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Research Article

On System of Generalized Vector Quasiequilibrium Problems with Applications

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We introduce a new system of generalized vector quasiequilibrium problems which includes system of vector quasiequilibrium problems, system of vector equilibrium problems, and vector equilibrium problems, and so forth in literature as special cases. We prove the existence of solutions for this system of generalized vector quasi-equilibrium problems. Consequently, we derive some existence results of a solution for the system of generalized quasi-equilibrium problems and the generalized Debreu-type equilibrium problem for both vector-valued functions and scalar-valued functions.

1. Introduction and Formulations

In the resent years, the vector equilibrium problems have been studied in [1–7] and the references therein which is a unified model of several problems, for instance, vector variational inequality, vector variational-like inequality, vector complementarity problems, vector optimization problems. A comprehensive bibliography on vector equilibrium problems, vector variational inequalities, vector variational-like inequalities and their generalizations can be found in a recent volume [1]. Ansari and Yao [8] and Chiang et al. [9] introduced and studied some vector quasi-equilibrium problems which generalized those quasi-equilibrium problems in [10–17] to the case of vector-valued function. Very recently, the system of vector equilibrium problems was introduced by Ansari et al. [18] with applications in Nash-type equilibrium problem for vector-valued functions. The system of vector quasi-equilibrium problems was introduced by Ansari et al. [19] with applications in Debreu-type equilibrium problem for vector-valued functions. As a generalization of the above models, we introduce a new system of generalized vector quasi-equilibrium problems, that is, a family of generalized quasi-equilibrium problems for vector-valued maps defined on a product set.

Throughout this paper, for a set A in a topological space, we denote by $\operatorname{co} A$, $\operatorname{int} A$, $\overline{\operatorname{co}}\, A$ the convex hull, interior, and the convex closure of A, respectively.

Let I be an index set. For each $i \in I$, let Z_i, E_i and let F_i be topological vector spaces. Consider two family of nonempty convex subsets $\{X_i\}_{i\in I}$ with $X_i \subseteq E_i$ and $\{Y_i\}_{i\in I}$ with $Y_i \subseteq F_i$. Let

$$E = \prod_{i\in I} E_i, \qquad X = \prod_{i\in I} X_i, \qquad F = \prod_{i\in I} F_i, \qquad Y = \prod_{i\in I} Y_i. \tag{1.1}$$

An element of the set $X^i = \prod_{j\in I\setminus i} X_i$ will be denoted by x^i, therefore, $x \in X$ will be written as $x = (x^i, x_i) \in X^i \times X_i$. Similarly, an element of the set Y will be denoted by $y = (y^i, y_i) \in Y^i \times Y_i$. For each $i \in I$, let $C_i : X \to 2^{Z_i}$, $D_i : X \times Y \to 2^{X_i}$ and $T_i : X \times Y \to 2^{Y_i}$ be set-valued maps with nonempty values, and let $f_i : X \times Y \times X_i \to Z_i$ be a the vector-valued function. Then the system of generalized vector quasi-equilibrium problems (in Short, SGVQEP) is to find $(\overline{x}, \overline{y}) = (\overline{x}^i, \overline{x}_i, \overline{y}^i, \overline{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\overline{x}_i \in D_i(\overline{x}, \overline{y}), \qquad \overline{y}_i \in T_i(\overline{x}, \overline{y}) : f_i(\overline{x}, \overline{y}, z_i) \notin -\operatorname{int} C_i(\overline{x}), \quad \forall z_i \in D_i(\overline{x}, \overline{y}). \tag{1.2}$$

Here are some special cases of the (SGVQEP).

(i) For each $i \in I$, let $\phi_i : X \times Y \to Z_i$ be a vector-valued function. We define a trifunction $f_i : X \times Y \times X_i \to Z_i$ as $f_i(x, y, u_i) = \phi_i(x^i, y, u_i) - \phi_i(x, y), \forall (x, y, u_i) \in X \times Y \times X_i$. Then the (SGVQEP) reduces to the generalized Debreu-type equilibrium problem for vector-valued functions (in short, G-Debreu VEP), which is to find $(\overline{x}, \overline{y}) = (\overline{x}^i, \overline{x}_i, \overline{y}^i, \overline{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\overline{x}_i \in D_i(\overline{x}, \overline{y}), \qquad \overline{y}_i \in T_i(\overline{x}, \overline{y}) : \phi_i\left(\overline{x}^i, \overline{y}, z_i\right) - \phi_i(\overline{x}, \overline{y}) \notin -\operatorname{int} C_i(\overline{x}), \quad \forall z_i \in D_i(\overline{x}, \overline{y}). \tag{1.3}$$

(ii) We denote by R and R^+ the set of real numbers and the set of real nonnegative numbers, respectively. For each $i \in I$, if $Z_i = R$, and $C_i(x) = R^+$ for all $x \in X$, then the (SGVQEP) reduces to the system of generalized quasi-equilibrium problems (in short, SGQEP), which is to find $(\overline{x}, \overline{y}) = (\overline{x}^i, \overline{x}_i, \overline{y}^i, \overline{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\overline{x}_i \in D_i(\overline{x}, \overline{y}), \qquad \overline{y}_i \in T_i(\overline{x}, \overline{y}) : f_i(\overline{x}, \overline{y}, z_i) \geq 0, \quad \forall z_i \in D_i(\overline{x}, \overline{y}). \tag{1.4}$$

