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Research Article

The Elliptic $GL(n)$ Dynamical Quantum Group as an $\mathfrak{h}$-Hopf Algebroid

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Using the language of $\mathfrak{h}$-Hopf algebroids which was introduced by Etingof and Varchenko, we construct a dynamical quantum group, $\mathcal{F}_{\text{ell}}(GL(n))$, from the elliptic solution of the quantum dynamical Yang-Baxter equation with spectral parameter associated to the Lie algebra $\mathfrak{sl}_n$. We apply the generalized FRST construction and obtain an $\mathfrak{h}$-bialgebroid $\mathcal{F}_{\text{ell}}(M(n))$. Natural analogs of the exterior algebra and their matrix elements, elliptic minors, are defined and studied. We show how to use the cobraiding to prove that the elliptic determinant is central. Localizing at this determinant and constructing an antipode we obtain the $\mathfrak{h}$-Hopf algebroid $\mathcal{F}_{\text{ell}}(GL(n))$.

1. Introduction

The quantum dynamical Yang-Baxter (QDYB) equation was introduced by Gervais and Neveu [1]. It was realized by Felder [2] that this equation is equivalent to the Star-Triangle relation in statistical mechanics. It is a generalization of the quantum Yang-Baxter equation, involving an extra, so-called dynamical, parameter. In [2] an interesting elliptic solution to the QDYB equation with spectral parameter was given, adapted from the $A_n^{(1)}$ solution to the Star-Triangle relation constructed in [3]. Felder also defined a tensor category, which he suggested that it should be thought of as an elliptic analog of the category of representations of quantum groups. This category was further studied in [4] in the $\mathfrak{sl}_2$ case.

In [5], the authors considered objects in Felder's category which were proposed as analogs of exterior and symmetric powers of the vector representation of $\mathfrak{gl}_n$. To each object in the tensor category they associate an algebra of vector-valued difference operators and prove that a certain operator, constructed from the analog of the top exterior power, commutes with all other difference operators. This is also proved in [6, Appendix B] in more detail and in [7] using a different approach.

An algebraic framework for studying dynamical R-matrices without spectral parameter was introduced in [8]. There the authors defined the notion of $\mathfrak{h}$-bialgebroids and $\mathfrak{h}$-Hopf algebroids, a special case of the Hopf algebroids defined by Lu [9]. See [10, Remark 2.1] for a comparison of Hopf algebroids to related structures. In [8] the authors also show, using a generalized version of the FRST construction, how to associate to every solution R of the nonspectral quantum dynamical Yang-Baxter equation an $\mathfrak{h}$-bialgebroid. Under some extra condition they get an $\mathfrak{h}$-Hopf algebroid by adjoining formally the matrix elements of the inverse L-matrix. This correspondence gives a tensor equivalence between the category of representations of the R-matrix and the category of so-called dynamical representations of the $\mathfrak{h}$-bialgebroid.

In this paper we define an $\mathfrak{h}$-Hopf algebroid associated to the elliptic R-matrix from [2] with both dynamical and spectral parameters for $\mathfrak{g} = \mathfrak{sl}_n$. This generalizes the spectral elliptic dynamical $GL(2)$ quantum group from [11] and the nonspectral trigonometric dynamical $GL(n)$ quantum group from [12]. As in [11], this is done by first using the generalized FRST construction, modified to also include spectral parameters. In addition to the usual RLL relation, residual relations must be added "by hand" to be able to prove that different expressions for the determinant are equal.

Instead of adjoining formally all the matrix elements of the inverse L-matrix, we adjoin only the inverse of the determinant, as in [11]. Then we express the antipode using this inverse. The main problem is to find the correct formula for the determinant, to prove that it is central and to provide row and column expansion formulas for the determinant in the setting of $\mathfrak{h}$-bialgebroids.

It would be interesting to develop harmonic analysis for the elliptic $GL(n)$ quantum group, similarly to [13]. In this context it is valuable to have an abstract algebra to work with and not only a tensor category analogous to a category of representations. For example, the analog of the Haar measure seems most naturally defined as a certain linear functional on the algebra.

The plan of this paper is as follows. After introducing some notation in Section 2.1, we recall the definition of the elliptic R-matrix in Section 2.2. In Section 3 we review the definition of $\mathfrak{h}$-bialgebroids and the generalized FRST construction with special emphasis on how to treat residual relations for a general R-matrix. We write down the relations explicitly in Section 4 for the algebra $\mathcal{F}_{\mathrm{ell}}(M(n))$ obtained from the elliptic R-matrix. In particular we show that only one family of residual identities is needed.

Left and right analogs of the exterior algebra over $\mathbb{C}^n$ are defined in Section 5 in a similar way as in [12]. They are certain comodule algebras over $\mathcal{F}_{\mathrm{ell}}(M(n))$ and arise naturally from a single relation analogous to $v \wedge v = 0$. The matrix elements of these corepresentations are generalized minors depending on a spectral parameter. Their properties are studied in Section 6. In particular we show that the left and right versions of the minors in fact coincide. In Section 6.3 we prove Laplace expansion formulas for these elliptic quantum minors.

In Section 7 we show that the $\mathfrak{h}$-bialgebroid $\mathcal{F}_{\mathrm{ell}}(M(n))$ can be equipped with a cobraiding, in the sense of [14], extending the $n = 2$ case from [10]. We use this and the ideas as in [5, 6] to prove that the determinant is central for all values of the spectral parameters. This implies that the determinant is central in the operator algebra as shown in [5].

Finally, in Section 7.4 we define $\mathcal{F}_{\mathrm{ell}}(GL(n))$ to be the localization of $\mathcal{F}_{\mathrm{ell}}(M(n))$ at the determinant and show that it has an antipode giving it the structure of an $\mathfrak{h}$-Hopf algebroid.

2. Preliminaries

2.1. Notation

Let $p, q \in \mathbb{R}$, $0 < p, q < 1$. We assume p, q are generic in the sense that if $p^a q^b = 1$ for some $a, b \in \mathbb{Z}$, then $a = b = 0$.

Denote by θ the normalized Jacobi theta function:

$$\theta(z) = \theta(z;p) = \prod_{j=0}^{\infty}\left(1 - zp^j\right)\left(1 - \frac{p^{j+1}}{z}\right). \tag{2.1}$$

It is holomorphic on $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ with zero set $\{p^k : k \in \mathbb{Z}\}$ and satisfies

$$\theta\left(z^{-1}\right) = \theta(pz) = -z^{-1}\theta(z) \tag{2.2}$$

and the addition formula

$$\theta\left(xy, \frac{x}{y}, zw, \frac{z}{w}\right) = \theta\left(xw, \frac{x}{w}, zy, \frac{z}{y}\right) + \left(\frac{z}{y}\right)\theta\left(xz, \frac{x}{z}, yw, \frac{y}{w}\right), \tag{2.3}$$

where we use the notation

$$\theta(z_1, \ldots, z_n) = \theta(z_1)\cdots\theta(z_n). \tag{2.4}$$

Recall also the Jacobi triple product identity, which can be written

$$\sum_{k\in\mathbb{Z}}(-z)^k p^{k(k-1)/2} = \theta(z)\prod_{j=1}^{\infty}\left(1 - p^j\right). \tag{2.5}$$

It will sometimes be convenient to use the auxiliary function E given by

$$E : \mathbb{C} \longrightarrow \mathbb{C}, \qquad E(s) = q^s\theta\left(q^{-2s}\right). \tag{2.6}$$

Relation (2.2) implies that $E(-s) = -E(s)$.

The set $\{1, 2, \ldots, n\}$ will be denoted by $[1, n]$.

2.2. The Elliptic R*-Matrix*

Let $\mathfrak{h}$ be a complex vector space, viewed as an abelian Lie algebra, $\mathfrak{h}^*$ its dual space, and let $V = \oplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ a diagonalizable $\mathfrak{h}$-module. A dynamical R-matrix is by definition a meromorphic function

$$R : \mathfrak{h}^* \times \mathbb{C}^\times \longrightarrow \operatorname{End}_{\mathfrak{h}}(V \otimes V) \tag{2.7}$$

satisfying the quantum dynamical Yang-Baxter equation with spectral parameter (QDYBE):

$$\begin{aligned} &R\left(\lambda, \frac{z_2}{z_3}\right)^{(23)} R\left(\lambda - h^2, \frac{z_1}{z_3}\right)^{(13)} R\left(\lambda, \frac{z_1}{z_2}\right)^{(12)} \\ &\quad = R\left(\lambda - h^3, \frac{z_1}{z_2}\right)^{(12)} R\left(\lambda, \frac{z_1}{z_3}\right)^{(13)} R\left(\lambda - h^1, \frac{z_2}{z_3}\right)^{(23)}. \end{aligned} \tag{2.8}$$

Equation (2.8) is an equality in the algebra of meromorphic functions $\mathfrak{h}^* \times \mathbb{C}^\times \to \mathrm{End}(V^{\otimes 3})$. Upper indices are leg-numbering notation, and h indicates the action of $\mathfrak{h}$. For example,

$$R\left(\lambda - h^3, \frac{z_1}{z_2}\right)^{(12)} (u \otimes v \otimes w) = R\left(\lambda - \alpha, \frac{z_1}{z_2}\right)(u \otimes v) \otimes w, \quad \text{if } w \in V_\alpha. \tag{2.9}$$

An R-matrix R is called *unitary* if

$$R(\lambda, z) R\left(\lambda, z^{-1}\right)^{(21)} = \mathrm{Id}_{V \otimes V} \tag{2.10}$$

as meromorphic functions on $\mathfrak{h}^* \times \mathbb{C}^\times$ with values in $\mathrm{End}_{\mathfrak{h}}(V \otimes V)$.

In the example we study, $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}(n)$. Thus $\mathfrak{h}$ is the abelian Lie algebra of all traceless diagonal complex $n \times n$ matrices. Let V be the $\mathfrak{h}$-module $\mathbb{C}^n$ with standard basis $e_1, \ldots, e_n$. Define $\omega(i) \in \mathfrak{h}^*$ $(i = 1, \ldots, n)$ by

$$\omega(i)(h) = h_i, \quad \text{if } h = \mathrm{diag}(h_1, \ldots, h_n) \in \mathfrak{h}. \tag{2.11}$$

We have $V = \oplus_{i=1}^n V_{\omega(i)}$ and $V_{\omega(i)} = \mathbb{C} e_i$. Define

$$R : \mathfrak{h}^* \times \mathbb{C}^\times \longrightarrow \mathrm{End}(V \otimes V) \tag{2.12}$$

by

$$R(\lambda, z) = \sum_{i=1}^n E_{ii} \otimes E_{ii} + \sum_{i \neq j} \alpha\left(\lambda_{ij}, z\right) E_{ii} \otimes E_{jj} + \sum_{i \neq j} \beta\left(\lambda_{ij}, z\right) E_{ij} \otimes E_{ji}, \tag{2.13}$$

where $E_{ij} \in \mathrm{End}(V)$ are the matrix units, λ_{ij} $(\lambda \in \mathfrak{h}^*)$ is an abbrevation for $\lambda(E_{ii} - E_{jj})$, and

$$\alpha, \beta : \mathbb{C} \times \mathbb{C}^\times \longrightarrow \mathbb{C} \tag{2.14}$$

are given by

$$\alpha(\lambda, z) = \alpha(\lambda, z; p, q) = \frac{\theta(z)\theta(q^{2(\lambda+1)})}{\theta(q^2 z)\theta(q^{2\lambda})}, \tag{2.15}$$

$$\beta(\lambda, z) = \beta(\lambda, z; p, q) = \frac{\theta(q^2)\theta(q^{-2\lambda} z)}{\theta(q^2 z)\theta(q^{-2\lambda})}. \tag{2.16}$$

Proposition 2.1 (see [2]). *The map R is a unitary R-matrix.*

For the reader's convenience, we give the explicit relationship between the R-matrix (2.13) and Felders R-matrix as written in [5] which we denote by R_1. Thus $R_1 : \mathfrak{h}_1^* \times \mathbb{C} \to \mathrm{End}(V \otimes V)$, where $\mathfrak{h}_1$ is the Cartan subalgebra of $\mathfrak{gl}(n)$, is defined as in (2.13) with α, β replaced by $\alpha_1, \beta_1 : \mathbb{C}^2 \to \mathbb{C}$,

$$\begin{aligned} \alpha_1(\lambda, x) &= \alpha_1(\lambda, x; \tau, \gamma) = \frac{\theta_1(x;\tau)\theta_1(\lambda+\gamma;\tau)}{\theta_1(x-\gamma;\tau)\theta_1(\lambda;\tau)}, \\ \beta_1(\lambda, x) &= \beta_1(\lambda, x; \tau, \gamma) = -\frac{\theta_1(x+\lambda;\tau)\theta_1(\gamma;\tau)}{\theta_1(x-\gamma;\tau)\theta_1(\lambda;\tau)}. \end{aligned} \tag{2.17}$$

Here $\tau, \gamma \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$, and θ_1 is the first Jacobi theta function:

$$\theta_1(x;\tau) = -\sum_{j \in \mathbb{Z}+1/2} e^{\pi i j^2 \tau + 2\pi i j (x+1/2)}. \tag{2.18}$$

As proved in [2], R_1 satisfies the following version of the QDYBE:

$$\begin{aligned} &R_1\left(\lambda - \gamma h^3, x_1 - x_2\right)^{(12)} R_1(\lambda, x_1 - x_3)^{(13)} R_1\left(\lambda - \gamma h^1, x_2 - x_3\right)^{(23)} \\ &\quad = R_1(\lambda, x_2 - x_3)^{(23)} R_1(\lambda - \gamma h^2, x_1 - x_3)^{(13)} R_1(\lambda, x_1 - x_2)^{(12)} \end{aligned} \tag{2.19}$$

and the unitarity condition

$$R_1(\lambda, x) R_1^{21}(\lambda, -x) = \mathrm{Id}_{V \otimes V}. \tag{2.20}$$

We can identify $\mathfrak{h}^* \simeq \mathfrak{h}_1^* / \mathbb{C}\,\mathrm{tr}$ where $\mathrm{tr} \in \mathfrak{h}_1^*$ is the trace. Since R_1 has the form (2.13), it is constant, as a function of $\lambda \in \mathfrak{h}_1^*$, on the cosets modulo $\mathbb{C}\,\mathrm{tr}$. So R_1 induces a map $\mathfrak{h}^* \times \mathbb{C} \to \mathrm{End}(V \otimes V)$, which we also denote by R_1, still satisfying (2.19), (2.20).

