overline{n}|)} \exp\left(-\frac{\phi^2(|\overline{n}|)}{2\sigma^2_{|\overline{n}|}}\right) < \infty, \tag{1.8}$$

$$\sum_{\overline{n}} \frac{\psi^2(|\overline{n}|)}{|\overline{n}|\phi(|\overline{n}|)} \exp\left(-\frac{1}{2}\phi^2(|\overline{n}|)\right) < \infty, \tag{1.9}$$

where $\phi(t)$ and $\psi(t)$ are the same as those in Theorem 1.1.

Gut [9] has obtained some equivalent conditions of (1.7) for special functions $\phi(|\overline{n}|) = \epsilon(\log|\overline{n}|)^{1/2}$, and $\psi(|\overline{n}|) = (\log|\overline{n}|)$. For $\phi(|\overline{n}|) = \log\log|\overline{n}|$, and $\psi(|\overline{n}|) = \log|\overline{n}|$, he also obtained sufficient conditions of (1.7). Recently, many researchers focus on the precise asymptotics in some strong limit theorems for multidimensionally indexed random variables (see Gut and Spătaru [10], Jiang et al. [11], Jiang and Yang [12], and Su [13]). However for the general form of functions $\phi(x)$ and $\psi(x)$ there is no discussion on the equivalences of (1.7) and (1.8) or (1.7) and (1.9) for the multidimensionally indexed random variables.

Therefore the aim of the present paper is to discuss the equivalences of (1.7) and (1.8) or of (1.7) and (1.9). In addition, for general ϕ and ψ, we will also consider the equivalence of the following statements:

$$\sum_{\overline{n}} b(\overline{n})\psi^2(a(\overline{n}))P\{|S_{\overline{n}}| \geq a(\overline{n})\phi(a(\overline{n}))\} < \infty, \tag{1.10}$$

$$\sum_{\overline{n}} \frac{b(\overline{n})\psi^2(a(\overline{n}))|\overline{n}|^{1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a^2(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|\sigma^2_{a(\overline{n})}}\right) < \infty, \tag{1.11}$$

$$\sum_{\overline{n}} \frac{b(\overline{n})\psi^2(a(\overline{n}))|\overline{n}|^{1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a^2(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|}\right) < \infty. \tag{1.12}$$

The equivalence of (1.7) and (1.8) (or (1.7) and (1.9)) follows from that of (1.10) and (1.11) (or (1.10) and (1.12)). Thus the main results of this paper not only generalize the result of Li et al. [6], but also improve the theorems of Gut [9]. The remainder of the paper is organized as follows. In Section 2, the main results are stated; proofs of which are presented in Section 4. In Section 3, we give some auxiliary results needed in the proofs of the theorems in Section 2.

2. Main Results

Let $\{X, X_{\overline{n}}; \overline{n} \in Z_+^d\}$ be a sequence of real-valued random variables with $EX = 0$, $EX^2 = 1$, and let $S_{\overline{n}} = \sum_{\overline{k} \leq \overline{n}} X_{\overline{k}}$, where $\overline{n} \in Z_+^d$. $\phi(t), \varphi(t)$ and $\sigma^2(t)$ are the same as in Theorem 1.1. For convenience, we use the symbols $\sigma_{\overline{n}}^2$ to denote $\sigma^2(|\overline{n}|\phi^2(|\overline{n}|))$ and $\sigma_{a(\overline{n})}^2$ to denote $\sigma^2(a(\overline{n})\phi^2(a(\overline{n})))$. Let $Lx = L_1 x = \log \max(e, x)$ and $L_k x = L(L_{k-1} x)$ for $k \geq 2$. We use Lx and $\log x$ interchangeably. We do the same for $L_2 x$ and $\log\log x$. $\log^+ x$ stands for $\max\{1, \log x\}$.

The following is a general result, which improves a number of existing results in the literature.

Theorem 2.1. *If $E|X|^r (\log^+|X|)^{q-1+p} < \infty$ and $\alpha = 2$, then* (1.10) *and* (1.11) *are equivalent. If, in addition, $EX^2 I\{|X| > a(\overline{n})\phi(a(\overline{n}))\} = O(|\overline{n}|/(a^2(\overline{n})\phi^2(a(\overline{n}))))$, then* (1.10) *and* (1.12) *are equivalent.*

By taking $a(\overline{n}) = |\overline{n}|^{1/2}$ and $b(\overline{n}) = |\overline{n}|^{-1}$ from Theorem 2.1, respectively, we obtain the following Theorems 2.2 and 2.3.