And the G-Debreu VEP reduces to the generalized Debreu-type equilibrium problem for scalar-valued functions (in short, G-Debreu EP), which is to find $(\overline{x}, \overline{y}) = (\overline{x}^i, \overline{x}_i, \overline{y}^i, \overline{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\overline{x}_i \in D_i(\overline{x}, \overline{y}), \qquad \overline{y}_i \in T_i(\overline{x}, \overline{y}) : \phi_i\left(\overline{x}^i, \overline{y}, z_i\right) \geq \phi_i(\overline{x}, \overline{y}), \quad \forall z_i \in D_i(\overline{x}, \overline{y}). \tag{1.5}$$

(iii) Let $Y = \{\overline{y}\}$. For each $i \in I$, let $D_i(x, \overline{y}) = A_i(x)$, $T_i(x, \overline{y}) = \{\overline{y}_i\}$ for all $x \in X$, where $A_i : X \to 2^{X_i}$ is a set-valued map. We define a function $\varphi_i : X \times X_i \to Z_i$ and a function $h_i : X \times Y \to Z_i$ as $\varphi_i(x, z_i) = f_i(x, \overline{y}, z_i)$, for all $(x, z_i) \in X \times X_i$, and $h_i(x) = \phi_i(x, \overline{y})$, for all $x \in X$, then (SGVQEP) and (G-Debreu VEP), respectively, reduce to the system of vector quasi-equilibrium problems and the (Debreu VEP) introduced by Ansari et al. [19] which contain those mathematical in [18, 20] as special cases. The (SGQEP) reduces to the mathematical models in [21, page 286] and [22, pages 152-153] and the (G-Debreu EP) reduces to the abstract economy in [23, page 345] which contains the noncooperative game in [24] as a special case.

(iv) If the index set I is singleton, $D(x, y) = D_i(x)$, $T(x, y) = T(x)$, and $C(x) = C$, then the (SGVQEP) becomes the implicit vector variational inequality in [9] and the (SGQEP) reduces to the quasi-equilibrium problem investigated in [14–17].

The rest of this paper is arranged in the following manner. The following section deals with some preliminary definitions, notations and results which will be used in the sequel. In Section 3, we establish existence results for a solution to the (SGVQEP) and the (SGQEP) with or without involving Φ-condensing maps by using similar techniques in [19]. In Section 4, as applications of the results of Section 3, we derive some existence results of a solution for the (G-Debreu VEP) and the the (G-Debreu EP).

2. Preliminaries

In order to prove the main results, we need the following definitions.

Definition 2.1 ([19, 25]). Let M be a nonempty convex subset of a topological vector space E and Z a real topological space with a closed and convex cone P with apex at the origin. A vector-valued function $\varphi : M \to Z$ is called

(i) P-quasifunction if and only if, for all $z \in Z$, the set $\{x \in M : \varphi(x) \in z - P\}$ is convex,

(ii) natural P-quasifunction if and only if, $\forall x, y \in M$, and $\lambda \in [0, 1]$, $\varphi(\lambda x + (1 - \lambda)y) \in \mathrm{co}\{\varphi(x), \varphi(y)\} - P$.

Definition 2.2 ([13]). Let X and Y be two topological spaces. $T : X \to 2^Y$ be a set-valued map. Then T is said to be upper semicontinuous if the set $\{x \in X : T(x) \subseteq V\}$ is open in X for every open subset V of Y. Also T is said to be lower semicontinuous if the set $\{x \in X : T(x) \cap V\}$ is open in X for every open subset V of $Y \cdot T$ is said to have open lower sections if the set $T^{-1}(y) = \{x \in X : y \in T(x)\}$ is open in X for each $y \in Y$.

Definition 2.3 ([26]). Let E be a Hausdorff topological space and L a lattice with least element, denoted by 0. A map $\Phi : 2^E \to L$ is a measure of noncompactness provided that the following conditions hold $\forall M, N \in 2^E$:

(i) $\Phi(M) = 0$ iff M is precompact (i.e., it is relatively compact),

(ii) $\Phi(\overline{\mathrm{co}}M) = \Phi(M)$,

(iii) $\Phi(M \cup N) = \max\{\Phi(M), \Phi(N)\}$.

Definition 2.4 ([26]). Let $\Phi : 2^E \to L$ be a measure of noncompactness on E and $X \subseteq E$. A set-valued map $T : X \to 2^E$ is called Φ-condensing provided that, if $M \subseteq X$ with $\Phi(T(M)) \geq \Phi(M)$, then M is relatively compact.

Remark 2.5. Note that every set-valued map defined on a compact set is Φ-condensing for any measure of noncompactness Φ. If E is locally convex and $T : X \to 2^E$ is a compact set-valued map (i.e., $T(X)$ is precompact), then T is Φ-condensing for any measure of noncompactness Φ. It is clear that if $T : X \to 2^E$ is Φ-condensing and $T^* : X \to 2^E$ satisfies $T^*(x) \subseteq T(x)$ $\forall x \in X$, then T^* is also Φ-condensing.

We will use the following particular forms of two maximal element theorems for a family of set-valued maps due to Deguire et al. [27, Theorem 7] and Chebbi and Florenzano [28, Corollary 4].