Let $\tau, \gamma \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$ be such that $p = e^{\pi i \tau}$, $q = e^{\pi i \gamma}$. Then, as meromorphic functions of $(\lambda, x) \in \mathfrak{h}^* \times \mathbb{C}$,

$$R_1\left(\gamma\lambda, -x; \frac{\tau}{2}, \gamma\right) = R(\lambda, z; p, q), \tag{2.21}$$

where $z = e^{2\pi i x}$. Indeed, using the Jacobi triple product identity (2.5) we have

$$\theta_1\left(x; \frac{\tau}{2}\right) = i e^{\pi i(\tau/2 - x)} \theta(z) \prod_{j=1}^{\infty}\left(1 - p^j\right), \tag{2.22}$$

and substituting this into (2.17) gives $\alpha_1(\gamma\lambda, -x; \tau/2, \gamma) = \alpha(\lambda, z; p, q)$ and $\beta_1(\gamma\lambda, -x; \tau/2, \gamma) = \beta(\lambda, z; p, q)$ which proves (2.21).

By replacing λ, x_i in (2.19) by $\gamma\lambda$, $-x_i$ and using (2.21) we obtain (2.8) with $z_i = e^{2\pi i x_i}$. Similarly the unitarity (2.10) of R is obtained from (2.20).

2.3. Useful Identities

We end this section by recording some useful identities. Recall the definitions of α, β in (2.15). It is immediate that

$$\alpha\left(\lambda, q^2\right) = \beta\left(-\lambda, q^2\right). \tag{2.23}$$

By induction, one generalizes (2.2) to

$$\theta(p^s z) = (-1)^s \left(p^{s(s-1)/2} z^s\right)^{-1} \theta(z), \quad \text{for } s \in \mathbb{Z}. \tag{2.24}$$

Applying (2.24) to the definitions of α, β we get

$$\alpha\left(\lambda, p^k z\right) = q^{2k} \alpha(\lambda, z), \qquad \beta\left(\lambda, p^k z\right) = q^{2k(\lambda+1)} \beta(\lambda, z), \tag{2.25}$$

and, using also $\theta(z^{-1}) = -z^{-1}\theta(z)$,

$$\begin{aligned} \lim_{z \to p^{-k} q^{-2}} \frac{q^{-1}\theta(q^2 z)}{q\theta(q^{-2} z)} \alpha(\lambda, z) &= \alpha\left(\lambda, p^k q^2\right), \\ \lim_{z \to p^{-k} q^{-2}} \frac{q^{-1}\theta(q^2 z)}{q\theta(q^{-2} z)} \beta(\lambda, z) &= -\beta\left(-\lambda, p^k q^2\right), \end{aligned} \tag{2.26}$$

for $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^{\times}$, and $k \in \mathbb{Z}$. By the addition formula (2.3) with

$$(x, y, z, w) = \left(z^{1/2} q^{-\lambda+1}, z^{1/2} q^{\lambda-1}, z^{1/2} q^{\lambda+1}, z^{1/2} q^{-\lambda-1}\right), \tag{2.27}$$

we have

$$\alpha(\lambda, z)\alpha(-\lambda, z) - \beta(\lambda, z)\beta(-\lambda, z) = q^2 \frac{\theta(q^{-2} z)}{\theta(q^2 z)}. \tag{2.28}$$

3. $\mathfrak{h}$-Bialgebroids

3.1. Definitions

We recall some definitions from [8]. Let $\mathfrak{h}^*$ be a finite-dimensional complex vector space (e.g., the dual space of an abelian Lie algebra), and let $M_{\mathfrak{h}^*}$ be the field of meromorphic functions on $\mathfrak{h}^*$.

Definition 3.1. An $\mathfrak{h}$-*algebra* is a complex associative algebra A with 1 which is bigraded over $\mathfrak{h}^*$, $A = \oplus_{\alpha,\beta\in\mathfrak{h}^*} A_{\alpha\beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : M_{\mathfrak{h}^*} \to A$, called the left and right moment maps, such that

$$\mu_l(f)a = a\mu_l(T_\alpha f), \quad \mu_r(f)a = a\mu_r(T_\beta f), \quad \text{for } a \in A_{\alpha\beta},\ f \in M_{\mathfrak{h}^*}, \tag{3.1}$$

where T_α denotes the automorphism $(T_\alpha f)(\zeta) = f(\zeta+\alpha)$ of $M_{\mathfrak{h}^*}$. A morphism of $\mathfrak{h}$-algebras is an algebra homomorphism preserving the bigrading and the moment maps.

The *matrix tensor product* $A\widetilde{\otimes}B$ of two $\mathfrak{h}$-algebras A, B is the $\mathfrak{h}^*$-bigraded vector space with $(A\widetilde{\otimes}B)_{\alpha\beta} = \oplus_{\gamma\in\mathfrak{h}^*}(A_{\alpha\gamma}\otimes_{M_{\mathfrak{h}^*}} B_{\gamma\beta})$, where $\otimes_{M_{\mathfrak{h}^*}}$ denotes tensor product over $\mathbb{C}$ modulo the relations:

$$\mu_r^A(f)a \otimes b = a \otimes \mu_l^B(f)b, \quad \forall a \in A,\ b \in B,\ f \in M_{\mathfrak{h}^*}. \tag{3.2}$$

The multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$ for $a, c \in A$ and $b, d \in B$ and the moment maps $\mu_l(f) = \mu_l^A(f) \otimes 1$ and $\mu_r(f) = 1 \otimes \mu_r^B(f)$ make $A\widetilde{\otimes}B$ into an $\mathfrak{h}$-algebra.

Example 3.2. Let $D_{\mathfrak{h}}$ be the algebra of operators on $M_{\mathfrak{h}^*}$ of the form $\sum_i f_i T_{\alpha_i}$ with $f_i \in M_{\mathfrak{h}^*}$ and $\alpha_i \in \mathfrak{h}^*$. It is an $\mathfrak{h}$-algebra with bigrading $fT_{-\alpha} \in (D_{\mathfrak{h}})_{\alpha\alpha}$, and both moment maps equal to the natural embedding.

For any $\mathfrak{h}$-algebra A, there are canonical isomorphisms $A \simeq A\widetilde{\otimes}D_{\mathfrak{h}} \simeq D_{\mathfrak{h}}\widetilde{\otimes}A$ defined by

$$x \simeq x \otimes T_{-\beta} \simeq T_{-\alpha} \otimes x, \quad \text{for } x \in A_{\alpha\beta}. \tag{3.3}$$

Definition 3.3. An $\mathfrak{h}$-*bialgebroid* is an $\mathfrak{h}$-algebra A equipped with two $\mathfrak{h}$-algebra morphisms, the comultiplication $\Delta : A \to A\widetilde{\otimes}A$ and the counit $\varepsilon : A \to D_{\mathfrak{h}}$ such that $(\Delta\otimes\mathrm{Id})\circ\Delta = (\mathrm{Id}\otimes\Delta)\circ\Delta$ and $(\varepsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id} = (\mathrm{Id} \otimes \varepsilon) \circ \Delta$, under the identifications (3.3).

3.2. The Generalized FRST Construction

In [8] the authors gave a generalized FRST construction which attaches an $\mathfrak{h}$-bialgebroid to each solution of the quantum dynamical Yang-Baxter equation without spectral parameter. One way of extending to the case including a spectral parameter is described in [11]. However, when specifying the R-matrix to (2.13) with $n = 2$, they had to impose in addition certain so-called *residual relations* in order to prove, for example, that the determinant is central. Such relations were also required in [4] in a different algebraic setting. In the setting of operator algebras, where the algebras consist of linear operators on a vector space depending meromorphically on the spectral variables, as in [5], such relations are consequences of the ordinary RLL relations by taking residues.

Another motivation for our procedure is that $\mathfrak{h}$-bialgebroids associated to gauge equivalent R-matrices should be isomorphic. In particular one should be allowed to multiply the R-matrix by any nonzero meromorphic function of the spectral variable without changing the isomorphism class of the associated algebra (for the full definition of gauge equivalent R-matrices see [8]).

These considerations suggest the following procedure for constructing an $\mathfrak{h}$-bialgebroid from a quantum dynamical R-matrix with spectral parameter.

Let $\mathfrak{h}$ be a finite-dimensional abelian Lie algebra, $V = \oplus_{\alpha\in\mathfrak{h}^*} V_\alpha$ a finite-dimensional diagonalizable $\mathfrak{h}$-module, and $R : \mathfrak{h}^* \times \mathbb{C}^\times \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$ a meromorphic function. We attach to this data an $\mathfrak{h}$-bialgebroid A_R as follows. Let $\{e_x\}_{x\in X}$ be a homogeneous basis of V, where X is an index set. The matrix elements $R^{ab}_{xy} : \mathfrak{h}^* \times \mathbb{C}^\times \to \mathbb{C}$ of R are given by

$$R(\zeta, z)(e_a \otimes e_b) = \sum_{x,y\in X} R^{ab}_{xy}(\zeta, z) e_x \otimes e_y. \tag{3.4}$$

They are meromorphic on $\mathfrak{h}^* \times \mathbb{C}^\times$. Define $\omega : X \to \mathfrak{h}^*$ by $e_x \in V_{\omega(x)}$. Let $\widetilde{A}_R$ be the complex associative algebra with 1 generated by $\{L_{xy}(z) : x, y \in X, z \in \mathbb{C}^\times\}$ and two copies of $M_{\mathfrak{h}^*}$, whose elements are denoted by $f(\lambda)$ and $f(\rho)$, respectively, with defining relations $f(\lambda)g(\rho) = g(\rho)f(\lambda)$ for $f, g \in M_{\mathfrak{h}^*}$ and

$$f(\lambda)L_{xy}(z) = L_{xy}(z)f(\lambda + \omega(x)), \qquad f(\rho)L_{xy}(z) = L_{xy}(z)f(\rho + \omega(y)), \tag{3.5}$$

for all $x, y \in X$, $z \in \mathbb{C}^\times$ and $f \in M_{\mathfrak{h}^*}$. The bigrading on $\widetilde{A}_R$ is given by $L_{xy}(z) \in (\widetilde{A}_R)_{\omega(x),\omega(y)}$ for $x, y \in X$, $z \in \mathbb{C}^\times$ and $f(\lambda), f(\rho) \in (\widetilde{A}_R)_{00}$ for $f \in M_{\mathfrak{h}^*}$. The moment maps are defined by $\mu_l(f) = f(\lambda)$, $\mu_r(f) = f(\rho)$. The counit and comultiplication are defined by

$$\begin{aligned}
\varepsilon(L_{ab}(z)) &= \delta_{ab} T_{-\omega(a)}, \\
\varepsilon(f(\lambda)) &= \varepsilon(f(\rho)) = f T_0, \\
\Delta(L_{ab}(z)) &= \sum_{x\in X} L_{ax}(z) \otimes L_{xb}(z), \\
\Delta(f(\lambda)) &= f(\lambda) \otimes 1, \\
\Delta(f(\rho)) &= 1 \otimes f(\rho).
\end{aligned} \tag{3.6}$$

This makes $\widetilde{A}_R$ into an $\mathfrak{h}$-bialgebroid.