Theorem 2.2. *Let $E|X|^{2(\beta+2)} (\log^+|X|)^{q_1-1} < \infty$, where $\beta = \max\{\beta_i, 1 \leq i \leq d\} > -1$, and $q_1 = \mathrm{card}\{i : \beta_i = \beta\}$. Then the following results are equivalent:*

$$\sum_{\overline{n}} b(\overline{n})\varphi^2(|\overline{n}|) P\left\{|S_{\overline{n}}| \geq |\overline{n}|^{1/2}\phi(|\overline{n}|)\right\} < \infty, \tag{2.1}$$

$$\sum_{\overline{n}} \frac{b(\overline{n})\varphi^2(|\overline{n}|)}{\phi(|\overline{n}|)} \exp\left(-\frac{\phi^2(|\overline{n}|)}{2\sigma^2_{|\overline{n}|^{1/2}}}\right) < \infty. \tag{2.2}$$

Moreover, if $EX^2 I\{|X| > |\overline{n}|^{1/2}\phi(|\overline{n}|)\} = O(\phi^{-2}(|\overline{n}|))$, then (2.1) *is equivalent to*

$$\sum_{\overline{n}} \frac{b(\overline{n})\varphi^2(|\overline{n}|)}{\phi(|\overline{n}|)} \exp\left(-\frac{1}{2}\phi^2(|\overline{n}|)\right) < \infty. \tag{2.3}$$

Theorem 2.3. *Suppose that $\alpha = 2$ and $E|X|^2 (\log^+|X|)^{q_2-1} < +\infty$, where $q_2 = \mathrm{card}\{i : \alpha_i = 2\}$, then the following are equivalent:*

$$\sum_{\overline{n}} |\overline{n}|^{-1} \varphi^2(a(\overline{n})) P\{|S_{\overline{n}}| \geq a(\overline{n})\phi(a(\overline{n}))\} < \infty, \tag{2.4}$$

$$\sum_{\overline{n}} \frac{\varphi^2(a(\overline{n}))|\overline{n}|^{-1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a^2(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|\sigma^2_{a(\overline{n})}}\right) < \infty. \tag{2.5}$$

If, in addition, $EX^2I\{|X| \geq a(\overline{n})\phi(a(\overline{n}))\} = O(|\overline{n}|/a^2(\overline{n})\phi^2(a(\overline{n})))$*, then* (2.4) *is equivalent to*

$$\sum_{\overline{n}} \frac{\psi^2(a(\overline{n}))|\overline{n}|^{-1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a^2(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|}\right) < \infty. \tag{2.6}$$

In particular, we obtain the equivalence of (1.7) and (1.9).

Theorem 2.4. *Let* $E|X|^2(\log^+|X|)^{d-1} < \infty$ *and* $d \geq 2$*. Then,* (1.7) *and* (1.9) *are equivalent.*

Remarks. (i)If $r = 2$, then the condition that $EX^2I\{|X| > a(\overline{n})\phi(a(\overline{n}))\} = O(|\overline{n}|/(a^2(\overline{n})\phi^2(a(\overline{n}))))$ can be replaced by $EX^2I\{|X| > a(\overline{n})\phi(a(\overline{n}))\} = O(|\overline{n}|/(a^2(\overline{n})\log\log a(\overline{n})))$ in Theorem 2.1. This leads to the equivalence of the following two results:

$$\sum_{\overline{n}} \frac{b(\overline{n})\psi^2(a(\overline{n}))|\overline{n}|^{1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|\sigma^2_{a(\overline{n})}}\right) < \infty, \tag{2.7}$$

$$\sum_{\overline{n}} \frac{b(\overline{n})\psi_1^2(a(\overline{n}))|\overline{n}|^{1/2}}{a(\overline{n})\phi_1(a(\overline{n}))} \exp\left(-\frac{a(\overline{n})\phi_1^2(a(\overline{n}))}{2|\overline{n}|\sigma^2_{a(\overline{n})}}\right) < \infty, \tag{2.8}$$

where

$$\begin{aligned} \phi_1(t) &= \begin{cases} 2(d+1)\left(\log_2 t\right)^{1/2}, & \text{if } \phi(t) \geq 2(d+1)\left(\log_2 t\right)^{1/2}, \\ \phi(t), & \text{otherwise}, \end{cases} \\ \psi_1(t) &= \begin{cases} 2(d+1)\left(\log_2 t\right)^{1/2}, & \text{if } \psi(t) \geq 2(d+1)\left(\log_2 t\right)^{1/2}, \\ \psi(t), & \text{otherwise}. \end{cases} \end{aligned} \tag{2.9}$$

To see this, we note that

$$\sum_{\overline{n}} \frac{b(\overline{n})\left(2(d+1)\log_2 a(\overline{n})\right)^{1/2}|\overline{n}|^{1/2}}{a(\overline{n})} \exp\left(-\frac{a(\overline{n})2(d+1)\log_2 a(\overline{n})}{2|\overline{n}|\sigma^2_{a(\overline{n})}}\right) < \infty. \tag{2.10}$$

The equivalence of (2.7) and (2.8) follows immediately from (2.10) and $\sigma^2_{a(\overline{n})} \to 1$, $\phi_1(t) \leq \phi(t)$, and $\psi_1(t) \leq \psi(t)$. Equation (2.10) does not converge if $r > 2$. Therefore, the assumption $EX^2I\{|X| > |\overline{n}|^{1/2}\phi(\overline{n})\} = O(\phi^{-2}(|\overline{n}|))$ in Theorem 2.2 cannot be relaxed. Under the assumption that $\phi(t) = O(t^{-(\beta+1)/2\beta})$, $EX^2I\{|X| \geq a(\overline{n})\phi(a(\overline{n}))\} = O(\phi^{-2}(|\overline{n}|))$ can be removed. Similarly, $EX^2I\{|X| \geq a(\overline{n})\phi(a(\overline{n}))\} = O(|\overline{n}|/a^2(\overline{n})\phi^2(a(\overline{n})))$ can be replaced by $EX^2I\{|X| \geq a(\overline{n})\phi(a(\overline{n}))\} = O(|\overline{n}|/(a^2(\overline{n})\log_2 a(\overline{n})))$ in Theorem 2.3.