Lemma 2.6 ([19, 27]). *Let $\{X_i\}_{i\in I}$ be a family of nonempty convex subsets where each X_i is contained in a Hausdorff topological vector space E_i, For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a set-valued map such that*

(i) *for each $i \in I$, $S_i(x)$ is convex,*

(ii) *for each $x \in X$, $x_i \notin S_i(x)$,*

(iii) *for each $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in X.*

(iv) *there exist a nonempty compact subset N of X and a nonempty compact convex subset B_i of X_i for each $i \in I$ such that for each $x \in X \setminus N$ there exists $i \in I$ satisfying $S_i(x) \cap B_i \neq \emptyset$. Then there exists $\overline{x} \in X$ such that $S_i(\overline{x}) = \emptyset$ for all $i \in I$.*

Lemma 2.7 ([19, 28]). *Let I be any index set and $\{X_i\}_{i\in I}$ be a family of nonempty, closed and convex subsets where each X_i is contained in a locally convex Hausdorff topological vector space E_i. For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a set-valued map. Assume that the set-valued map $S : X \to 2^X$ defined as $S(x) = \prod_{i\in I} S_i(x)$, $\forall x \in X$, is Φ-condensing and the conditions (i), (ii), (iii) of Lemma 2.6 hold. Then there exists $\overline{x} \in X$ such that $S_i(\overline{x}) = \emptyset$ for all $i \in I$.*

3. Existence Results

An existence result of a solution for the system of generalized vector quasi-equilibrium problems with or without Φ-condensing maps are will shown in this section.

Theorem 3.1. *Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, let E_i and F_i be two Hausdorff topological vector spaces, let $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty and convex subsets, let $D_i : X \times Y \to 2^{X_i}$ and $T_i : X \times Y \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, and the set $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x, y) \text{ and } y_i \in T_i(x, y)\}$ be closed in $X \times Y$ and let $f_i : X \times Y \times X_i \to Z_i$ be a vector-valued function. For each $i \in I$, let $C_i : X \to 2^{Z_i}$ be a set-valued map such that $C_i(x)$ be a proper closed and convex cone with apex at the origin and $\operatorname{int} C_i(x) \neq \emptyset$ for all $x \in X$ and $P_i = \cap_{x\in X} C_i(x)$. Assume that*

(i) *for all $x = (x^i, x_i) \in X$, for all $y \in Y$, $f_i(x, y, x_i) \notin -\operatorname{int} C_i(x)$;*

(ii) *for each $(x, y) \in X \times Y$, $z_i \mapsto f_i(x, y, z_i)$ is natural P_i-quasifunction;*

(iii) *for all $z_i \in X_i$, the set $\{(x, y) \in X \times Y : f_i(x, y, z_i) \notin -\operatorname{int} C_i(x)\}$ is closed in $X \times Y$;*

(iv) *there exist nonempty and compact subsets $N \subseteq X$ and $K \subseteq Y$ and nonempty, compact and convex subsets $B_i \subseteq X_i$, $A_i \subseteq Y_i$ for each $i \in I$ such that $\forall (x, y) = (x^i, x_i, y) \in X \times Y \setminus N \times K$ $\exists i \in I$ and $\exists \overline{u}_i \in B_i$, $\overline{v}_i \in A_i$ satisfying $\overline{u}_i \in D_i(x, y)$, $\overline{v}_i \in T_i(x, y)$ and $f_i(x, y, \overline{u}_i) \in -\operatorname{int} C_i(x)$.*

Then, there exists $(\overline{x}, \overline{y}) = (\overline{x}^i, \overline{x}_i, \overline{y}^i, \overline{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\overline{x}_i \in D_i(\overline{x}, \overline{y}), \qquad \overline{y}_i \in T_i(\overline{x}, \overline{y}) : f_i(\overline{x}, \overline{y}, z_i) \notin -\operatorname{int} C_i(\overline{x}), \quad \forall z_i \in D_i(\overline{x}, \overline{y}). \tag{3.1}$$

That is, the solution set of the (SGVQEP) is nonempty.

Proof. For each $i \in I$, let us define a set-valued map $Q_i : X \times Y \to 2^{X_i}$ by

$$Q_i(x,y) = \{z_i \in X_i : f_i(x,y,z_i) \in -\operatorname{int} C_i(x)\}, \quad \forall (x,y) \in X \times Y. \tag{3.2}$$

Then, $\forall i \in I$ and $\forall (x,y) \in X \times Y$, $Q_i(x,y)$ is a convex set.

To prove it, let us fix arbitrary $i \in I$ and $(x,y) \in X \times Y$. Let $z_{i1}, z_{i2} \in Q_i(x,y)$ and $\lambda \in [0,1]$, then we have

$$f_i\left(x,y,z_{i_j}\right) \in -\operatorname{int} C_i(x), \quad \text{for } j = 1,2. \tag{3.3}$$

Since $f_i(x,y,\cdot)$ is natural P_i-quasifunction, $\exists \mu \in [0,1]$ such that

$$f_i(x,y,\lambda z_{i_1} + (1-\lambda)z_{i_2}) \in \mu f_i(x,y,z_{i_1}) + (1-\mu) f_i(x,y,z_{i_2}) - P_i. \tag{3.4}$$

From (3.3) and (3.4), we get

$$f_i(x,y,\lambda z_{i_1} + (1-\lambda)z_{i_2}) \in -\operatorname{int} C_i(x) - \operatorname{int} C_i(x) - P_i \subseteq -\operatorname{int} C_i(x). \tag{3.5}$$

Hence $\lambda z_{i_1} + (1-\lambda)z_{i_2} \in Q_i(x,y)$ and, therefore, $Q_i(x,y)$ is convex.