Consider the ideal in $\widetilde{A}_R$ generated by the RLL relations:

$$\sum_{x,y\in X} R^{xy}_{ac}\left(\lambda, \frac{z_1}{z_2}\right) L_{xb}(z_1) L_{yd}(z_2) = \sum_{x,y\in X} R^{bd}_{xy}\left(\rho, \frac{z_1}{z_2}\right) L_{cy}(z_2) L_{ax}(z_1), \tag{3.7}$$

where $a, b, c, d \in X$, and $z_1, z_2 \in \mathbb{C}^\times$. More precisely, to account for possible singularities of R, we let I_R be the ideal in $\widetilde{A}_R$ generated by all relations of the form

$$\sum_{x,y\in X} \lim_{w\to z_1/z_2} \Big(\varphi(w) R_{ac}^{xy}(\lambda, w)\Big) L_{xb}(z_1) L_{yd}(z_2) = \sum_{x,y\in X} \lim_{w\to z_1/z_2} \Big(\varphi(w) R_{xy}^{bd}(\rho, w)\Big) L_{cy}(z_2) L_{ax}(z_1), \tag{3.8}$$

where $a, b, c, d \in X$, $z_1, z_2 \in \mathbb{C}^\times$, and $\varphi : \mathbb{C}^\times \to \mathbb{C}$ is a meromorphic function such that the limits exist.

We define A_R to be $\widetilde{A}_R / I_R$. The bigrading descends to A_R because (3.8) is homogeneous, of bidegree $\omega(a) + \omega(c)$, $\omega(b) + \omega(d)$, by the $\mathfrak{h}$-invariance of R. One checks that $\Delta(I_R) \subseteq \widetilde{A}_R \widetilde{\otimes} I_R + I_R \widetilde{\otimes} \widetilde{A}_R$ and $\varepsilon(I_R) = 0$. Thus A_R is an $\mathfrak{h}$-bialgebroid with the induced maps.

Remark 3.4. Objects in Felder's tensor category associated to an R-matrix R are certain meromorphic functions $L : \mathfrak{h}^* \times \mathbb{C}^\times \to \mathrm{End}_{\mathfrak{h}}(\mathbb{C}^n \otimes W)$ where W is a finite-dimensional $\mathfrak{h}$-module [2]. After regularizing L with respect to the spectral parameter it will give rise to a dynamical representation of the $\mathfrak{h}$-bialgebroid A_R in the same way as in the nonspectral case treated in [8]. The residual relations incorporated in (3.8) are crucial for this fact to be true in the present, spectral, case.

3.3. Operator form of the RLL Relations

It is well known that the RLL relation (3.7) can be written as a matrix relation. We show how this is done in the present setting. It will be used later in Section 6.2.

Assume $R_{xy}^{ab}(\zeta, z)$ are defined, as meromorphic functions of $\zeta \in \mathfrak{h}^*$ for any $z \in \mathbb{C}^\times$. Define $\mathsf{R}(\lambda, z), \mathsf{R}(\rho, z) \in \mathrm{End}(V \otimes V \otimes A_R)$ by

$$\mathsf{R}(\lambda, z)(e_a \otimes e_b \otimes u) = \sum_{x,y\in X} e_x \otimes e_y \otimes R_{xy}^{ab}(\lambda, z) u,$$
$$\mathsf{R}(\rho, z)(e_a \otimes e_b \otimes u) = \sum_{x,y\in X} e_x \otimes e_y \otimes R_{xy}^{ab}(\rho, z) u, \tag{3.9}$$

for $a, b \in X$, $u \in A_R$. Note that the λ and ρ in the left-hand side are not variables but merely indicate which moment map is to be used. For $z \in \mathbb{C}^\times$ we also define $\mathsf{L}(z) \in \mathrm{End}(V \otimes A_R)$ by

$$\mathsf{L}(z) = \sum_{x,y\in X} E_{xy} \otimes L_{xy}(z). \tag{3.10}$$

Here E_{xy} are the matrix units in $\mathrm{End}(V)$, and A_R acts on itself by left multiplication. The RLL relation (3.7) is equivalent to

$$\mathsf{R}\left(\lambda, \frac{z_1}{z_2}\right) \mathsf{L}^1(z_1) \mathsf{L}^2(z_2) = \mathsf{L}^2(z_2) \mathsf{L}^1(z_1) \mathsf{R}\left(\rho + h^1 + h^2, \frac{z_1}{z_2}\right) \tag{3.11}$$

in $\operatorname{End}(V \otimes V \otimes A_R)$, where $\mathsf{L}^i(z) = \mathsf{L}(z)^{(i,3)} \in \operatorname{End}(V \otimes V \otimes A_R)$ for $i = 1,2$ and the operator $\mathsf{R}(\rho + h^1 + h^2, z_1/z_2) \in \operatorname{End}(V \otimes V \otimes A_R)$ is given by

$$e_a \otimes e_b \otimes u \longmapsto \sum_{x,y\in X} e_x \otimes e_y \otimes R^{ab}_{xy}\left(\rho + \omega(a) + \omega(b), \frac{z_1}{z_2}\right)u, \tag{3.12}$$

where $R^{ab}_{xy}(\rho + \omega(a) + \omega(b), z_1/z_2)$ means the image in A_R of the meromorphic function $\mathfrak{h}^* \ni \zeta \mapsto R^{ab}_{xy}(\zeta+\omega(a)+\omega(b), z_1/z_2)$ under the right moment map μ_r. This equivalence can be seen by acting on $e_b \otimes e_d \otimes 1$ in both sides of (3.11) and collecting and equating terms of the form $e_a \otimes e_c \otimes u$. The matrix elements of the R-matrix in the right-hand side can then be moved to the left using that R is $\mathfrak{h}$-invariant and using relation (3.5).

4. The Algebra $\mathcal{F}_{\text{ell}}(M(n))$

We now specialize to the case where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}(n)$, $V = \mathbb{C}^n$, and R is given by (2.13)–(2.16). The case $n = 2$ was considered in [11]. We will show that (3.8) contains precisely one additional family of relations, as compared to (3.7), and we write down all relations explicitly.

When we apply the generalized FRST construction to this data we obtain an $\mathfrak{h}$-bialgebroid which we denote by $\mathcal{F}_{\text{ell}}(M(n))$. The generators $L_{ij}(z)$ will be denoted by $e_{ij}(z)$. Thus $\mathcal{F}_{\text{ell}}(M(n))$ is the unital associative $\mathbb{C}$-algebra generated by $e_{ij}(z)$, $i,j \in [1,n]$, $z \in \mathbb{C}^\times$, and two copies of $M_{\mathfrak{h}^*}$, whose elements are denoted by $f(\lambda)$ and $f(\rho)$ for $f \in M_{\mathfrak{h}^*}$, subject to the following relations:

$$f(\lambda)e_{ij}(z) = e_{ij}(z)f(\lambda + \omega(i)), \qquad f(\rho)e_{ij}(z) = e_{ij}(z)f(\rho + \omega(j)), \tag{4.1}$$

for all $f \in M_{\mathfrak{h}^*}$, $i,j \in [1,n]$, and $z \in \mathbb{C}^\times$, and

$$\sum_{x,y=1}^{n} R^{xy}_{ac}\left(\lambda, \frac{z_1}{z_2}\right)e_{xb}(z_1)e_{yd}(z_2) = \sum_{x,y=1}^{n} R^{bd}_{xy}\left(\rho, \frac{z_1}{z_2}\right)e_{cy}(z_2)e_{ax}(z_1), \tag{4.2}$$

for all $a,b,c,d \in [1,n]$. More explicitly, from (2.13) we have

$$R^{ab}_{xy}(\zeta, z) = \begin{cases} 1, & a = b = x = y, \\ \alpha(\zeta_{xy}, z), & a \neq b,\ x = a,\ y = b, \\ \beta(\zeta_{xy}, z), & a \neq b,\ x = b,\ y = a, \\ 0, & \text{otherwise}, \end{cases} \tag{4.3}$$

which substituted into (4.2) yields four families of relations:

$$e_{ab}(z_1)e_{ab}(z_2) = e_{ab}(z_2)e_{ab}(z_1), \tag{4.4a}$$

$$e_{ab}(z_1)e_{ad}(z_2) = \alpha\left(\rho_{bd}, \frac{z_1}{z_2}\right)e_{ad}(z_2)e_{ab}(z_1) + \beta\left(\rho_{db}, \frac{z_1}{z_2}\right)e_{ab}(z_2)e_{ad}(z_1), \tag{4.4b}$$

$$\alpha\left(\lambda_{ac}, \frac{z_1}{z_2}\right)e_{ab}(z_1)e_{cb}(z_2) + \beta\left(\lambda_{ac}, \frac{z_1}{z_2}\right)e_{cb}(z_1)e_{ab}(z_2) = e_{cb}(z_2)e_{ab}(z_1), \tag{4.4c}$$

$$\begin{aligned} &\alpha\left(\lambda_{ac}, \frac{z_1}{z_2}\right)e_{ab}(z_1)e_{cd}(z_2) + \beta\left(\lambda_{ac}, \frac{z_1}{z_2}\right)e_{cb}(z_1)e_{ad}(z_2) \\ &\quad = \alpha\left(\rho_{bd}, \frac{z_1}{z_2}\right)e_{cd}(z_2)e_{ab}(z_1) + \beta\left(\rho_{db}, \frac{z_1}{z_2}\right)e_{cb}(z_2)e_{ad}(z_1), \end{aligned} \tag{4.4d}$$

where $a, b, c, d \in [1, n]$, $a \neq c$, and $b \neq d$. Since θ has zeros precisely at $p^k, k \in \mathbb{Z}$, α and β have poles at $z = q^{-2}p^k$, $k \in \mathbb{Z}$. Thus (4.4b)–(4.4d) are to hold for $z_1, z_2 \in \mathbb{C}^\times$ with $z_1/z_2 \notin \{p^k q^{-2} : k \in \mathbb{Z}\}$.

In (3.8), assuming $a \neq c$, $b \neq d$, taking $z_1 = z$, $z_2 = p^k q^2 z$, $\varphi(w) = q^{-1}\theta(q^2 w)/q\theta(q^{-2}w)$, and using the limit formulas (2.26), we obtain the relation

$$\begin{aligned} &\alpha\left(\lambda_{ac}, q^2\right)\left(e_{ab}(z)e_{cd}\left(p^k q^2 z\right) - q^{2k\lambda_{ca}}e_{cb}(z)e_{ad}\left(p^k q^2 z\right)\right) \\ &\quad = \alpha\left(\rho_{bd}, q^2\right)e_{cd}\left(p^k q^2 z\right)e_{ab}(z) - q^{2k\rho_{bd}}\beta\left(\rho_{bd}, q^2\right)e_{cb}\left(p^k q^2 z\right)e_{ad}(z). \end{aligned} \tag{4.5}$$

This identity does not follow from (4.4a)–(4.4d) in an obvious way. It will be called the *residual RLL relation*.

Proposition 4.1. *Relations* (4.4a)–(4.4d)*, and* (4.5) *generate the ideal* I_R*. Hence* (4.1)*,* (4.4a)–(4.4d)*, and* (4.5) *consitute the defining relations of the algebra* $\mathcal{F}_{\text{ell}}(M(n))$*.*

Proof. Assume that we have a relation of the form (3.8) and that a limit in one of the terms, $\lim_{w \to z}\varphi(w)R^{ab}_{xy}(\lambda, w)$, say, exists and is nonzero. Then one of the following cases occurs.

(1) At $w = z$, $\varphi(w)$ and $R^{ab}_{xy}(\lambda, w)$ are both regular. If this holds for all terms, then the relation is just a multiple of one of (4.4a)–(4.4d).

(2) At $w = z$, $\varphi(w)$ has a pole while $R^{ab}_{xy}(\lambda, w)$ is regular. Then $R^{ab}_{xy}(\lambda, w)$ must vanish identically at $w = z$. The only case where this is possible is when $x \neq y$ and $R^{ab}_{xy}(\lambda, w) = \alpha(\lambda_{xy}, w)$, and $z = p^k$. But then there is another term containing β which is never identically zero for any z, and hence the limit in that term does not exist.

(3) At $w = z$, $\varphi(w)$ is regular while $R^{ab}_{xy}(\lambda, w)$ has a pole. Since these poles are simple and occur only when $z \in q^{-2}p^{\mathbb{Z}}$, the function φ must have a zero of multiplicity one there. We can assume without loss of generality that φ has the specific form

$$\varphi(w) = \frac{q^{-1}\theta(q^2 w)}{q\theta(q^{-2}w)}. \tag{4.6}$$

Then, if $a \neq c$ and $b \neq d$, (3.8) becomes the residual RLL relation (4.5).