(ii) In Theorem 2.4, $EX^2(\log^+|X|)^{d-1} < +\infty$ implies $EX^2I\{|X| \geq \sqrt{t}\} = O(1/(\log\log t)$. From (i), $EX^2I\{|X| > |\overline{n}|^{1/2}\phi(|\overline{n}|)\} = O(\phi^{-2}(|\overline{n}|))$ may be discarded.

(iii) If $d = 1$, we can obtain Theorem 2.1 of Li et al. [6] from Theorem 2.1.

In the previous theorems, we assumed that $\psi(t) = O(\phi(t))$. Now we relax this assumption and obtain the following better result.

Theorem 2.5. *Let $\phi(t)$ and $\psi(t)$ be slowly varying functions, and let $\phi(t)$ be nondecreasing with $\phi(t) \to \infty$ $(t \to \infty)$. Then the following are equivalent:*

$$E(H(|X|))^r (\log^+ H(|X|))^{q-1+p} \psi(H(|X|)) < \infty,$$
$$\sum_{\overline{n}} \frac{b(\overline{n})\psi(a(\overline{n}))|\overline{n}|^{1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a^2(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|\sigma^2_{a(\overline{n})}}\right) < \infty, \tag{2.11}$$

$$\sum_{\overline{n}} b(\overline{n})\psi(a(\overline{n}))P\{|S_{\overline{n}}| \geq a(\overline{n})\phi(a(\overline{n}))\} < \infty. \tag{2.12}$$

If, in addition, $EX^2 I\{|X| > a(\overline{n})\phi(a(\overline{n}))\} = O(|\overline{n}|/a^2(\overline{n})\phi^2(a(\overline{n})))$, then (2.12) *is equivalent to the following:*

$$E(H(|X|))^r (\log^+ H(|X|))^{q-1+p} \psi(H(|X|)) < \infty,$$
$$\sum_{\overline{n}} \frac{b(\overline{n})\psi(a(\overline{n}))|\overline{n}|^{1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a^2(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|}\right) < \infty, \tag{2.13}$$

where $H(X)$ is the reverse function of $t\phi(t)$ and α, r, q, and p are the same as in Theorem 2.1.

Similar to Theorems 2.2 and 2.3, by choosing specific $\phi(\overline{n})$ and $\psi(\overline{n})$, we get the convergence rate for the LIL of random variables with multidimensional indices.

Corollary 2.6. *Under the conditions of Theorem 2.5, if $r = 2$ and $q > 1$, then the following are equivalent:*

$$EX^2 (\log^+|X|)^{q-1} (\log^+\log^+|X|)^{\gamma-1} < \infty,$$
$$EX = 0, \quad EX^2 = 1, \tag{2.14}$$

$$\sum_{\overline{n}} b(\overline{n})(\log_2 a(\overline{n}))^{\gamma} P\left\{|S_{\overline{n}}| \geq a(\overline{n})\sqrt{\delta \log_2 a(\overline{n})}\right\} < \infty \quad \text{for } \delta > 2q_3, \tag{2.15}$$

where $q_3 = \operatorname{card}\{i : \beta_i = 1\}$. In particular, for $d > 1$, the following are equivalent:

$$EX^2 (\log^+|X|)^{d-1} (\log^+\log^+|X|)^{\gamma-1} < \infty,$$
$$EX = 0, \quad EX^2 = 1, \tag{2.16}$$

$$\sum_{\overline{n}} |\overline{n}|^{-1} (\log_2|\overline{n}|)^{\gamma} P\left\{|S_{\overline{n}}| \geq \sqrt{2(d+\epsilon)|\overline{n}|\log_2|\overline{n}|}\right\} < \infty \quad \text{for each } \epsilon > 0. \tag{2.17}$$

Let $\phi_1,\ldots,\phi_d;\psi_1,\ldots,\psi_d$ be functions satisfying the previous assumptions. We now consider the equivalence of the following statements:

$$\sum_{\overline{n}}\left(\prod_{i=1}^{d}\frac{\psi_i^2(n_i)}{n_i}\right)P\left\{|S_{\overline{n}}|\geq|\overline{n}|^{1/2}\phi_1(n_1)\cdots\phi_d(n_d)\right\}<\infty, \tag{2.18}$$

$$\sum_{\overline{n}}\left(\prod_{i=1}^{d}\frac{\psi_i^2(n_i)}{n_i\phi_i(n_i)}\right)\exp\left(-\frac{1}{2\sigma_{\overline{n}}^2}\prod_{i=1}^{d}\phi_i^2(n_i)\right)<\infty, \tag{2.19}$$

$$\sum_{\overline{n}}\left(\prod_{i=1}^{d}\frac{\psi_i^2(n_i)}{n_i\phi_i(n_i)}\right)\exp\left(-\frac{1}{2}\prod_{i=1}^{d}\phi_i^2(n_i)\right)<\infty \tag{2.20}$$

and obtain the following theorem.