It follows from condition (i) that, for each $i \in I$ and for all $(x,y) = (x^i, x_i, y) \in X \times Y$,

$$x_i \notin Q_i(x,y). \tag{3.6}$$

It follows from condition (iii) that for each $i \in I$ and each $z_i \in X_i$, the set

$$Q_i^{-1}(z_i) = \{(x,y) \in X \times Y : f_i(x,y,z_i) \in -\operatorname{int} C_i(x)\} \tag{3.7}$$

is open in X_i. That is, Q_i has open lower sections on $X \times Y$. For each $i \in I$, we also define another set-valued map $S_i : X \times Y \to 2^{X_i \times Y_i}$ by

$$S_i(x,y) = \begin{cases} [D_i(x,y) \cap Q_i(x,y)] \times T_i(x,y), & \text{if } (x,y) \in W_i, \\ D_i(x,y) \times T_i(x,y), & \text{if } (x,y) \notin W_i. \end{cases} \tag{3.8}$$

Then, it is clear that $\forall i \in I$ and $\forall (x,y) \in X \times Y$, $S_i(x,y)$ is convex, and $(x_i, y_i) \notin S_i(x,y)$. Since $\forall i \in I$ and $\forall (u_i, v_i) \in X_i \times Y_i$,

$$\begin{aligned} S_i^{-1}(u_i,v_i) = {} & \left[Q_i^{-1}(u_i) \cap \left(D_i^{-1}(u_i)\right) \cap \left(T_i^{-1}(v_i)\right)\right] \\ & \cup \left[(X \times Y \setminus W_i) \cap \left(D_i^{-1}(u_i)\right) \cap \left(T_i^{-1}(v_i)\right)\right], \end{aligned} \tag{3.9}$$

and $D_i^{-1}(u_i)$, $T_i^{-1}(v_i)$, $Q_i^{-1}(u_i)$, and $X \times Y \setminus W_i$ are open in $X \times Y$, we have $S_i^{-1}(u_i,v_i)$ is open in $X \times Y$.

From condition (iv), there exist a nonempty and compact subset $N \times K \subseteq X \times Y$ and a nonempty, compact, and convex subset $B_i \times A_i \subseteq X_i \times Y_i$ for each $i \in I$ such that $\forall (x, y) = (x^i, x_i, y) \in X \times Y \setminus N \times K$ $\exists i \in I$ and $\exists(\overline{u}_i, \overline{v}_i) \in S_i(x, y) \cap (B_i \times A_i)$. Hence, by Lemma 2.6, $\exists(\overline{x}, \overline{y}) \in X \times Y$ such that $S_i(\overline{x}, \overline{y}) = \emptyset, \forall i \in I$. Since $\forall i \in I$ and $\forall (x, y) \in X \times X$, $D_i(x, y)$ and $T_i(x, y)$ are nonempty, we have $(\overline{x}, \overline{y}) \in W_i$ and $D_i(\overline{x}, \overline{y}) \cap Q_i(\overline{x}, \overline{y}) = \emptyset$, $\forall i \in I$. This implies $(\overline{x}, \overline{y}) \in W_i$ and $D_i(\overline{x}, \overline{y}) \cap Q_i(\overline{x}, \overline{y}) = \emptyset$, $\forall i \in I$. Therefore, $\forall i \in I$,

$$\overline{x}_i \in D_i(\overline{x}, \overline{y}), \qquad \overline{y}_i \in T_i(\overline{x}, \overline{y}), \qquad f_i(\overline{x}, \overline{y}, z_i) \notin -\operatorname{int} C_i(\overline{x}), \quad \forall z_i \in D_i(\overline{x}, \overline{y}). \tag{3.10}$$

That is, the solution set of the (SGVQEP) is nonempty. □

Remark 3.2. (1) The condition (iii) of Theorem 3.1 is satisfied if the following conditions hold $\forall i \in I$:

(a) $C_i : X \to 2^{Z_i}$ is a set-valued map such that $\operatorname{int} C_i(x) \neq \emptyset$ for each $x \in X$ and the set-valued map $M_i = Z_i \setminus (-\operatorname{int} C_i) : X \to 2^{Z_i}$ is upper semicontinuous;

(b) for all $z_i \in X_i$, the map $(x, y) \mapsto f_i(x, y, z_i)$ is continuous on $X \times Y$;

(2) If $\forall i \in I$, and $\forall x \in X$, $C_i(x) = C_i$, a (fixed) proper, closed and convex cone in Z_i, then the condition (ii) and (iii) of Theorem 3.1 can be replaced, respectively, by the following conditions:

(c) $\forall i \in I$, the vector-valued function $\forall (x, y) \in X \times Y$, $z_i \mapsto f_i(x, y, z_i)$ is C_i-quasifunction;

(d) $\forall i \in I, \forall z_i \in X_i$, the map $(x, y) \mapsto f_i(x, y, z_i)$ is C_i-upper semicontinuous on $X \times Y$;

(3) Theorem 3.1 extends and generalizes in [19, Theorem 2], [20, Theorem 2.1] and [18, Theorem 2.1] in several ways.