If instead $c = a$, $b \neq d$, and we take $z_1 = z$, $z_2 = p^k q^2 z$ in (3.8), we get, using (2.26),

$$0 = \alpha\left(\rho_{bd}, p^k q^2\right) e_{ad}\left(p^k q^2 z\right) e_{ab}(z) - \beta\left(\rho_{bd}, p^k q^2\right) e_{ab}\left(p^k q^2 z\right) e_{ad}(z), \tag{4.7}$$

or, rewritten,

$$e_{ad}\left(p^k q^2 z\right) e_{ab}(z) = q^{2k\rho_{bd}} \frac{E(\rho_{bd} - 1)}{E(\rho_{bd} + 1)} e_{ab}\left(p^k q^2 z\right) e_{ad}(z). \tag{4.8}$$

However this relation is already derivable from (4.4b) as follows. Take $z_1 = p^k q^2 z$ and $z_2 = z$ in (4.4b) multiply both sides by $q^{2k\rho_{bd}}(E(\rho_{bd} - 1)/E(\rho_{bd} + 1))$, and then use (4.4b) on the right-hand side.

Similarly, if $a \neq c$, $d = b$, $z_1 = z$, $z_2 = p^k q^2 z$, $\varphi(w) = q^{-1}\theta(q^2 w)/q\theta(q^{-2} w)$ in (3.8), and using (2.26) we get

$$\alpha\left(\lambda_{ac}, p^k q^2\right) e_{ab}(z) e_{cb}\left(p^k q^2 z\right) - \beta\left(\lambda_{ca}, p^k q^2\right) e_{cb}(z) e_{ab}\left(p^k q^2 z\right) = 0, \tag{4.9}$$

or

$$e_{ab}(z) e_{cb}\left(p^k q^2 z\right) = q^{2k\lambda_{ca}} e_{cb}(z) e_{ab}\left(p^k q^2 z\right). \tag{4.10}$$

Similarly to the previous case, this identity follows already from (4.4c). □

5. Left and Right Elliptic Exterior Algebras

5.1. Corepresentations of $\mathfrak{h}$-Bialgebroids

We recall the definition of corepresentations of an $\mathfrak{h}$-bialgebroid given in [13].

Definition 5.1. An $\mathfrak{h}$-space V is an $\mathfrak{h}^*$-graded vector space over $M_{\mathfrak{h}^*}$, $V = \oplus_{\alpha\in\mathfrak{h}^*} V_\alpha$, where each V_α is $M_{\mathfrak{h}^*}$-invariant. A morphism of $\mathfrak{h}$-spaces is a degree-preserving $M_{\mathfrak{h}^*}$-linear map.

Given an $\mathfrak{h}$-space V and an $\mathfrak{h}$-bialgebroid A, we define $A\widetilde{\otimes}V$ to be the $\mathfrak{h}^*$-graded space with $(A\widetilde{\otimes}V)_\alpha = \oplus_{\beta\in\mathfrak{h}^*}(A_{\alpha\beta} \otimes_{M_{\mathfrak{h}^*}} V_\beta)$, where $\otimes_{M_{\mathfrak{h}^*}}$ denotes $\otimes_{\mathbb{C}}$ modulo the relations

$$\mu_r(f) a \otimes v = a \otimes f v, \tag{5.1}$$

for $f \in M_{\mathfrak{h}^*}$, $a \in A$, $v \in V$. $A\widetilde{\otimes}V$ becomes an $\mathfrak{h}$-space with the $M_{\mathfrak{h}^*}$-action $f(a\otimes v) = \mu_l(f) a \otimes v$. Similarly we define $V\widetilde{\otimes}A$ as an $\mathfrak{h}$-space by $(V\widetilde{\otimes}A)_\beta = \oplus_\alpha V_\alpha \otimes_{M_{\mathfrak{h}^*}} A_{\alpha\beta}$, where $\otimes_{M_{\mathfrak{h}^*}}$ here means $\otimes_{\mathbb{C}}$ modulo the relation $v \otimes \mu_l(f) a = f v \otimes a$ and $M_{\mathfrak{h}^*}$-action given by $f(v \otimes a) = v \otimes \mu_r(f) a$.

For any $\mathfrak{h}$-space V we have isomorphisms $D_{\mathfrak{h}}\widetilde{\otimes}V \simeq V \simeq V\widetilde{\otimes}D_{\mathfrak{h}}$ given by

$$T_{-\alpha} \otimes v \simeq v \simeq v \otimes T_\alpha, \quad \text{for } v \in V_\alpha, \tag{5.2}$$

extended to $\mathfrak{h}$-space morphisms.

Definition 5.2. A left corepresentation V of an $\mathfrak{h}$-bialgebroid A is an $\mathfrak{h}$-space equipped with an $\mathfrak{h}$-space morphism $\Delta_V : V \to A\widetilde{\otimes}V$ such that $(\Delta_V\otimes 1)\circ\Delta_V = (1\otimes\Delta)\circ\Delta_V$ and $(\varepsilon\otimes 1)\circ\Delta_V = \mathrm{Id}_V$ (under the identification (5.2)).

Definition 5.3. A left $\mathfrak{h}$-comodule algebra V over an $\mathfrak{h}$-bialgebroid A is a left corepresentation $\Delta_V : V \to A\widetilde{\otimes}V$ and in addition a $\mathbb{C}$-algebra such that $V_\alpha V_\beta \subseteq V_{\alpha+\beta}$ and such that Δ_V is an algebra morphism, when $A\widetilde{\otimes}V$ is given the natural algebra structure.

Right corepresentations and comodule algebras are defined analogously.

5.2. The Comodule Algebras Λ and Λ'.

We define in this section an elliptic analog of the exterior algebra, following [12], where it was carried out in the trigonometric nonspectral case. It will lead to natural definitions of elliptic minors as certain elements of $\mathcal{F}_{\mathrm{ell}}(M(n))$. One difference between this approach and the one in [5] is that the elliptic exterior algebra in our setting is really an algebra and not just a vector space. Another one is that the commutation relations in our elliptic exterior algebras are completely determined by requiring the natural relations (5.3a), (5.3b), and (5.5) and that the coaction is an algebra homomorphism. This fact can be seen from the proof of Proposition 5.4. Since the proof does not depend on the particular form of α and β, we can obtain exterior algebras for any $\mathfrak{h}$-bialgebroid obtained through the generalized FRST construction from an R-matrix in the same manner. In particular the method is independent of the gauge equivalence class of R.

Let Λ be the unital associative $\mathbb{C}$-algebra generated by $v_i(z)$, $1 \le i \le n$, $z \in \mathbb{C}^\times$ and a copy of $M_{\mathfrak{h}^*}$ embedded as a subalgebra subject to the relations

$$f(\zeta)v_i(z) = v_i(z)f(\zeta+\omega(i)), \tag{5.3a}$$

$$v_i(z)v_i(w) = 0, \tag{5.3b}$$

$$\alpha\Big(\zeta_{kj}, \frac{z}{w}\Big)v_k(z)v_j(w) + \beta\Big(\zeta_{kj}, \frac{z}{w}\Big)v_j(z)v_k(w) = 0, \tag{5.3c}$$

for $i, j, k \in [1, n]$, $j \ne k$, $z, w \in \mathbb{C}^\times$, $z/w \notin \{p^k q^{-2} : k \in \mathbb{Z}\}$ and $f \in M_{\mathfrak{h}^*}$. We require also the residual relation of (5.3c) obtained by multiplying by $\varphi(z/w) = q^{-1}\theta(q^2 z/w)/q\theta(q^{-2}z/w)$ and letting $z/w \to p^{-k}q^{-2}$. After simplification using (2.26), we get

$$v_k(z)v_j\Big(p^k q^2 z\Big) = q^{2k\zeta_{jk}} v_j(z)v_k\Big(p^k q^2 z\Big). \tag{5.3d}$$

Λ becomes an $\mathfrak{h}$-space by

$$\mu_\Lambda(f)v = f(\zeta)v \tag{5.4}$$

and requiring $v_i(z) \in \Lambda_{\omega(i)}$ for each i, z.

Proposition 5.4. *Λ is a left comodule algebra over $\mathcal{F}_{\mathrm{ell}}(M(n))$ with left coaction $\Delta_\Lambda : \Lambda \to \mathcal{F}_{\mathrm{ell}}(M(n))\widetilde{\otimes}\Lambda$ satisfying*

$$\Delta_\Lambda(v_i(z)) = \sum_j e_{ij}(z) \otimes v_j(z), \tag{5.5}$$

$$\Delta_\Lambda\big(f(\zeta)\big) = f(\lambda) \otimes 1. \tag{5.6}$$

Proof. We have

$$\begin{aligned}\Delta_\Lambda(v_i(z))\Delta_\Lambda(v_i(w)) &= \sum_{jk} e_{ij}(z)e_{ik}(w) \otimes v_j(z)v_k(w)\\ &= \sum_{j\neq k}\Big(\alpha\Big(\mu_{jk}, \frac{z}{w}\Big)e_{ik}(w)e_{ij}(z) + \beta\Big(\mu_{kj}, \frac{z}{w}\Big)e_{ij}(w)e_{ik}(z)\Big) \otimes v_j(z)v_k(w)\\ &= \sum_{j\neq k} e_{ij}(w)e_{ik}(z) \otimes \Big(\alpha\Big(\zeta_{kj}, \frac{z}{w}\Big)v_k(z)v_j(w) + \beta\Big(\zeta_{kj}, \frac{z}{w}\Big)v_j(z)v_k(w)\Big)\\ &= 0.\end{aligned} \tag{5.7}$$

Similarly one proves that (5.3c), (5.3d) are preserved. □

Relation (5.3c) is not symmetric under interchange of j and k. We now derive a more explicit, independent, set of relations for Λ. We will use the function E, defined in (2.6).

Proposition 5.5. (i) *The following is a complete set of relations for Λ:*

$$f(\zeta)v_i(z) = v_i(z)f(\zeta + \omega(i)), \tag{5.8a}$$

$$v_k\Big(p^s q^2 z\Big)v_j(z) = -q^{2s\zeta_{kj}}\frac{E(\zeta_{kj}-1)}{E(\zeta_{kj}+1)}v_j\Big(p^s q^2 z\Big)v_k(z), \quad \forall s \in \mathbb{Z},\ k \neq j, \tag{5.8b}$$

$$v_k(z)v_j\Big(p^s q^2 z\Big) = q^{2s\zeta_{jk}}v_j(z)v_k\Big(p^s q^2 z\Big), \tag{5.8c}$$

$$v_k(z)v_j(w) = 0 \quad \text{if } \frac{z}{w} \notin \Big\{p^s q^{\pm 2} \mid s \in \mathbb{Z}\Big\} \text{ or if } k = j. \tag{5.8d}$$

(ii) *The set*

$$\Big\{v_{i_d}(z_d)\cdots v_{i_1}(z_1) : 1 \leq i_1 < \cdots < i_d \leq n,\ \frac{z_{i+1}}{z_i} \in p^{\mathbb{Z}}q^{\pm 2}\Big\} \tag{5.9}$$

is a basis for Λ over $M_{\mathfrak{h}^*}$.

Proof. (i) Elimination of the $v_j(z)v_k(w)$-term in (5.3c) yields

$$\Big(\alpha\Big(\zeta_{jk}, \frac{z}{w}\Big)\alpha\Big(\zeta_{kj}, \frac{z}{w}\Big) - \beta\Big(\zeta_{kj}, \frac{z}{w}\Big)\beta\Big(\zeta_{jk}, \frac{z}{w}\Big)\Big)v_k(z)v_j(w) = 0. \tag{5.10}$$

Combining (5.10), (2.28), and the fact that the $\theta(z)$ is zero precisely for $z \in \{p^k \mid k \in \mathbb{Z}\}$ we deduce that in Λ,

$$v_k(z)v_j(w) \neq 0 \Longrightarrow \frac{z}{w} = p^s q^2 \quad \text{for some } s \in \mathbb{Z}. \tag{5.11}$$

Using (2.25) we obtain from (5.11), (5.3b), and (5.3c) that relations (5.8b), (5.8d) hold in the left elliptic exterior algebra Λ. Relations (5.8a), (5.8c) are just repetitions of (5.3a), (5.3d).