Theorem 2.7. *If $EX^2(\log^+|X|)^{\delta}<\infty$ for $\delta>d-1$, then* (2.18) *and* (2.19) *are equivalent. If, in addition, $EX^2I\{|X|\geq|\overline{n}|^{1/2}\prod_{i=1}^{d}\phi_i(n_i)\}=O(\prod_{i=1}^{d}\phi^{-2}(n_i))$, then* (2.18) *and* (2.20) *are equivalent.*

3. Auxiliary Results

In this section we give some lemmas that will be of use later. Again $\{X,X_{\overline{n}};\overline{n}\in Z_+^d\}$ are i.i.d. random variables with mean zero. C denotes a generic positive constant which varies from line to line. The notation $\simeq$ between sums and/or integrals will be used to denote that the quantities on either of the sign converge simultaneously and $f(x)\approx g(x)$ means that there are constants C_1 and C_2 such that for every x greater than some $x_0, C_1g(x)\leq f(x)\leq C_2g(x)$.

Lemma 3.1. *Let $\{\alpha_i;\ 1\leq i\leq d\},\{\tau_i;\ 1\leq i\leq d\}$ be real numbers with $\alpha_i\geq 0$, and let $a(\overline{n})=n_1^{1/\alpha_1}\cdots n_d^{1/\alpha_d}$, $t(\overline{n})=n_1^{\tau_1}\cdots n_d^{\tau_d}$, and $s(x)$ be a nondecreasing varying function. Define $\alpha=\max\{\alpha_i;\ 1\leq i\leq d\}$, $r=\max\{\alpha_i(\tau_i+1);\ 1\leq i\leq d\}$, $q=\operatorname{card}\{i:\alpha_i(\tau_i+1)=r\}$, $p=\operatorname{card}\{i:(\tau_i+1)=0\}$, and $\tau=\max\{\tau_i;\ 1\leq i\leq d\}$. For each $x\geq 0$, define*

$$\begin{aligned} g_1(x)&=\sum_{a(\overline{n})\geq x}t(\overline{n}), \quad g_2(x)=\sum_{a(\overline{n})\geq x}t(\overline{n})s(|\overline{n}|),\\ f_1(x)&=\sum_{a(\overline{n})\leq x}t(\overline{n}), \quad f_2(x)=\sum_{a(\overline{n})\leq x}t(\overline{n})s(|\overline{n}|). \end{aligned} \tag{3.1}$$

One has the following conclusions.

(i) *If $\tau+1\leq 0$, then $f_1(x)\approx(\log^+x)^p$, $f_2(x)\approx s(x)(\log^+x)^p$.*

(ii) *If $\tau+1>0$, then $f_1(x)\approx x^r(\log^+x)^{p+q-1}$, $f_2(x)\approx x^r(\log^+x)^{q+p-1}s(x)$.*

(iii) *If $\tau+1<0$, then $g_1(x)\approx x^r(\log^+x)^{q-1}$, $g_2(x)\approx x^r(\log^+x)^{q-1}s(x)$.*

This lemma extends Lemma 1.1 of Giang [14]. Since the proof is similar to that of Lemma 1.1 in Giang [14], the details are omitted.

Lemma 3.2. *Let $\tau_i\leq -1,(1\leq i\leq d)$, $t(\overline{n})=n_1^{\tau_1}\cdots n_d^{\tau_d}$, and $k=\operatorname{card}\{i:\tau_i=-1\}$. Then $\sum_{\overline{n}}t(\overline{n})(\log|\overline{n}|)^{-\beta}$ converges if $\beta>k$ and diverges if $\beta\leq k$.*

Proof. Without loss of generality, let $\tau_1 = \cdots = \tau_k = -1$, and $\tau_{k+1} < -1, \ldots, \tau_d < -1$. For large $|\overline{n}|$, we have

$$(\log n_1 \cdots n_k)^{-\beta} (\log n_{k+1} \cdots n_d)^{-\beta} \leq (\log|\overline{n}|)^{-\beta} \leq (\log n_1 \cdots n_k)^{-\beta}. \tag{3.2}$$

Thus,

$$\sum_{\overline{n}} t(\overline{n}) (\log|\overline{n}|)^{-\beta} \approx \sum_{n_1 \cdots n_k} (n_1 \cdots n_k)^{-1} (\log n_1 \cdots n_k)^{-\beta}. \tag{3.3}$$

The lemma follows from Lemma 5.1 of Gut [9]. □

Remark 3.3. From Lemma 3.2, one may show that $\sum_{\overline{n}} t(\overline{n})(\log|\overline{n}|)^{-\beta}(\log_2|\overline{n}|)^r$ converges for $-\infty < r < \infty$ if $\beta > k$.

The following lemma gives the estimate of the remainder term in the central limit theorem [15, page 125, Theorem 15] which plays an important role in the proofs of theorems in this paper.

Lemma 3.4. *Let $\{X, X_n; n \geq 1\}$ be i.i.d. random variables with $EX = 0, EX^2 = 1$, and $E|X|^3 < +\infty$. Then for all x,*

$$|F_n(x) - \Phi(x)| \leq C \frac{E|X|^3}{\sqrt{n}\left(1 + |x|^3\right)}, \tag{3.4}$$

where $F_n(x) = P((1/\sqrt{n}) \sum_{i=1}^{n} X_i \leq x), \Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} dt$.