(4) If $\forall i \in I$, X_i is a nonempty, compact and convex subset of a Hausdorff topological vector space E_i, then the conclusion of Theorem 3.1 holds without condition (iv).

Theorem 3.3. *Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, let E_i and F_i be two locally convex Hausdorff topological vector spaces, let $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty, closed and convex subsets, let $D_i : X \times Y \to 2^{X_i}$ and $T_i : X \times Y \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, the set $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x, y)$ and $y_i \in T_i(x, y)\}$ be closed in $X \times Y$ and $f_i : X \times Y \times X_i \to Z_i$ be a vector-valued function. For each $i \in I$, let $C_i : X \to 2^{Z_i}$ be a set-valued map such that $C_i(x)$ be a proper closed and convex cone with apex at the origin and $\operatorname{int} C_i(x) \neq \emptyset$ for all $x \in X$ and $P_i = \cap_{x \in X} C_i(x)$. Assume that the set-valued map $D \times T = (\prod_{i \in I} D_i \times \prod_{i \in I} T_i) : X \times Y \to 2^{X \times Y}$ defined as $(D \times T)(x, y) = \prod_{i \in I} D_i(x, y) \times \prod_{i \in I} T_i(x, y)$, $\forall (x, y) \in X \times Y$, is Φ-condensing and for each $i \in I$, the conditions (i), (ii) and (iii) of Theorem 3.1 hold. Then the solution set of the (SGVQEP) is nonempty.*

Proof. In view of Lemma 2.7 and the proof of Theorem 3.1, it is sufficient to show that the set-valued map $S : X \times Y \to 2^{X \times Y}$ defined as $S(x, y) = \prod_{i \in I} S_i(x, y)$, for all $(x, y) \in X \times Y$, is Φ-condensing, where S_i's are the same as in the proof of Theorem 3.1. By the definition of S_i, $S_i(x, y) \subseteq D_i(x, y) \times T_i(x, y)$ for all $(x, y) \in X \times Y$ and for each $i \in I$, and therefore $S(x, y) \subseteq D(x, y) \times T(x, y)$ for all $(x, y) \in X \times Y$. Since $D \times T$ is Φ-condensing, by Remark 2.5, we have S is also Φ-condensing. □

By Theorem 3.1 and Remark 3.2, we can easily get the following result.

Corollary 3.4. *Let I be any index set. For each $i \in I$, let E_i and F_i be two Hausdorff topological vector spaces, let $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty and convex subsets, let $D_i : X \times Y \to 2^{X_i}$ and $T_i : X \times Y \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, let the set $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x, y)$ and $y_i \in T_i(x, y)\}$ be closed in $X \times Y$, and $f_i : X \times Y \times X_i \to R$ be a function. Assume that*

(i) *for all $x = (x^i, x_i) \in X$, for all $y \in Y$, $f_i(x, y, x_i) \geq 0$;*

(ii) *for each $(x, y) \in X \times Y$, $z_i \mapsto f_i(x, y, z_i)$ is quasiconvex;*

(iii) *for all $z_i \in X_i$, the set $\{(x, y) \in X \times Y : f_i(x, y, z_i) \geq 0\}$ is closed in $X \times Y$;*

(iv) *there exist nonempty and compact subsets $N \subseteq X$ and $K \subseteq Y$ and nonempty, compact and convex subsets $B_i \subseteq X_i$, $A_i \subseteq Y_i$ for each $i \in I$ such that $\forall (x, y) = (x^i, x_i, y) \in X \times Y \setminus N \times K$ $\exists i \in I$ and $\exists \overline{u}_i \in B_i$, $\overline{v}_i \in A_i$ satisfying $\overline{u}_i \in D_i(x, y)$, $\overline{v}_i \in T_i(x, y)$ and $f_i(x, y, \overline{u}_i) < 0$.*

Then, there exists $(\overline{x}, \overline{y}) = (\overline{x}^i, \overline{x}_i, \overline{y}^i, \overline{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\overline{x}_i \in D_i(\overline{x}, \overline{y}), \qquad \overline{y}_i \in T_i(\overline{x}, \overline{y}) : f_i(\overline{x}, \overline{y}, z_i) \geq 0, \quad \forall z_i \in D_i(\overline{x}, \overline{y}). \tag{3.11}$$

That is, the solution set of the (SGQEP) is nonempty.

By Theorem 3.3, we can easily get the following result.

Corollary 3.5. *Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, let E_i and F_i be two locally convex Hausdorff topological vector spaces, let $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty, closed and convex subsets, let $D_i : X \times Y \to 2^{X_i}$ and $T_i : X \times Y \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, the set $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x, y)$ and $y_i \in T_i(x, y)\}$ be closed in $X \times Y$ and $f_i : X \times Y \times X_i \to R$ be a function. Assume that the set-valued map $D \times T = (\prod_{i \in I} D_i \times \prod_{i \in I} T_i) : X \times Y \to 2^{X \times Y}$ defined as $(D \times T)(x, y) = \prod_{i \in I} D_i(x, y) \times \prod_{i \in I} T_i(x, y)$, $\forall (x, y) \in X \times Y$, is Φ-condensing and for each $i \in I$, the conditions (i), (ii) and (iii) of Corollary 3.4 hold. Then the solution set of the (SGQEP) is nonempty.*

Remark 3.6. Theorem 3.3 is a generalization of [19, Theorem 3]. Corollaries 3.4 and 3.5 extend and generalize the main results in [10–17].