(ii) It follows from the relations that each monomial in Λ can be uniquely written as $f(\zeta)v_{i_d}(z_d)\cdots v_{i_1}(z_1)$, where $1 \leq i_1 < \cdots < i_d \leq n$ and $f \in M_{\mathfrak{h}^*}$. It remains to show that the set (5.9) is linearly independent over $M_{\mathfrak{h}^*}$. Assume that a linear combination of basis elements is zero and that the sum has minimal number of terms. By multiplying from the right or left by $v_j(w)$ for appropriate j, w we can assume that the sum is of the form

$$f_1(\zeta)v_{i_d}\left(z_d^1\right)\cdots v_{i_1}\left(z_1^1\right) + \cdots + f_r(\zeta)v_{i_d}(z_d^r)\cdots \mathrm{v}_{i_1}(z_1^r) = 0, \tag{5.12}$$

for some fixed set $\{i_1,\ldots,i_d\}$. By the relations, a monomial $v_{i_d}(z_d)\cdots v_{i_1}(z_1)$ can be given the "degree" $\sum_{i=1}^d z_i t^{i-1} \in \mathbb{C}[t]$, where t is an indeterminate. Formally, consider $\mathbb{C}(t)\otimes\Lambda$, the tensor product (over $\mathbb{C}$) of Λ by the field of rational functions in t. We identify Λ with its image under $\Lambda \ni v \mapsto 1\otimes v \in \mathbb{C}(t)\otimes\Lambda$ and view $\mathbb{C}(t)\otimes\Lambda$ naturally as a vector space over $\mathbb{C}(t)$. By relations (5.8a)–(5.8d), there is a $\mathbb{C}$-algebra automorphism T of $\mathbb{C}(t)\otimes\Lambda$ satisfying $T(v_j(z)) = tv_j(z)$, $T(f(\zeta)) = f(\zeta)$, and $T(p\otimes 1) = p\otimes 1$. Define

$$D(v_i(z)) = zv_i(z), \qquad D(f(\zeta)) = 0, \qquad D(p\otimes 1) = 0, \tag{5.13}$$

for $f \in M_{\mathfrak{h}^*}$, $p \in \mathbb{C}(t)$ and $i \in [1,n]$, $z \in \mathbb{C}^\times$, and extend D to a $\mathbb{C}$-linear map $D : \mathbb{C}(t)\otimes\Lambda \to \mathbb{C}(t)\otimes\Lambda$ by requiring

$$D(ab) = D(a)T(b) + aD(b), \tag{5.14}$$

for $a, b \in \mathbb{C}(t)\otimes\Lambda$. The point is that the requirement (5.14) respects relations (5.8a)–(5.8d), making D well defined. Write $u_j = f_j(\zeta)v_{i_d}(z_d^j)\cdots v_{i_1}(z_1^j)$. Then one checks that $D(u_j) = p_j(t)u_j$, where $p_j(t) = \sum_{i=1}^d z_i^j t^{i-1}$. By applying D repeatedly we get

$$\begin{aligned} u_1\left(z^1\right) + \cdots + u_r(z^r) &= 0,\\ p_1(t)u_1\left(z^1\right) + \cdots + p_r(t)u_r(z^r) &= 0,\\ &\vdots\\ p_1(t)^{r-1}u_1\left(z^1\right) + \cdots + p_r(t)^{r-1}u_r(z^r) &= 0. \end{aligned} \tag{5.15}$$

Inverting the Vandermonde matrix $(p_j(t)^{i-1})_{ij}$ we obtain $u_j(z^j) = 0$ for each j, that is, $f_j(\zeta) = 0$ for each j. This proves linear independence of (5.9). □

Analogously one defines a right comodule algebra Λ' with generators $w^i(z)$ and $f(\zeta) \in M_{\mathfrak{h}^*}$. The following relations will be used:

$$w^k(z)w^j\left(p^s q^2 z\right) = -q^{2s\zeta_{kj}} w^j(z) w^k\left(p^s q^2 z\right), \quad \forall s \in \mathbb{Z},\ k \neq j,$$
$$w^k(z_1)w^j(z_2) = 0, \quad \text{if } \frac{z_2}{z_1} \notin \left\{p^s q^{\pm 2} \mid s \in \mathbb{Z}\right\}, \text{ or if } k = j. \tag{5.16}$$

Λ' has also $M_{\mathfrak{h}^*}$-basis of the form (5.9). In fact Λ and Λ' are isomorphic as algebras.

5.3. Action of the Symmetric Group

From (4.4a)–(4.4d), and (4.5) we see that $S_n \times S_n$ acts by $\mathbb{C}$-algebra automorphisms on $\mathcal{F}_{\text{ell}}(M(n))$ as follows:

$$(\sigma,\tau)\big(f(\lambda)\big) = f(\lambda \circ L_\sigma), \qquad (\sigma,\tau)\big(f(\mu)\big) = f(\mu \circ L_\tau),$$
$$(\sigma,\tau)\big(e_{ij}(z)\big) = e_{\sigma(i)\tau(j)}(z), \tag{5.17}$$

where $L_\sigma : \mathfrak{h} \to \mathfrak{h}$ $(\sigma \in S_n)$ is given by permutation of coordinates:

$$L_\sigma\big(\operatorname{diag}(h_1,\ldots,h_n)\big) = \operatorname{diag}\big(h_{\sigma(1)},\ldots,h_{\sigma(n)}\big). \tag{5.18}$$

Also, S_n acts on Λ by $\mathbb{C}$-algebra automorphisms via

$$\sigma\big(f(\zeta)\big) = f(\zeta \circ L_\sigma), \qquad \sigma(v_i(z)) = v_{\sigma(i)}(z). \tag{5.19}$$

Similarly we define an S_n action on Λ'.

Lemma 5.6. *For each $v \in \Lambda$, $w \in \Lambda'$, and any $\sigma, \tau \in S_n$ we have*

$$\Delta_\Lambda(\sigma(v)) = ((\sigma,\tau)\otimes\tau)(\Delta_\Lambda(v)), \tag{5.20}$$

$$\Delta_{\Lambda'}(\tau(w)) = (\sigma \otimes (\sigma,\tau))(\Delta_{\Lambda'}(w)). \tag{5.21}$$

Proof. By multiplicativity, it is enough to prove these claims on the generators, which is easy. □

6. Elliptic Quantum Minors

6.1. Definition

For $I \subseteq [1,n]$ we set

$$F_I(\zeta) = \prod_{i,j\in I, i<j} E\big(\zeta_{ij}+1\big), \qquad F^I(\zeta) = \prod_{i,j\in I, i<j} E\big(\zeta_{ij}\big), \tag{6.1}$$

and define the left and right elliptic sign functions:

$$\operatorname{sgn}_I(\sigma;\zeta) = \frac{\sigma(F_I(\zeta))}{F_{\sigma(I)}(\zeta)} = \prod_{i,j\in I, i<j, \sigma(i)>\sigma(j)} \frac{E\left(\zeta_{\sigma(i)\sigma(j)}+1\right)}{E\left(\zeta_{\sigma(j)\sigma(i)}+1\right)},$$
$$\operatorname{sgn}^I(\sigma;\zeta) = \frac{F^{\sigma(I)}(\zeta)}{\sigma(F^I(\zeta))} = \prod_{i,j\in I, i<j, \sigma(i)>\sigma(j)} \frac{E\left(\zeta_{\sigma(j)\sigma(i)}\right)}{E\left(\zeta_{\sigma(i)\sigma(j)}\right)}, \tag{6.2}$$

for $\sigma \in S_n$. In fact, $E(\zeta_{ij})/E(\zeta_{ji}) = -1$ so $\operatorname{sgn}^{[1,n]}(\sigma;\zeta)$ is just the usual sign $\operatorname{sgn}(\sigma)$. However we view this as a "coincidence" depending on the particular choice of R-matrix from its gauge equivalence class. We keep our notation to emphasize that the methods do not depend on this choice of R-matrix.

We will denote the elements of a subset $I \subseteq [1,n]$ by $i_1 < i_2 < \cdots$.

Proposition 6.1. *Let $I \subseteq [1,n]$, $d = \#I$, $\sigma \in S_n$, and $J = \sigma(I)$. Then for $z \in \mathbb{C}^\times$,*

$$v_{\sigma(i_d)}\left(q^{2(d-1)}z\right)\cdots v_{\sigma(i_1)}(z) = \operatorname{sgn}_I(\sigma;\zeta)v_{j_d}\left(q^{2(d-1)}z\right)\cdots v_{j_1}(z), \tag{6.3}$$

$$w^{\sigma(i_1)}(z)\cdots w^{\sigma(i_d)}\left(q^{2(d-1)}z\right) = \operatorname{sgn}^I(\sigma;\zeta)w^{j_1}(z)\cdots w^{j_d}\left(q^{2(d-1)}z\right). \tag{6.4}$$

Proof. We prove (6.3). The proof of (6.4) is analogous. We proceed by induction on $\#I = d$, the case $d = 1$ being clear. If $d > 1$, set $I' = \{i_1,\ldots,i_{d-1}\}$, $J' = \sigma(I')$. Let $1 \le j_1' < \cdots < j_{d-1}' \le n$ be the elements of J'. By the induction hypothesis, the left hand side of (6.3) equals

$$v_{\sigma(i_d)}\left(q^{2(d-1)}z\right)\operatorname{sgn}_{I'}(\sigma,\zeta)v_{j_{d-1}'}\left(q^{2(d-2)}z\right)\cdots v_{j_1'}(z). \tag{6.5}$$

Now $v_{\sigma(i_d)}(q^{2(d-1)}z)$ commutes with $\operatorname{sgn}_{I'}(\sigma,\zeta)$ since the latter only involves ζ_{ij} with $i,j \neq \sigma(i_d)$. Using the commutation relations (5.8b) we obtain

$$\operatorname{sgn}_{I'}(\sigma,\zeta)\cdot \prod_{j\in J', j>\sigma(i_d)} \frac{E\left(\zeta_{j\sigma(i_d)}+1\right)}{E\left(\zeta_{\sigma(i_d)j}+1\right)}\cdot v_{j_d}\left(q^{2(d-1)}z\right)\cdots v_{j_1}(z). \tag{6.6}$$

Replace $j \in J'$ such that $j > \sigma(i_d)$ by $\sigma(i)$, where $i \in I$, $i < i_d$, $\sigma(i) > \sigma(i_d)$. □

Introduce the normalized monomials

$$v_I(z) = F_I(\zeta)^{-1}v_{i_r}\left(q^{2(d-1)}z\right)v_{i_{r-1}}\left(q^{2(d-2)}z\right)\cdots v_{i_1}(z) \in \Lambda, \tag{6.7}$$

$$w^I(z) = F^I(\zeta)w^{i_1}(z)w^{i_2}\left(q^2z\right)\cdots w^{i_d}\left(q^{2(d-1)}z\right) \in \Lambda'. \tag{6.8}$$

Corollary 6.2. *Let $I \subseteq [1,n]$. For any permutation $\sigma \in S_n$,*

$$\sigma(v_I(z)) = v_{\sigma(I)}(z), \qquad \sigma\left(w^I(z)\right) = w^{\sigma(I)}(z) \tag{6.9}$$

for any $z \in \mathbb{C}^\times$. In particular $v_I(z)$ and $w^I(z)$ are fixed by any permutation which preserves the subset I.

Proof. Let $J = \sigma(I)$. Then

$$\begin{aligned}\sigma(v_I(z)) &= \sigma\left(F_I(\zeta)^{-1}\right)v_{\sigma(i_d)}\left(q^{2(d-1)}z\right)\cdots v_{\sigma(i_1)}(z)\\ &= \sigma(F_I(\zeta))^{-1}\operatorname{sgn}_I(\sigma;\zeta)v_{j_d}\left(q^{2(d-1)}z\right)\cdots v_{j_1}(z)\\ &= v_{\sigma(I)}(z).\end{aligned} \tag{6.10}$$

The proof for $w^I(z)$ is analogous. □

For any $I \subseteq [1,n]$, let S_I denote the group of all permutations of the set I. We are now ready to define certain elements of the $\mathfrak{h}$-bialgebroid $\mathcal{F}_{\mathrm{ell}}(M(n))$ which are analogs of minors.

Proposition 6.3. *For $I, J \subseteq [1,n]$ and $z \in \mathbb{C}^\times$, the left and right elliptic minors, $\overleftarrow{\xi}_I^J(z)$ and $\overrightarrow{\xi}_I^J(z)$, respectively, can be defined by*

$$\Delta_\Lambda(v_I(z)) = \sum_J \overleftarrow{\xi}_I^J(z) \otimes v_J(z), \tag{6.11}$$

$$\Delta_{\Lambda'}\left(w^J(z)\right) = \sum_I w^I(z) \otimes \overrightarrow{\xi}_I^J(z), \tag{6.12}$$

where the sums are taken over all subsets of $[1,n]$.