4. Proofs of Main Results

Proof of Theorem 2.1. We first show that

$$\sum_{\overline{n}} b(\overline{n}) \psi^2(a(\overline{n})) \left| P\{S_{\overline{n}} \geq a(\overline{n})\phi(a(\overline{n}))\} - \Phi\left(-\frac{a(\overline{n})\phi(a(\overline{n}))}{|\overline{n}|^{1/2} \sigma_{a(\overline{n})}} \right) \right| < \infty, \tag{4.1}$$

where $\sigma^2_{a(\overline{n})} = \sigma^2(a^2(\overline{n})\phi^2(a(\overline{n}))) = EX^2 I\{|X| \leq a(\overline{n})\phi(a(\overline{n}))\} - (EXI\{|X| \leq a(\overline{n})\phi(a(\overline{n}))\})^2$.

For convenience, let $H(\overline{n}) = a(\overline{n})\phi(a(\overline{n}))$. For $\overline{k} \leq \overline{n}$, define

$$X'_k = X_k I\{|X_k| < H(\overline{n})\}. \tag{4.2}$$

We have that

$$\left| P\{S_{\overline{n}} > H(\overline{n})\} - P\left\{ \sum_{\overline{k}\le\overline{n}} X'_{\overline{k}} > H(\overline{n}) \right\} \right| \le |\overline{n}| P\{|X| \ge H(\overline{n})\}. \tag{4.3}$$

Now, put $M_{\overline{n}} = EXI\{|X| < H(\overline{n})\}$. Then we easily yield that

$$\begin{gathered} |M_{\overline{n}}|^3 \le E|X|^3 I\{|X| < H(\overline{n})\}, \\ |\overline{n}|^{-1/2} M_{\overline{n}} = -|\overline{n}| EXI\{|X| \ge H(\overline{n})\} \longrightarrow 0 \quad (|\overline{n}| \longrightarrow \infty), \\ 1 \ge \sigma^2_{a(\overline{n})} \longrightarrow 1 \quad (|\overline{n}| \longrightarrow \infty). \end{gathered} \tag{4.4}$$

By using Lemma 3.4, we have that

$$\begin{aligned} &\left| P\{S_{\overline{n}} > H(\overline{n})\} - \Phi\left\{ -\frac{H(\overline{n})}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}} \right\} \right| \\ &\le \left| P\{S_{\overline{n}} > H(\overline{n})\} - P\left\{ \sum_{\overline{k}\le\overline{n}} X'_{\overline{k}} > H(\overline{n}) \right\} \right| \\ &\quad + \left| P\left\{ -\frac{\sum_{\overline{k}\le\overline{n}} (X_{\overline{k}} - M_{\overline{k}})}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}} < -\frac{H(\overline{n}) - |\overline{n}| M_{\overline{n}}}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}} \right\} - \Phi\left(-\frac{H(\overline{n}) - |\overline{n}| M_{\overline{n}}}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}} \right) \right| \\ &\quad + \left| \Phi\left(-\frac{H(\overline{n}) - |\overline{n}| M_{\overline{n}}}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}} \right) - \Phi\left(-\frac{H(\overline{n})}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}} \right) \right| \\ &\le |\overline{n}| P\{|X| \ge H(\overline{n})\} + C \frac{E|X|^3 I\{|X| < H(\overline{n})\} + |M_{\overline{n}}|^3}{|\overline{n}|^{1/2}\left(1 + \left| (H(\overline{n}) - |\overline{n}| M_{\overline{n}}) / |\overline{n}|^{1/2}\sigma_{a(\overline{n})} \right|^3 \right)} \\ &\quad + C \frac{|\overline{n}||M_{\overline{n}}|}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}} \exp\left\{ -\frac{H^2(\overline{n})}{|\overline{n}|\sigma^2_{a(\overline{n})}} \right\} \\ &\le |\overline{n}| P\{|X| \ge H(\overline{n})\} + C \frac{|\overline{n}|}{H^3(\overline{n})} E|X|^3 I\{|X| < H(\overline{n})\} + C \frac{|\overline{n}|}{H(\overline{n})} E|X| I\{|X| \ge H(\overline{n})\} \\ &\le C \frac{|\overline{n}|}{H^3(\overline{n})} E|X|^3 I\{|X| < H(\overline{n})\} + C \frac{|\overline{n}|}{H(\overline{n})} E|X| I\{|X| \ge H(\overline{n})\}. \end{aligned} \tag{4.5}$$

Hence,

$$
\begin{aligned}
&\sum_{\overline{n}} b(\overline{n})\psi^2(a(\overline{n}))\left|P\{S_{\overline{n}} \geq a(\overline{n})\phi(a(\overline{n}))\} - \Phi\left(-\frac{a(\overline{n})\phi(a(\overline{n}))}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}}\right)\right| \\
&\quad\leq C\sum_{\overline{n}} \frac{b(\overline{n})\psi^2(a(\overline{n}))|\overline{n}|}{H(\overline{n})} E|X|I\{|X| \geq H(\overline{n})\} + C\sum_{\overline{n}} \frac{b(\overline{n})\psi^2(a(\overline{n}))|\overline{n}|}{H^3(\overline{n})} E|X|^3 I\{|X| < H(\overline{n})\} \\
&\quad\equiv I_1 + I_2.
\end{aligned}
\tag{4.6}
$$