4. Applications

In this section, we present some existence of a solution for the (G-Debreu VEP) and the (G-Debreu EP).

Theorem 4.1. *Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, let E_i and F_i be two Hausdorff topological vector spaces, let $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty and convex subsets, let $C_i : X \to 2^{Z_i}$ be a set-valued map such that $C_i(x)$ is a proper, closed and convex cone with apex at the origin and $\operatorname{int} C_i(x) \neq \emptyset$ for each $x \in X$ and $P_i = \cap_{x \in X} C_i(x)$, $D_i : X \times Y \to 2^{X_i}$ and $T_i : X \times Y \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, the set $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x, y)$ and $y_i \in T_i(x, y)\}$ be closed in $X \times Y$ and ϕ_i be a bifunction* *from $X \times Y$ into Z_i. For each $i \in I$, assume that*

(i) *$M_i = Z_i \setminus (-\operatorname{int} C_i) : X \to 2^{Z_i}$ is upper semicontinuous;*

(ii) *For all $x^i \in X^i$ and $y \in Y$, $z_i \mapsto \phi_i(x^i, y, z_i)$ is natural P_i-quasifunction, where $P_i = \cap_{x \in X} C_i(x)$;*

(iii) *ϕ_i is continuous on $X \times Y$;*

(iv) *there exist nonempty and compact subsets $N \subseteq X$ and $K \subseteq Y$ and nonempty, compact and convex subsets $B_i \subseteq X_i$, $A_i \subseteq Y_i$ for each $i \in I$ such that $\forall (x, y) = (x^i, x_i, y) \in X \times Y \setminus N \times K$ $\exists i \in I$ and $\exists \overline{u}_i \in B_i$, $\overline{v}_i \in A_i$ satisfying $\overline{u}_i \in D_i(x, y)$, $\overline{v}_i \in T_i(x, y)$ and $\phi_i(x^i, y, \overline{u}_i) - \phi_i(x, y) \in -\operatorname{int} C_i(x)$.*

Then, there exists $(\overline{x}, \overline{y}) = (\overline{x}^i, \overline{x}_i, \overline{y}^i, \overline{y}_i)$ in $X \times Y$ such that for each $i \in I$,

$$\overline{x}_i \in D_i(\overline{x}, \overline{y}), \qquad \overline{y}_i \in T_i(\overline{x}, \overline{y}) : \phi_i\left(\overline{x}^i, \overline{y}, z_i\right) - \phi_i(\overline{x}, \overline{y}) \notin -\operatorname{int} C_i(\overline{x}), \quad \forall z_i \in D_i(\overline{x}, \overline{y}). \tag{4.1}$$

That is, the solution set of the (G-Debreu VEP) is nonempty.

Proof. For each $i \in I$, we define a trifunction $f_i : X \times Y \times X_i$ as

$$f_i(x, y, u_i) = \phi_i\left(x^i, y, u_i\right) - \phi_i(x, y), \quad \forall (x, y, u_i) \in X \times Y \times X_i. \tag{4.2}$$

Since $\phi_i(x^i, y, \cdot)$ is natural P_i quasi-function, by [19, Remark 2], for all $u_{i_1}, u_{i_2} \in X_i$ and for all $\lambda \in [0,1]$, $\exists \alpha \in [0,1]$ such that

$$\phi_i\left(x^i, y, \lambda u_{i_1} + (1-\lambda) u_{i_2}\right) \in \alpha \phi_i\left(x^i, y, u_{i_1}\right) + (1-\alpha) \phi_i\left(x^i, y, u_{i_2}\right) - P_i, \tag{4.3}$$

Hence

$$f_i(x, y, \lambda u_{i_1} + (1-\lambda) u_{i_2}) \in \alpha f_i(x, y, u_{i_1}) + (1-\alpha) f_i(x, y, u_{i_2}) - P_i. \tag{4.4}$$

Hence, for all $(x, y) \in X \times Y$, $f_i(x, y, \cdot)$ is natural P_i quasifunction.

By condition (iii), we know that for all $z_i \in X_i$, the map $(x, y) \mapsto f_i(x, y, z_i)$ is continuous on $X \times Y$. So it follows from Remark 3.2 that condition (iii) of Theorem 3.1 holds. It is easy to verify that the other conditions of Theorem 3.1 are satisfied. By Theorem 3.1, we know that the conclusion holds. □

Similarly, by Theorem 3.3, Corollaries 3.4 and 3.5, respectively, we have the following results.