If $\#I \neq \#J$, then $\overleftarrow{\xi}_I^J(z) = 0 = \overrightarrow{\xi}_I^J(z)$, for all z. If $\#I = \#J = d$, they are explicitly given by

$$\overleftarrow{\xi}_I^J(z) = \frac{F_J(\rho)}{F_I(\lambda)} \sum_{\tau \in S_J} \frac{\operatorname{sgn}_J(\tau;\rho)}{\operatorname{sgn}_I(\sigma;\lambda)} e_{\sigma(i_d)\tau(j_d)}\left(q^{2(d-1)}z\right)e_{\sigma(i_{d-1})\tau(j_{d-1})}\left(q^{2(d-2)}z\right)\cdots e_{\sigma(i_1)\tau(j_1)}(z) \tag{6.13}$$

for any $\sigma \in S_I$, and

$$\overrightarrow{\xi}_I^J(z) = \frac{F^J(\rho)}{F^I(\lambda)} \sum_{\sigma \in S_I} \frac{\operatorname{sgn}^J(\tau;\rho)}{\operatorname{sgn}^I(\sigma;\lambda)} e_{\sigma(i_1)\tau(j_1)}(z)e_{\sigma(i_2)\tau(j_2)}\left(q^2 z\right)\cdots e_{\sigma(i_d)\tau(j_d)}\left(q^{2(d-1)}z\right) \tag{6.14}$$

for any $\tau \in S_J$. Moreover,

$$(\sigma,\tau)\left(\overleftarrow{\xi}_I^J(z)\right) = \overleftarrow{\xi}_{\sigma(I)}^{\tau(J)}(z), \qquad (\sigma,\tau)\left(\overrightarrow{\xi}_I^J(z)\right) = \overrightarrow{\xi}_{\sigma(I)}^{\tau(J)}(z) \tag{6.15}$$

for any $(\sigma,\tau) \in S_n \times S_n$ and $z \in \mathbb{C}^\times$.

Remark 6.4. In Theorem 6.10 we will prove that, in fact, $\overleftarrow{\xi}_I^J(z) = \overrightarrow{\xi}_I^J(z)$.

Proof. We prove the statements concerning the left elliptic minor $\overleftarrow{\xi}_I^J(z)$. We have

$$\begin{aligned}\Delta_\Lambda(v_I(z)) &= \sum_{1\le k_1,\dots,k_d\le n} F_I(\lambda)^{-1} e_{i_d k_d}\left(q^{2(d-1)}z\right)\cdots e_{i_1k_1}(z)\otimes v_{k_d}\left(q^{2(d-1)}z\right)\cdots v_{k_1}(z)\\ &= \sum_{J,\#J=d}\sum_{\tau\in S_J} F_I(\lambda)^{-1} e_{i_d\tau(j_d)}\left(q^{2(d-1)}z\right)\cdots e_{i_1\tau(j_1)}(z)\otimes v_{\tau(j_d)}\left(q^{2(d-1)}z\right)\cdots v_{\tau(j_1)}(z)\\ &= \sum_{J,\#J=d}\left(\sum_{\tau\in S_J}\frac{\tau(F_J(\rho))}{F_I(\lambda)} e_{i_d\tau(j_d)}\left(q^{2(d-1)}z\right)\cdots e_{i_1\tau(j_1)}(z)\right)\otimes v_J(z).\end{aligned} \tag{6.16}$$

Thus (6.11) holds when $\overleftarrow{\xi}_I^J(z)$ is defined by (6.13) with $\sigma = \mathrm{Id}$. Then the right hand side of (6.13) equals $(\sigma,\mathrm{Id})(\overleftarrow{\xi}_I^J(z))$ for any $\sigma\in S_I$. Thus only (6.15) remains. Using (5.20) and Corollary 6.2 we have

$$\Delta_\Lambda(\sigma(v_I(z))) = ((\sigma,\tau)\otimes\tau)(\Delta_\Lambda(v_I(z))) = \sum_J(\sigma,\tau)\left(\left(\overleftarrow{\xi}_I^J(z)\right)\otimes v_{\tau(J)}(z)\right). \tag{6.17}$$

On the other hand, again by Corollary 6.2,

$$\Delta_\Lambda(\sigma(v_I(z))) = \Delta_\Lambda\left(v_{\sigma(I)}(z)\right) = \sum_J\left(\overleftarrow{\xi}_{\sigma(I)}^{\tau(J)}(z)\otimes v_{\tau(J)}(z)\right), \tag{6.18}$$

where we made the substitution $J\mapsto\tau(J)$. This proves the first equality in (6.15). The statements concerning the right elliptic minors are proved analogously. □

6.2. Equality of Left and Right Minors

The goal of this section is to prove Theorem 6.10 stating that the left and right elliptic minors coincide. We use ideas from Section 3 of [5], where the authors study the objects of Felder's tensor category [2] and associate a linear operator (product of R-matrices) on $V^{\otimes n}$ to each diagram of a certain form, a kind of braid group representation. Then they consider the operator associated to the longest permutation, in [7] called the Cherednik operator. Instead of working with representations, we proceed inside the $\mathfrak{h}$-bialgebroid $\mathcal{F}_{\mathrm{ell}}(M(n))$ and consider certain operators on $V^{\otimes n}\otimes\mathcal{F}_{\mathrm{ell}}(M(n))$ depending on n spectral parameters. Using the analog of the Cherednik operator we prove an extended RLL relation (6.38). Theorem 6.10 then follows by extracting matrix elements from both sides of this matrix equation.

In this section, we set $\mathcal{F} = \mathcal{F}_{\mathrm{ell}}(M(n))$. Recall the operators from Section 3.3, defined for any $\mathfrak{h}$-bialgebroid A_R obtained from the FRST construction. When specializing to $\mathcal{F}$ we get operators $\mathsf{R}(\lambda, z)$, $\mathsf{R}(\rho, z) \in \operatorname{End}(V \otimes V \otimes \mathcal{F})$, where $V = \mathbb{C}^n$. For $z \in \mathbb{C}^\times$, define the following linear operators on $V^{\otimes n} \otimes \mathcal{F}$:

$$\mathsf{R}^{ij}(\lambda, z) := \lim_{w \to z} \theta\left(q^2 w\right) \mathsf{R}(\lambda, w)^{(i,j,n+1)}, \qquad \mathsf{R}^{ij}(\rho, z) := \lim_{w \to z} \theta\left(q^2 w\right) \mathsf{R}(\rho, w)^{(i,j,n+1)}. \tag{6.19}$$

The upper indices in parenthesis are tensor leg numbering and indicate the tensor factors the operator should act on. The limits are taken in the sense of taking limits of each matrix element. These operators are well defined for any z, since we multiply away the singularities in z of α and β (2.15), (2.16).

Let $\mathcal{E}_n$ denote the algebra of all functions

$$F : \left(\mathbb{C}^\times\right)^n \longrightarrow \operatorname{End}\left(V^{\otimes n} \otimes \mathcal{F}\right). \tag{6.20}$$

The symmetric group S_n acts on $\mathcal{E}_n$ by

$$\sigma(F(z)) = (\sigma \otimes \operatorname{Id}_{\mathcal{F}}) \circ F(\sigma(z)) \circ \left(\sigma^{-1} \otimes \operatorname{Id}_{\mathcal{F}}\right), \tag{6.21}$$

for $F \in \mathcal{E}_n$ and $\sigma \in S_n$. In the right hand side of (6.21), σ acts on $(\mathbb{C}^\times)^n$ by permuting coordinates, and on $V^{\otimes n}$ by permuting the tensor factors. For example, we have

$$(23)\left(\mathsf{R}^{12}\left(\lambda, \frac{z_1}{z_2}\right)\right) = \mathsf{R}^{13}\left(\lambda, \frac{z_1}{z_3}\right). \tag{6.22}$$

Consider the skew group algebra $\mathcal{E}_n * S_n$, defined as the algebra with underlying space $\mathcal{E}_n \otimes \mathbb{C}S_n$, where $\mathbb{C}S_n$ is the group algebra, with the multiplication

$$(F(z) \otimes \sigma)(G(z) \otimes \tau) = F(z)\sigma(G(z)) \otimes \sigma\tau, \tag{6.23}$$

for $\sigma, \tau \in S_n$, $F, G \in \mathcal{E}_n$. Since σ acts on $\mathcal{E}_n$ by automorphisms, $\mathcal{E}_n * S_n$ is an associative algebra. The constant function $z \mapsto \operatorname{Id}_{V^{\otimes n} \otimes \mathcal{F}} \otimes (1)$ is the unit element. Let B_n be the monoid (set with unital associative multiplication) generated by $\{s_1, \dots, s_{n-1}\}$ and relations

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \quad \text{for } 1 \le i \le n-2, \\ s_i s_j &= s_j s_i, \quad \text{if } |i-j| > 1. \end{aligned} \tag{6.24}$$

Let $\sigma_i = (i\,i+1) \in S_n$. We have an epimorphism $\pi : B_n \to S_n$ given by $\pi(s_i) = \sigma_i$, $\pi(1) = (1)$. Define

$$\begin{aligned} W(1) &= \operatorname{Id}_{V^{\otimes n} \otimes \mathcal{F}} \otimes (1), \\ W(s_i) &= \mathsf{R}^{i,i+1}\left(\lambda - h^{\geq i+2}, \frac{z_i}{z_{i+1}}\right) \otimes \sigma_i. \end{aligned} \tag{6.25}$$

Here and below we use $h^{\geq k}$ to denote the expression $\sum_{j=k}^{n} h^j$, and operators involving shifts h^i such as $\mathsf{R}^{n-2,n-1}(\lambda - h^n, z_i/z_{i+1})$ are defined as in Section 3.3.

Proposition 6.5. *W extends to a well-defined morphism of monoids, that is, a map*

$$W : B_n \longrightarrow \mathcal{E}_n * S_n \tag{6.26}$$

satisfying $W(b_1 b_2) = W(b_1)W(b_2)$ *for any* $b_1, b_2 \in B_n$.

Proof. We have to show the relations

$$W(s_i)W(s_{i+1})W(s_i) = W(s_{i+1})W(s_i)W(s_{i+1}), \tag{6.27}$$

$$W(s_i)W(s_j) = W(s_j)W(s_i) \quad \text{if } |i-j| > 1. \tag{6.28}$$

Relation (6.27) follows from the QDYBE (2.8). For example, $W(s_i)W(s_{i+1})W(s_i)$ equals

$$\mathsf{R}^{i,i+1}\left(\lambda - h^{\geq i+2}, \frac{z_i}{z_{i+1}}\right)\mathsf{R}^{i,i+2}\left(\lambda - h^{\geq i+3}, \frac{z_i}{z_{i+2}}\right)\mathsf{R}^{i+1,i+2}\left(\lambda - h^i - h^{\geq i+3}, \frac{z_{i+1}}{z_{i+2}}\right) \otimes \sigma_i\sigma_{i+1}\sigma_i. \tag{6.29}$$

Relation (6.28) is easy to check, using the $\mathfrak{h}$-invariance of R. □

For $b \in B_n$ we define $W_b(\lambda, z) \in \mathcal{E}_n$ by

$$W(b) = W_b(\lambda, z) \otimes \pi(b). \tag{6.30}$$

From this and the product rule (6.23) it follows that

$$W_{b_1 b_2}(\lambda, z) = W_{b_1}(\lambda, z) \cdot \pi(b_1)(W_{b_2}(\lambda, z)), \tag{6.31}$$

for $b_1, b_2 \in B_n$. By replacing λ by ρ we get similarly operators $W_b(\rho, z)$.

Recall the operators $\mathsf{L}(z) \in \mathrm{End}(V \otimes \mathcal{F})$ from Section 3.3. Define for $z \in \mathbb{C}^\times$, $i \in [1, n]$,

$$\mathsf{L}^i(z) = \mathsf{L}(z)^{(i,n+1)} \in \mathrm{End}\left(V^{\otimes n} \otimes \mathcal{F}\right). \tag{6.32}$$

If i, j, k are distinct, then one can check that

$$\mathsf{R}^{ij}\left(\lambda - h^k, z\right)\mathsf{L}^k(w) = \mathsf{L}^k(w)\mathsf{R}^{ij}(\lambda, z), \tag{6.33}$$

$$\mathsf{R}^{ij}(\rho, z)\mathsf{L}^k(w) = \mathsf{L}^k(w)\mathsf{R}^{ij}\left(\rho + h^k, z\right). \tag{6.34}$$

Due to the RLL relations (3.8) we have

$$\mathsf{R}^{12}\left(\lambda, \frac{z_1}{z_2}\right)\mathsf{L}^1(z_1)\mathsf{L}^2(z_2) = \mathsf{L}^2(z_2)\mathsf{L}^1(z_1)\mathsf{R}^{12}\left(\rho + h^1 + h^2, \frac{z_1}{z_2}\right) \tag{6.35}$$

for any $z_1, z_2 \in \mathbb{C}^\times$.