By Lemma 3.1 and $\max(\beta_i + 1 - 1/\alpha_i + 1) \geq r/2 - 1/2 \geq 1/2 > 0$, we set

$$
\begin{aligned}
I_1 &= C\sum_{\overline{n}} \frac{b(\overline{n})\psi^2(a(\overline{n}))|\overline{n}|}{a(\overline{n})\phi(a(\overline{n}))} E|X|I\{|X| \geq a(\overline{n})\phi(a(\overline{n}))\} \\
&\leq C\sum_{\overline{n}} \frac{b(\overline{n})\phi(a(\overline{n}))|\overline{n}|}{a(\overline{n})} E|X|I\{|X| \geq a(\overline{n})\phi(a(\overline{n}))\} \\
&\leq C\sum_{i=0}^{\infty}\left(\sum_{i\leq a(\overline{n})<i+1} \frac{b(\overline{n})\phi(a(\overline{n}))|\overline{n}|}{a(\overline{n})}\right) E|X|I\{|X| \geq i\phi(i)\} \\
&\leq C\sum_{\overline{n}}\left(\sum_{i\leq a(\overline{n})<i+1} \frac{b(\overline{n})\phi(a(\overline{n}))|\overline{n}|}{a(\overline{n})}\right) \sum_{j=i}^{\infty} E|X|I\{j\phi(j) \leq |X| < (j+1)\phi(j+1)\} \\
&\leq C\sum_{j=0}^{\infty}\sum_{i=0}^{j}\left(\sum_{i\leq a(\overline{n})<i+1} \frac{b(\overline{n})\phi(a(\overline{n}))|\overline{n}|}{a(\overline{n})}\right) E|X|I\{j\phi(j) \leq |X| < (j+1)\phi(j+1)\} \\
&\leq C\sum_{j=0}^{\infty}\left(\sum_{0\leq a(\overline{n})<j+1} \frac{b(\overline{n})\phi(a(\overline{n}))|\overline{n}|}{a(\overline{n})}\right) E|X|I\{j\phi(j) \leq |X| < (j+1)\phi(j+1)\} \\
&\leq C\sum_{j=0}^{\infty}\phi(j+1)\left(\sum_{0\leq a(\overline{n})\leq j+1} \frac{b(\overline{n})|\overline{n}|}{a(\overline{n})}\right) E|X|I\{j\phi(j) \leq |X| < (j+1)\phi(j+1)\} \\
&\leq C\sum_{j=0}^{\infty}\phi(j+1)(j+1)^{r-1}(\log j+1)^{p+q-1} E|X|I\{j\phi(j) \leq |X| < (j+1)\phi(j+1)\} \\
&\leq C\sum_{j=0}^{\infty}\phi^2(j+1)(j+1)^{r}(\log j+1)^{p+q-1} P\{j\phi(j) \leq |X| < (j+1)\phi(j+1)\} \\
&\leq C\sum_{j=0}^{\infty}(\phi(j+1)(j+1))^{r}(\log(j+1)\phi(j+1))^{p+q-1} P\{j\phi(j) \leq |X| < (j+1)\phi(j+1)\}.
\end{aligned}
\tag{4.7}
$$

Because $\limsup_{x\to\infty}\phi(x+1)/\phi(x) < +\infty$ and $\phi(x) \to \infty$, the last expression is finite if and only if $E|X|^r(\log^+|X|)^{p+q-1} < +\infty$.

Similarly, because $-1 \le \beta_i < -1/2$, $\beta_i + 2 - 3/\alpha_i < 0$, we have that

$$\begin{aligned}
I_2 &= C\sum_{\overline{n}}\frac{b(\overline{n})\varphi^2(a(\overline{n}))|\overline{n}|}{H^3(\overline{n})}E|X|^3 I\{|X| < H(\overline{n})\}\\
&\le C\sum_{i=1}^{\infty}\left(\sum_{i-1<a(\overline{n})\le i}\frac{b(\overline{n})\varphi^2(a(\overline{n}))|\overline{n}|}{a^3(\overline{n})\phi^3(a(\overline{n}))}\right)E|X|^3 I\{|X| \le i\phi(i)\}\\
&\le C\sum_{i=1}^{\infty}\left(\sum_{i-1<a(\overline{n})\le i}\frac{b(\overline{n})\phi^2(a(\overline{n}))|\overline{n}|}{a^3(\overline{n})\phi^3(a(\overline{n}))}\right)\sum_{j=1}^{i}E|X|^3 I\{(j-1)\phi(j-1) < |X| \le j\phi(j)\}\\
&\le C\sum_{j=1}^{\infty}\sum_{i=j}^{\infty}\left(\sum_{i-1<a(\overline{n})\le i}\frac{b(\overline{n})|\overline{n}|}{a^3(\overline{n})\phi(a(\overline{n}))}\right)E|X|^3 I\{(j-1)\phi(j-1) < |X| \le j\phi(j)\}\\
&\le C\sum_{j=1}^{\infty}\left(\sum_{a(\overline{n})\ge j-1}\frac{b(\overline{n})|\overline{n}|}{a^3(\overline{n})\phi(a(\overline{n}))}\right)E|X|^3 I\{(j-1)\phi(j-1) < |X| \le j\phi(j)\}\\
&\le C\sum_{j=1}^{\infty}\frac{1}{\phi(j-1)}\left(\sum_{a(\overline{n})\ge j-1}\frac{b(\overline{n})|\overline{n}|}{a^3(\overline{n})}\right)E|X|^3 I\{(j-1)\phi(j-1) < |X| \le j\phi(j)\}\\
&\le C\sum_{j=1}^{\infty}\frac{\phi^3(j)(j-1)^{r-3}(\log j)^{q-1}j^3}{\phi(j-1)}P\{(j-1)\phi(j-1) < |X| \le j\phi(j)\}\\
&\le C\sum_{j=1}^{\infty}(j\phi(j))^r(\log j\phi(j))^{p+q-1}P\{(j-1)\phi(j-1) < |X| \le j\phi(j)\}.
\end{aligned} \tag{4.8}$$