Theorem 4.2. *Let I be any index set. For each $i \in I$, let Z_i be a topological vector space, let E_i and F_i be two locally convex Hausdorff topological vector spaces, let $X_i \subseteq E_i$ and $Y_i \subseteq F_i$ be nonempty, closed and convex subsets, let $C_i : X \to 2^{Z_i}$ be a set-valued map such that $C_i(x)$ is a proper, closed and convex cone with apex at the origin and $\operatorname{int} C_i(x) \neq \emptyset$ for each $x \in X$ and $P_i = \cap_{x \in X} C_i(x)$, $D_i : X \times Y \to 2^{X_i}$ and $T_i : X \times Y \to 2^{Y_i}$ be set-valued maps with nonempty convex values and open lower sections, the set $W_i = \{(x, y) \in X \times Y : x_i \in D_i(x, y)$ and $y_i \in T_i(x, y)\}$ be* *closed in* $X \times Y$ *and* $\varphi_i : X \times Y \to Z_i$ *be a vector-valued function. Assume that the set-valued map* $D \times T = (\prod_{i\in I} D_i \times \prod_{i\in I} T_i) : X \times Y \to 2^{X\times Y}$ *defined as* $(D\times T)(x,y) = \prod_{i\in I} D_i(x,y) \times \prod_{i\in I} T_i(x,y)$, $\forall (x,y) \in X \times Y$, *is* Φ*-condensing and (i), (ii), and (iii) of Theorem 4.1 hold. Then, the solution set of the (G-Debreu VEP) is nonempty.*

Theorem 4.3. *Let* I *be any index set. For each* $i \in I$, *let* $X_i \subseteq E_i$ *and* $Y_i \subseteq F_i$ *be nonempty and convex subsets, let* $D_i : X \times Y \to 2^{X_i}$ *and* $T_i : X \times Y \to 2^{Y_i}$ *be set-valued maps with nonempty convex values and open lower sections, the set* $W_i = \{(x,y) \in X \times Y : x_i \in D_i(x,y) \text{ and } y_i \in T_i(x,y)\}$ *be closed in* $X \times Y$ *and* ϕ_i *be a bifunction from* $X \times Y$ *into* R. *For each* $i \in I$, *assume that*

(i) *for all* $x^i \in X^i$ *and* $y \in Y$, $z_i \mapsto \phi_i(x^i, y, z_i)$ *is quasiconvex;*

(ii) ϕ_i *is continuous on* $X \times Y$;

(iii) *there exist nonempty and compact subsets* $N \subseteq X$ *and* $K \subseteq Y$ *and nonempty, compact and convex subsets* $B_i \subseteq X_i$, $A_i \subseteq Y_i$ *for each* $i \in I$ *such that* $\forall (x,y) = (x^i, x_i, y) \in X \times Y \setminus N \times K$ $\exists i \in I$ *and* $\exists \overline{u}_i \in B_i$, $\overline{v}_i \in A_i$ *satisfying* $\overline{u}_i \in D_i(x,y)$, $\overline{v}_i \in T_i(x,y)$ *and* $\phi_i(x^i, y, \overline{u}_i) < \phi_i(x,y)$.

Then, there exists $(\overline{x}, \overline{y}) = (\overline{x}^i, \overline{x}_i, \overline{y}^i, \overline{y}_i)$ *in* $X \times Y$ *such that for each* $i \in I$,

$$\overline{x}_i \in D_i(\overline{x}, \overline{y}), \qquad \overline{y}_i \in T_i(\overline{x}, \overline{y}) : \phi_i\left(\overline{x}^i, \overline{y}, z_i\right) \geq \phi_i(\overline{x}, \overline{y}), \quad \forall z_i \in D_i(\overline{x}, \overline{y}). \tag{4.5}$$

That is, the solution set of the (G-Debreu EP) is nonempty.

Theorem 4.4. *Let* I *be any index set. For each* $i \in I$, *let* E_i *and* F_i *be two locally convex Hausdorff topological vector spaces,* $X_i \subseteq E_i$ *and* $Y_i \subseteq F_i$ *be nonempty, closed and convex subsets, let* $D_i : X \times Y \to 2^{X_i}$ *and* $T_i : X \times Y \to 2^{Y_i}$ *be set-valued maps with nonempty convex values and open lower sections, the set* $W_i = \{(x,y) \in X \times Y : x_i \in D_i(x,y) \text{ and } y_i \in T_i(x,y)\}$ *be closed in* $X \times Y$ *and* $\varphi_i : X \times Y \to R$ *be a function. Assume that the set-valued map* $D \times T = (\prod_{i\in I} D_i \times \prod_{i\in I} T_i) : X \times Y \to 2^{X\times Y}$ *defined as* $(D\times T)(x,y) = \prod_{i\in I} D_i(x,y) \times \prod_{i\in I} T_i(x,y)$, $\forall (x,y) \in X \times Y$, *is* Φ*-condensing and (i), and (ii) of Theorem 4.3 hold. Then, the solution set of the (G-Debreu EP) is nonempty.*

Remark 4.5. Theorem 4.1 extends and generalizes [19, Theorem 5] and [20, Theorems 3.1, 3.6 and Corollaries 3.2, 3.3, and 3.5]. Theorem 4.2 extends and generalizes [19, Theorem 6]. Theorems 4.3 and 4.4 are generalizations of [20, Corollaries 3.5 and 3.7] and the corresponding results in [21–24].

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References

[1] F. Giannessi, *Vector Variational Inequalities and Vector Equilibria. Mathematical Theories*, vol. 38 of *Nonconvex Optimization and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2000.