Define $t_d \in B_n$, $d \in [1,n]$, recursively by

$$t_d = \begin{cases} t_{d-1}s_{d-1}s_{d-2}\cdots s_1, & d > 1 \\ 1, & d = 1. \end{cases} \tag{6.36}$$

Let τ_d be the image of t_d in S_n:

$$\tau_d := \pi(t_d) = \begin{pmatrix} 1 & 2 & \cdots & d & d+1 & \cdots & n \\ d & d-1 & \cdots & 1 & d+1 & \cdots & n \end{pmatrix} \in S_n. \tag{6.37}$$

Proposition 6.6. *Let $1 \le d \le n$. For any $z = (z_1, \dots, z_d) \in (\mathbb{C}^\times)^d$ we have*

$$W_{t_d}(\lambda, z)\mathsf{L}^1(z_1)\cdots \mathsf{L}^d(z_d) = \mathsf{L}^d(z_d)\cdots \mathsf{L}^1(z_1)W_{t_d}\left(\rho + h^{\le d}, z\right). \tag{6.38}$$

Proof. We use induction on d. The case $d = 1$ is trivial, while $d = 2$ is the RLL relation (6.35). If $d > 2$, write $t_d = t_{d-1}u_d$, where $u_d = s_{d-1}s_{d-2}\cdots s_1$. Thus, by (6.31),

$$W_{t_d}(\lambda, z) = W_{t_{d-1}}(\lambda, z) \cdot \tau_{d-1}(W_{u_d}(\lambda, z)). \tag{6.39}$$

We claim that

$$\tau_{d-1}(W_{u_d}(\lambda, z))L^1(z_1)\cdots L^d(z_d) = L^d(z_d)L^1(z_1)\cdots L^{d-1}(z_{d-1})\tau_{d-1}\left(W_{u_d}\left(\rho + h^{\le d}, z\right)\right). \tag{6.40}$$

For notational simplicity, set $\lambda' = \lambda - h^{>d}$. A calculation using (6.30) shows that, compare the proof of Proposition 6.5,

$$W_{u_d}(\lambda, z) = \mathsf{R}^{d-1,d}\left(\lambda', \frac{z_{d-1}}{z_d}\right)\mathsf{R}^{d-2,d}\left(\lambda' - h^{d-1}, \frac{z_{d-2}}{z_d}\right)\cdots \mathsf{R}^{1,d}\left(\lambda' - h^{[2,d-1]}, \frac{z_1}{z_d}\right), \tag{6.41}$$

where $h^{[a,b]}$ means $\sum_{a \le j \le b} h^j$. Thus

$$\tau_{d-1}(W_{u_d}(\lambda, z)) = \mathsf{R}^{1,d}\left(\lambda', \frac{z_1}{z_d}\right)\mathsf{R}^{2,d}\left(\lambda' - h^1, \frac{z_2}{z_d}\right)\cdots \mathsf{R}^{d-1,d}\left(\lambda' - h^{\le d-2}, \frac{z_{d-1}}{z_d}\right). \tag{6.42}$$

Using (6.33) and the RLL relation (6.35) repeatedly, we obtain (6.40). Now the proposition follows by induction on d, using that

$$W_{t_{d-1}}(\lambda, z)L^d(z_d) = L^d(z_d)W_{t_{d-1}}\left(\lambda + h^d, z\right) \tag{6.43}$$

which follows from (6.33). □

The operator $\mathsf{C}(\lambda, z) := W_{t_n}(\lambda, z)$ is called the *Cherednik operator*. For an operator $F(z) \in \mathcal{E}_n$ we define its matrix elements $F(z)^{a_1,\dots,a_n}_{x_1,\dots,x_n} \in \mathcal{F}$ by

$$F(z)(e_{a_1} \otimes \cdots \otimes e_{a_n} \otimes 1) = \sum_{x_1,\dots,x_n} e_{x_1} \otimes \cdots \otimes e_{x_n} \otimes F(z)^{a_1,\dots,a_n}_{x_1,\dots,x_n}. \tag{6.44}$$

Proposition 6.7. *Let*

$$\tilde{\alpha}(\lambda, z) = \lim_{w \to z} \theta\left(q^2 w\right) \alpha(\lambda, w) = \frac{\theta(z)\theta\left(q^{2(\lambda+1)}\right)}{\theta(q^{2\lambda})}. \tag{6.45}$$

Then

$$\mathsf{C}(\lambda, z)^{1,\dots,n}_{1,\dots,n} = \prod_{i<j} \tilde{\alpha}\left(\lambda_{ij}, \frac{z_i}{z_j}\right) = \prod_{i<j} q\theta\left(\frac{z_i}{z_j}\right) \cdot \frac{F_{[1,n]}(\lambda)}{F^{[1,n]}(\lambda)}. \tag{6.46}$$

Proof. The second equality follows from the definition (6.1) of F_I and F^I. We prove by induction on d that $W_{t_d}(\lambda, z)^{1,\dots,n}_{1,\dots,n} = \prod_{i<j\le d} \tilde{\alpha}(\lambda_{ij}, z_i/z_j)$. For $d = 2$ we have $t_d = s_1$ and $W_{s_1}(\lambda, z)^{1,\dots,n}_{1,\dots,n} = \mathsf{R}^{12}(\lambda - h^{>2}, z_1/z_2)^{1,\dots,n}_{1,\dots,n} = \tilde{\alpha}(\lambda_{12}, z_1/z_2)$ as claimed. For $d > 2$, using factorization (6.39) we have

$$W_{t_d}(\lambda, z)^{1,\dots,n}_{1,\dots,n} = \sum_{x_1,\dots,x_n} W_{t_{d-1}}(\lambda, z)^{x_1,\dots,x_n}_{1,\dots,n} \tau_{d-1}(W_{u_d}(\lambda, z))^{1,\dots,n}_{x_1,\dots,x_n}. \tag{6.47}$$

Since $W_{t_{d-1}}(\lambda, z)$ is a product of operators of the form $\sigma(\mathsf{R}^{ii+1}(\lambda, z_i/z_{i+1}))$ where $1 \le i \le d-2$ and $\sigma \in S_n, \sigma(j) = j,\ j > d-1$, and each of these operators preserve the subspace spanned by $e_{\tau(1)} \otimes \cdots \otimes e_{\tau(d-1)} \otimes e_d \otimes \cdots \otimes e_n \otimes a$, where $\tau \in S_{d-1}$ and $a \in \mathcal{F}$; the operator $W_{t_{d-1}}(\lambda, z)$ also preserves this subspace. This means that $W_{t_{d-1}}(\lambda, z)^{x_1,\dots,x_n}_{1,\dots,n} = 0$ unless $x_j = j$ for $j \ge d$ and $\{x_1, \dots, x_{d-1}\} = \{1, \dots, d-1\}$. Furthermore, by (6.42),

$$\begin{aligned} &\tau_{d-1}(W_{u_d}(\lambda, z))^{1,\dots,n}_{x_1,\dots,x_{d-1},d,\dots,n} \\ &\quad = \sum_{y_2,\dots,y_{d-1}} \tilde{R}^{1y_2}_{x_1 d}\left(\lambda, \frac{z_1}{z_d}\right) \tilde{R}^{2y_3}_{x_2 y_2}\left(\lambda - \omega(1), \frac{z_2}{z_d}\right) \cdots \tilde{R}^{d-1,d}_{x_{d-1} y_{d-1}}\left(\lambda - \sum_{k \le d-2} \omega(k), \frac{z_{d-1}}{z_d}\right). \end{aligned} \tag{6.48}$$

Here $\tilde{R}^{ab}_{xy}(\lambda, z) = \lim_{w \to z} \theta(q^2 w) R^{ab}_{xy}(\lambda, w)$. Since $\tilde{R}^{ab}_{xy}(\lambda, z) = 0$ unless $\{x, y\} = \{a, b\}$, we deduce that, when $\{x_1, \dots, x_{d-1}\} = \{1, \dots, d-1\}$, the terms in the sum (6.48) are zero unless $x_i = i$ for all i and $y_j = d$ for all j. Substituting into (6.47) the claim follows by induction. □

Lemma 6.8. *Fix $2 \le d \le n$ and $i < d$. Then there are elements $b, c \in B_n$ such that $t_d = s_i b$ and $t_d = c s_i$.*

Proof. Since $t_2 = s_1$ and $t_3 = s_1 s_2 s_1 = s_2 s_1 s_2$, the statement clearly holds for $d = 2, 3$. Assuming $d > 3$, we first prove the existence of b. If $i < d-1$ then by induction there is a $b' \in B_n$ such that $t_{d-1} = s_i b'$. Hence $t_d = t_{d-1} s_{d-1} \cdots s_1 = s_i b' s_{d-1} \cdots s_1$. Thus we can take $b = b' s_{d-1} \cdots s_1$.

If $i = d-1$, write $t_d = t_{d-2}s_{d-2}\cdots s_1 s_{d-1}\cdots s_1$. Then move each of the $d-1$ rightmost factors $s_{d-1},\ldots,s_1$ as far to the left as possible, using that $s_j s_k = s_k s_j$ when $|j-k|>1$. This gives

$$t_d = t_{d-2}s_{d-2}s_{d-1}s_{d-3}s_{d-2}s_{d-4}\cdots s_2 s_3 s_1 s_2 s_1. \tag{6.49}$$

Then use $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$ repeatedly, working from right to left, to obtain

$$t_d = t_{d-2}s_{d-1}s_{d-2}s_{d-1}s_{d-3}s_{d-2}\cdots s_4 s_2 s_3 s_1 s_2. \tag{6.50}$$

Finally, s_{d-1} can be moved to the left of t_{d-2} since the latter is a product of s_j's with $j \le d-3$.

To prove the existence of c we note that B_n carries an involution $* : B_n \to B_n$ satisfying $(a_1a_2)^* = a_2^* a_1^*$ for any $a_1, a_2 \in B_n$, defined by $s_j^* = s_j$ for $j \in [1,n]$ and $1^* = 1$. Thus it suffices to show that $t_d^* = t_d$ for any d. This is trivial for $d = 2,3$. When $d > 3$ we have, by induction on d,

$$\begin{aligned} t_d^* &= (t_{d-1}s_{d-1}\cdots s_1)^* \\ &= s_1\cdots s_{d-1}t_{d-1} \\ &= s_1\cdots s_{d-1}t_{d-2}s_{d-2}\cdots s_1 \\ &= s_1\cdots s_{d-2}t_{d-2}s_{d-1}s_{d-2}\cdots s_1 \\ &= t_{d-1}^* s_{d-1}\cdots s_1 \\ &= t_d \quad (\text{since } s_{d-1} \text{ commutes with } t_{d-2}). \end{aligned} \tag{6.51}$$

□

Proposition 6.9. *Let $w = (z_0, q^2 z_0, \ldots, q^{2(n-1)}z_0)$, where $z_0 \neq 0$ is arbitrary, and let $\sigma, \tau \in S_n$. Then*

$$\mathsf{C}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma(1),\ldots,\sigma(n)} = \frac{\operatorname{sgn}_{[1,n]}(\sigma;\lambda)}{\operatorname{sgn}^{[1,n]}(\tau;\lambda)}\mathsf{C}(\lambda, w)^{1,\ldots,n}_{1,\ldots,n}. \tag{6.52}$$

Proof. First we claim that for all $\sigma, \tau \in S_n$ and each $i \in [1,n]$,

$$W_{s_i}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma\sigma_i(1),\ldots,\sigma\sigma_i(n)} = \sigma\Big(\operatorname{sgn}_{[1,n]}(\sigma_i;\lambda)\Big)W_{s_i}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma(1),\ldots,\sigma(n)}, \tag{6.53}$$

$$\begin{aligned} W_{s_i}(\lambda, w)^{\tau\sigma_i(1),\ldots,\tau\sigma_i(n)}_{\sigma(1),\ldots,\sigma(n)} &= \tau\Big(\operatorname{sgn}^{[1,n]}(\sigma_i;\lambda)\Big)W_{s_i}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma(1),\ldots,\sigma(n)} \\ &= -W_{s_i}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma(1),\ldots,\sigma(n)}. \end{aligned} \tag{6.54}$$

Indeed, assume that $z_i/z_{i+1} = q^{-2}$ and that $\{a_1,\ldots,a_n\} = \{b_1,\ldots,b_n\} = [1,n]$. Then $W_{s_i}(\lambda, z)^{b_1,\ldots,b_n}_{a_1,\ldots,a_n} \neq 0$ if and only if $\{a_i, a_{i+1}\} = \{b_i, b_{i+1}\}$ in which case

$$W_{s_i}(\lambda, z)^{b_1,\ldots,b_n}_{a_1,\ldots,a_n} = \frac{E(1)E(\lambda_{a_i a_{i+1}}+1)}{E(\lambda_{b_{i+1}b_i})}. \tag{6.55}$$

From this and the definitions of the sign functions, (6.2), the claims follow. Next, we prove (6.52) by induction on the sum ℓ of the lengths of σ and τ. If $\ell = 0$, it is trivial. Assuming (6.52) holds for (σ, τ) we prove it holds for $(\sigma\sigma_i, \tau)$ and $(\sigma, \tau\sigma_i)$ where i is arbitrary.