The last expression is finite if and only if $E|X|^r(\log^+|X|)^{p+q-1} < +\infty$.

As above, we have that

$$\sum_{\overline{n}}b(\overline{n})\varphi^2(a(\overline{n}))\left|P\{-S_{\overline{n}} \ge a(\overline{n})\phi(a(\overline{n}))\} - \Phi\left(-\frac{a(\overline{n})\phi(a(\overline{n}))}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}}\right)\right| < \infty. \tag{4.9}$$

By (4.1) and (4.9), we set

$$\sum_{\overline{n}}b(\overline{n})\varphi^2(a(\overline{n}))\left|P\{|S_{\overline{n}}| \ge a(\overline{n})\phi(a(\overline{n}))\} - \Phi\left(-\frac{a(\overline{n})\phi(a(\overline{n}))}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}}\right)\right| < \infty. \tag{4.10}$$

Therefore,

$$\sum_{\overline{n}} b(\overline{n})\psi^2(a(\overline{n}))P\{|S_{\overline{n}}| \geq a(\overline{n})\phi(a(\overline{n}))\} < \infty \tag{4.11}$$

is equivalent to

$$\sum_{\overline{n}} b(\overline{n})\psi^2(a(\overline{n}))\Phi\left(-\frac{a(\overline{n})\phi(a(\overline{n}))}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}}\right) < \infty \tag{4.12}$$

(4.12)is also equivalent to the following:

$$\sum_{\overline{n}} \frac{b(\overline{n})\psi^2(a(\overline{n}))|\overline{n}|^{1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a^2(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|\sigma^2_{a(\overline{n})}}\right) < \infty. \tag{4.13}$$

If $E|X|^2 I\{|X| > a(\overline{n})\phi(a(\overline{n}))\} = O(|\overline{n}|/a^2(\overline{n})\phi^2(a(\overline{n})))$, then (4.13) is equivalent to

$$\sum_{\overline{n}} \frac{b(\overline{n})\psi^2(a(\overline{n}))|\overline{n}|^{1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a^2(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|}\right) < \infty. \tag{4.14}$$

The theorem is proved. □

The proofs of Theorems 2.2, 2.3, and 2.4 are similar to those of Theorem 2.1 and are omitted. Before proving Theorem 2.5, we first give two lemmas.

Lemma 4.1. *Let $\phi(t)$ and $\psi(t)$ be slowly varying functions at infinity with $\phi(t)$ nondecreasing and $\phi(t) \to \infty$. The following are equivalent:*

$$E(H(|X|))^r(\log^+ H(|X|))^{q-1+p}\psi(H(|X|)) < \infty, \tag{4.15}$$

$$\sum_{\overline{n}} b(\overline{n})\psi(a(\overline{n}))|\overline{n}|P\{|X| \geq a(\overline{n})\phi(a(\overline{n}))\} < \infty, \tag{4.16}$$

where $H(x)$ is the reverse function of $x\phi(x)$.

Proof. Because $\alpha_i(\beta_i+2) \geq \alpha_i > 0$, by using Lemma 3.1, we have

$$\begin{aligned}
&\sum_{\overline{n}} b(\overline{n})\psi(a(\overline{n}))|\overline{n}|P\{|X| \geq a(\overline{n})\phi(a(\overline{n}))\} \\
&\quad \simeq \sum_{i=1}^{\infty}\left(\sum_{i\leq a(\overline{n})<i+1} b(\overline{n})\psi(a(\overline{n}))|\overline{n}|\right)P\{|X| \geq i\phi(i)\} \\
&\quad = \sum_{i=1}^{\infty}\left(\sum_{i\leq a(\overline{n})<i+1} b(\overline{n})\psi(a(\overline{n}))|\overline{n}|\right)\sum_{j=1}^{\infty}P\{j\phi(j) \leq |X| < (j+1)\phi(j+1)\} \\
&\quad = \sum_{j=1}^{\infty}\left(\sum_{1\leq a(\overline{n})<j+1} b(\overline{n})\psi(a(\overline{n}))|\overline{n}|\right)P\{j \leq H(|X|) < (j+1)\} \\
&\quad \simeq \sum_{j=1}^{\infty}\psi(j+1)(j+1)^r(\log(j+1))^{q-1+p}P\{j \leq H(|X|) < (j+1)\}.
\end{aligned} \tag{4.17}$$

The last expression is finite if and only if $E(H(|X|))^r(\log^+ H(|X|))^{q-1+p}\psi(H(|X|)) < +\infty$. This completes the proof of the lemma. □

Lemma 4.2. $\sum_{\overline{n}} b(\overline{n})\psi(a(\overline{n}))P\{|S_{\overline{n}}| \geq a(\overline{n})\phi(a(\overline{n}))\} < \infty$ *implies* (4.15).