[2] X. H. Gong, "Efficiency and Henig efficiency for vector equilibrium problems," *Journal of Optimization Theory and Applications*, vol. 108, no. 1, pp. 139–154, 2001.

[3] M. Bianchi, N. Hadjisavvas, and S. Schaible, "Vector equilibrium problems with generalized monotone bifunctions," *Journal of Optimization Theory and Applications*, vol. 92, no. 3, pp. 527–542, 1997.

[4] Q. H. Ansari, X. Q. Yang, and J.-C. Yao, "Existence and duality of implicit vector variational problems," *Numerical Functional Analysis and Optimization*, vol. 22, no. 7-8, pp. 815–829, 2001.

[5] W. Oettli, "A remark on vector-valued equilibria and generalized monotonicity," *Acta Mathematica Vietnamica*, vol. 22, no. 1, pp. 213–221, 1997.

[6] N. Hadjisavvas and S. Schaible, "From scalar to vector equilibrium problems in the quasimonotone case," *Journal of Optimization Theory and Applications*, vol. 96, no. 2, pp. 297–309, 1998.

[7] N. X. Tan and P. N. Tinh, "On the existence of equilibrium points of vector functions," *Numerical Functional Analysis and Optimization*, vol. 19, no. 1-2, pp. 141–156, 1998.

[8] Q. H. Ansari and J. C. Yao, "On vector quasi-quasi-equilibrium problems," in *Equilibrium Problems and Variational Models*, P. Daniele, F. Giannessi, and A. Maugeri, Eds., Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[9] Y. Chiang, O. Chadli, and J. C. Yao, "Existence of solutions to implicit vector variational inequalities," *Journal of Optimization Theory and Applications*, vol. 116, no. 2, pp. 251–264, 2003.

[10] J. W. Peng, "Quasi-equilibrium problem on W-spase," *Journal of Chongqiong Normal University*, vol. 17, no. 4, pp. 36–40, 2000 (Chinese).

[11] P. Cubiotti, "Existence of Nash equilibria for generalized games without upper semicontinuity," *International Journal of Game Theory*, vol. 26, no. 2, pp. 267–273, 1997.

[12] J. Zhou and G. Chen, "Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 132, no. 1, pp. 213–225, 1988.

[13] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Pure and Applied Mathematics, John Wiley & Sons, New York, NY, USA, 1984.

[14] S. Chang, B. S. Lee, X. Wu, Y. J. Cho, and G. M. Lee, "On the generalized quasi-variational inequality problems," *Journal of Mathematical Analysis and Applications*, vol. 203, no. 3, pp. 686–711, 1996.

[15] M.-P. Chen, L.-J. Lin, and S. Park, "Remarks on generalized quasi-equilibrium problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 52, no. 2, pp. 433–444, 2003.

[16] L.-J. Lin and Z.-T. Yu, "Fixed points theorems of KKM-type maps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 38, no. 2, pp. 265–275, 1999.

[17] L.-J. Lin and S. Park, "On some generalized quasi-equilibrium problems," *Journal of Mathematical Analysis and Applications*, vol. 224, no. 2, pp. 167–181, 1998.

[18] Q. H. Ansari, S. Schaible, and J. C. Yao, "System of vector equilibrium problems and its applications," *Journal of Optimization Theory and Applications*, vol. 107, no. 3, pp. 547–557, 2000.

[19] Q. H. Ansari, W. K. Chan, and X. Q. Yang, "The system of vector quasi-equilibrium problems with applications," *Journal of Global Optimization*, vol. 29, no. 1, pp. 45–57, 2004.

[20] J. W. Peng, X. M. Yang, and D. L. Zhu, "System of vector quasi-equilibrium problems and its applications," *Applied Mathematics and Mechanics*, vol. 27, no. 8, pp. 1107–1114, 2006.

[21] C. Ionescu Tulcea, "On the approximation of upper semi-continuous correspondences and the equilibriums of generalized games," *Journal of Mathematical Analysis and Applications*, vol. 136, no. 1, pp. 267–289, 1988.

[22] G. X.-Z. Yuan, G. Isac, K.-K. Tan, and J. Yu, "The study of minimax inequalities, abstract economics and applications to variational inequalities and Nash equilibria," *Acta Applicandae Mathematicae*, vol. 54, no. 2, pp. 135–166, 1998.

[23] W. Shafer and H. Sonnenschein, "Equilibrium in abstract economies without ordered preferences," *Journal of Mathematical Economics*, vol. 2, no. 3, pp. 345–348, 1975.

[24] J. Nash, "Non-cooperative games," *Annals of Mathematics*, vol. 54, pp. 286–295, 1951.

[25] D. T. Luc, *Theory of Vector Optimization*, vol. 319 of *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, Germany, 1989.

[26] P. M. Fitzpatrick and W. V. Petryshyn, "Fixed point theorems for multivalued noncompact acyclic mappings," *Pacific Journal of Mathematics*, vol. 54, no. 2, pp. 17–23, 1974.

[27] P. Deguire, K. K. Tan, and G. X.-Z. Yuan, "The study of maximal elements, fixed points for L_S-majorized mappings and their applications to minimax and variational inequalities in product topological spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 37, no. 7, pp. 933–951, 1999.

[28] S. Chebbi and M. Florenzano, "Maximal elements and equilibria for condensing correspondences," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 38, no. 8, pp. 995–1002, 1999.