Proof. We first prove the result for symmetric random variables. One may rearrange $\{X_{\overline{k}}, \overline{k} \leq \overline{n}\}$ and obtain $\{X_j, 1 \leq j \leq |\overline{n}|\}$. By Levy inequality, we set

$$\begin{aligned}
P\left\{\max_{\overline{k}\leq\overline{n}}|X_{\overline{k}}| \geq a(\overline{n})\phi(a(\overline{n}))\right\} &= P\left\{\max_{1\leq j\leq|\overline{n}|}|X_j| \geq a(\overline{n})\phi(a(\overline{n}))\right\} \\
&\leq 2P\left\{\left|\sum_{j=1}^{|\overline{n}|}X_j\right| \geq a(\overline{n})\phi(a(\overline{n}))\right\} \\
&= 2P\{|S_{\overline{n}}| \geq a(\overline{n})\phi(a(\overline{n}))\} < \infty,
\end{aligned} \tag{4.18}$$

but

$$\begin{aligned}
P\left\{\max_{\overline{k}\leq\overline{n}}|X_{\overline{k}}| \geq a(\overline{n})\phi(a(\overline{n}))\right\} &= 1-\left(1-P\{|X| \geq a(\overline{n})\phi(a(\overline{n}))\}\right)^{|\overline{n}|} \\
&\approx |\overline{n}|P\{|X| \geq a(\overline{n})\phi(a(\overline{n}))\}.
\end{aligned} \tag{4.19}$$

Thus for large $|\overline{n}|$,

$$|\overline{n}|P\{|X| \geq a(\overline{n})\phi(a\overline{n})\} \leq CP\{|S_{\overline{n}}| \geq a(\overline{n})\phi(a(\overline{n}))\}, \tag{4.20}$$

equation (4.15) follows from Lemma 4.1.

If $\{X_{\overline{n}}, \overline{n} \in Z_+^d\}$ are nonsymmetric random variables, then by using the standard symmetrization method, it is easy to prove that for some constant C,

$$|\overline{n}|P\{|X| \geq Ca(\overline{n})\phi(a\overline{n})\} \leq CP\{|S_{\overline{n}}| \geq a(\overline{n})\phi(a(\overline{n}))\}, \tag{4.21}$$

which implies that

$$\sum_{\overline{n}} b(\overline{n})\psi(a(\overline{n}))|\overline{n}|P\{|X| \geq Ca(\overline{n})\phi(a(\overline{n}))\} < \infty. \tag{4.22}$$

That is, (4.15) holds. □

Proof of Theorem 2.5. Similar to the proof of Theorem 2.2, and also by Lemma 4.1, $E(H(|X|))^r(\log^+ H(|X|))^{q-1+p}\psi(H(|X|)) < \infty$ implies that

$$\sum_{\overline{n}} b(\overline{n})\psi(a(\overline{n}))\left|P\{S_{\overline{n}} \geq a(\overline{n})\phi(a(\overline{n}))\} - \Phi\left(-\frac{a(\overline{n})\phi(a(\overline{n}))}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}}\right)\right| < \infty. \tag{4.23}$$

Therefore, if $E(H(|X|))^r(\log^+ H(|X|))^{q-1+p}\ \psi(H(|X|)) < \infty$, (2.12) is equivalent to

$$\sum_{\overline{n}} b(\overline{n})\psi(a(\overline{n}))\Phi\left(-\frac{a(\overline{n})\phi(a(\overline{n}))}{|\overline{n}|^{1/2}\sigma_{a(\overline{n})}}\right) < \infty, \tag{4.24}$$

which is equivalent to

$$\sum_{\overline{n}} \frac{b(\overline{n})\psi(a(\overline{n}))|\overline{n}|^{1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a^2(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|\sigma^2_{a(\overline{n})}}\right) < \infty. \tag{4.25}$$

On the other hand, (2.12) implies $E(H(|X|))^r\ (\log^+ H(|X|))^{q-1+p}\ \psi(H(|X|)) < \infty$ by Lemma 4.2. Hence, (2.11) and (2.12) are equivalent. If, in addition, $EX^2 I\{|X| > a(\overline{n})\phi(a(\overline{n}))\} = O(|\overline{n}|/a^2(\overline{n})\phi^2(a(\overline{n})))$, (4.24) is equivalent to the following:

$$\sum_{\overline{n}} \frac{b(\overline{n})\psi(a(\overline{n}))|\overline{n}|^{1/2}}{a(\overline{n})\phi(a(\overline{n}))} \exp\left(-\frac{a^2(\overline{n})\phi^2(a(\overline{n}))}{2|\overline{n}|}\right) < \infty. \tag{4.26}$$

Hence, (2.12) and (2.13) are equivalent. □

Proof of Theorem 2.7. Like the proof of Theorem 2.1, $EX^2(\log^+|X|)^{d-1} < +\infty$ and (2.17) imply (2.16). On the other hand, if (2.16) holds, then $EX^2(\log^+|X|)^{\delta} < \infty$ implies (2.17). The proof is complete. □

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