Let $i \in [1, n]$. By Lemma 6.8 we have $t_n = s_i b$ for some $b \in B_n$. We have

$$\begin{aligned} W_{t_n}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma\sigma_i(1),\ldots,\sigma\sigma_i(n)} &= \left(W_{s_i}(\lambda, w)\sigma_i(W_b(\lambda, w))\right)^{\tau(1),\ldots,\tau(n)}_{\sigma\sigma_i(1),\ldots,\sigma\sigma_i(n)} \\ &= \sum_{x_1,\ldots,x_n} W_{s_i}(\lambda, w)^{x_1,\ldots,x_n}_{\sigma\sigma_i(1),\ldots,\sigma\sigma_i(n)} \sigma_i(W_b(\lambda, w))^{\tau(1),\ldots,\tau(n)}_{x_1,\ldots,x_n}. \end{aligned} \tag{6.56}$$

As in the proof of Proposition 6.7, $W_{s_i}(\lambda, w)^{x_1,\ldots,x_n}_{\sigma\sigma_i(1),\ldots,\sigma\sigma_i(n)}$ is zero, if $x_1, \ldots, x_n$ is not a permutation of $1, \ldots, n$. Using (6.53) we obtain

$$\begin{aligned} &\sigma\Big(\operatorname{sgn}_{[1,n]}(\sigma_i; \lambda)\Big) \sum_{x_1,\ldots,x_n} W_{s_i}(\lambda, w)^{x_1,\ldots,x_n}_{\sigma(1),\ldots,\sigma(n)} \sigma_i(W_b(\lambda, w))^{\tau(1),\ldots,\tau(n)}_{x_1,\ldots,x_n} \\ &\quad = \sigma\Big(\operatorname{sgn}_{[1,n]}(\sigma_i; \lambda)\Big) W_{t_n}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma(1),\ldots,\sigma(n)}. \end{aligned} \tag{6.57}$$

Using the induction hypothesis and the relation $\operatorname{sgn}_{[1,n]}(\sigma; \lambda)\sigma(\operatorname{sgn}_{[1,n]}(\sigma_i; \lambda)) = \operatorname{sgn}_{[1,n]}(\sigma\sigma_i; \lambda)$ we obtain (6.52) for $(\sigma\sigma_i, \tau)$.

For the other case, let i be arbitrary, and set $j = \tau_n(i)$. By Lemma 6.8 there is a $c \in B_n$ such that $t_n = cs_j$. Recall the surjective morphism $\pi : B_n \to S_n$ sending s_i to $\sigma_i = (ii + 1)$. Then $\sigma_j \pi(c) = \pi(c)\sigma_i$. We have

$$\begin{aligned} W_{t_n}(\lambda, w)^{\tau\sigma_i(1),\ldots,\tau\sigma_i(n)}_{\sigma(1),\ldots,\sigma(n)} &= \Big(W_c(\lambda, w) \cdot \pi(c)\Big(W_{s_j}(\lambda, w)\Big)\Big)^{\tau\sigma_i(1),\ldots\tau\sigma_i(n)}_{\sigma(1),\ldots,\sigma(n)} \\ &= \sum_{x_1,\ldots,x_n} W_c(\lambda, w)^{x_1,\ldots,x_n}_{\sigma(1),\ldots,\sigma(n)} \pi(c)\Big(W_{s_j}(\lambda, w)\Big)^{\tau\sigma_i(1),\ldots,\tau\sigma_i(n)}_{x_1,\ldots,x_n}. \end{aligned} \tag{6.58}$$

It is easy to check that $\sigma(F(z))^{b_1,\ldots,b_n}_{a_1,\ldots,a_n} = F(\sigma(z))^{b_{\sigma(1)},\ldots,b_{\sigma(n)}}_{a_{\sigma(1)},\ldots,a_{\sigma(n)}}$ for any $F(z) \in \mathcal{E}_n$ and $\sigma \in S_n$. Define w_i by $(w_1, \ldots, w_n) = w = (z_0, q^2 z_0, \ldots, q^{2(n-1)} z_0)$. Then $w_i / w_{i+1} = q^{-2}$ for each i. Set $w' = (w_{\pi(c)(1)}, \ldots, w_{\pi(c)(n)})$. For each i, $w_{\pi(c)(i)} / w_{\pi(c)(i+1)} = w_{\tau_n(i+1)} / w_{\tau_n(i)} = q^{-2}$ also. Therefore

$$\begin{aligned} W_{t_n}(\lambda, w)^{\tau\sigma_i(1),\ldots,\tau\sigma_i(n)}_{\sigma(1),\ldots,\sigma(n)} &= \sum_{x_1,\ldots,x_n} W_c(\lambda, w)^{x_1,\ldots,x_n}_{\sigma(1),\ldots,\sigma(n)} W_{s_j}(\lambda, w')^{\tau\sigma_i\pi(c)(1),\ldots,\tau\sigma_i\pi(c)(n)}_{x_{\pi(c)(1)},\ldots,x_{\pi(c)(n)}} \\ &= \sum_{x_1,\ldots,x_n} W_c(\lambda, w)^{x_1,\ldots,x_n}_{\sigma(1),\ldots,\sigma(n)} W_{s_j}(\lambda, w')^{\tau\pi(c)\sigma_j(1),\ldots,\tau\pi(c)\sigma_j(n)}_{x_{\pi(c)(1)},\ldots,x_{\pi(c)(n)}} \\ &= \sum_{x_1,\ldots,x_n} W_c(\lambda, w)^{x_1,\ldots,x_n}_{\sigma(1),\ldots,\sigma(n)} (\operatorname{sgn}\sigma_j) W_{s_j}(\lambda, w')^{\tau\pi(c)(1),\ldots,\tau\pi(c)(n)}_{x_{\pi(c)(1)},\ldots,x_{\pi(c)(n)}} \\ &= \sum_{x_1,\ldots,x_n} W_c(\lambda, w)^{x_1,\ldots,x_n}_{\sigma(1),\ldots,\sigma(n)} (-1)\pi(c)\Big(W_{s_j}(\lambda, w)\Big)^{\tau(1),\ldots,\tau(n)}_{x_1,\ldots,x_n} \\ &= -W_{t_n}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma(1),\ldots,\sigma(n)}. \end{aligned} \tag{6.59}$$

By the induction hypothesis it follows that (6.52) holds for $(\sigma, \tau\sigma_i)$. This proves the formula (6.52). □

Theorem 6.10. *For any subsets $I, J \subseteq [1, n]$ and $z \in \mathbb{C}^\times$, the left and right elliptic minors coincide:*

$$\overleftarrow{\xi}_I^J(z) = \overrightarrow{\xi}_I^J(z). \tag{6.60}$$

We denote this common element by $\xi_I^J(z)$.

Proof. If $\#I \neq \#J$ then both sides are zero. Suppose $\#I = \#J = d$. By relation (6.15) we can, after applying a suitable automorphism, assume that $I = J = [1, d]$. Since the subalgebra of $\mathcal{F}$ generated by $e_{ij}(z)$, $i, j \in [1, d]$, $z \in \mathbb{C}^\times$ and $f(\lambda)$, $f(\rho)$ with $f \in M_{\mathfrak{h}_d^*} \subseteq M_{\mathfrak{h}^*}$, $\mathfrak{h}_d$ being the Cartan subalgebra of $\mathfrak{sl}(d)$, is isomorphic to $\mathcal{F}_{\text{ell}}(M(d))$, we can also assume $d = n$. Identifying the matrix element ${}^{1,\dots,n}_{1,\dots,n}$ on both sides of (6.38) we get

$$\sum_{x_1,\dots,x_n} \mathsf{C}(\lambda, z)^{x_1,\dots,x_n}_{1,\dots,n} e_{x_1,1}(z_1)\cdots e_{x_n,n}(z_n) = \sum_{x_1,\dots,x_n} e_{n,x_n}(z_n)\cdots e_{1,x_1}(z_1)\mathsf{C}\left(\rho + h^{\leq n}, z\right)^{1,\dots,n}_{x_1,\dots,x_n}. \tag{6.61}$$

As in the proof of Proposition 6.7, $\mathsf{C}(\lambda, z)^{x_1,\dots,x_n}_{1,\dots,n}$ is zero if $x_1, \dots, x_n$ is not a permutation of $1, \dots, n$. Taking $z = w = (z_0, q^2 z_0, \dots, q^{2(n-1)} z_0)$ and dividing both sides by $\prod_{i<j} q\theta(w_i/w_j) = \prod_{i<j} q\theta(q^{2(i-j)})$ we get

$$\begin{aligned} &\frac{F_{[1,n]}(\lambda)}{F^{[1,n]}(\lambda)} \sum_{\sigma \in S_n} \operatorname{sgn}^{[1,n]}(\sigma; \lambda)^{-1} e_{\sigma(1)1}(z_0)\cdots e_{\sigma(n)n}\left(q^{2(n-1)} z_0\right) \\ &\quad = \frac{F_{[1,n]}(\rho)}{F^{[1,n]}(\rho)} \sum_{\tau \in S_n} \operatorname{sgn}_{[1,n]}(\tau; \rho) e_{n\tau(n)}\left(q^{2(n-1)} z_0\right)\cdots e_{1\sigma(1)}(z_0). \end{aligned} \tag{6.62}$$

Multiplying by $F^{[1,n]}(\rho)/F_{[1,n]}(\lambda)$ and comparing with (6.13) and (6.14), we deduce that $\overrightarrow{\xi}^{[1,n]}_{[1,n]}(z_0) = \overleftarrow{\xi}^{[1,n]}_{[1,n]}(z_0)$, as desired. □

6.3. Laplace Expansions

Using the left (right) $\mathcal{F}_{\text{ell}}(M(n))$-comodule algebra structure of $\Lambda(\Lambda')$ it is straightforward to prove Laplace expansion formulas for the elliptic minors. For subsets $I, J \subseteq [1, n]$ we define $S_l(I, J; \zeta), S_r(I, J; \zeta) \in M_{\mathfrak{h}^*}$ by

$$\begin{aligned} v_I\left(q^{2\#J} z\right) v_J(z) &= S_l(I, J; \zeta) v_{I\cup J}(z), \\ w^I(z) w^J\left(q^{2\#I} z\right) &= S_r(I, J; \zeta) w^{I\cup J}(z). \end{aligned} \tag{6.63}$$

That this is possible follows from the definitions (6.7) and (6.8) of $v_I(z)$, $w^I(z)$, and the commutation relations (5.8b)–(5.8d), (5.16). In particular $S_l(I, J; \zeta) = 0 = S_r(I, J; \zeta)$, if $I \cap J \neq \emptyset$.

Theorem 6.11. (i) *Let* $I_1, I_2, J \subseteq [1,n]$*, and set* $I = I_1 \cup I_2$*. Then*

$$S_l(I_1, I_2; \lambda)\xi_I^J(z) = \sum_{J_1 \cup J_2 = J} S_l(J_1, J_2; \rho)\xi_{I_1}^{J_1}\left(q^{2\#I_2}z\right)\xi_{I_2}^{J_2}(z). \tag{6.64}$$

(ii) *Let* $J_1, J_2, I \subseteq [1,n]$ *and set* $J = J_1 \cup J_2$*. Then*

$$S_r(J_1, J_2; \rho)\xi_I^J(z) = \sum_{I_1 \cup I_2 = I} S_r(I_1, I_2; \lambda)\xi_{I_1}^{J_1}(z)\xi_{I_2}^{J_2}\left(q^{2\#J_1}z\right). \tag{6.65}$$

Proof. We have

$$\begin{aligned} \Delta_\Lambda\left(v_{I_1}\left(q^{2\#I_2}z\right)\right)\Delta_\Lambda(v_{I_2}(z)) &= \sum_{J_1, J_2} \xi_{I_1}^{J_1}\left(q^{2\#I_2}z\right)\xi_{I_2}^{J_2}(z) \otimes v_{J_1}\left(q^{2\#I_2}z\right)v_{J_2}(z) \\ &= \sum_{J_1, J_2} \xi_{I_1}^{J_1}\left(q^{2\#I_2}z\right)\xi_{I_2}^{J_2}(z) \otimes S_l(J_1, J_2; \zeta)v_J(z) \\ &= \sum_J \left(\sum_{J_1 \cup J_2 = J} S_l(J_1, J_2; \rho)\xi_{I_1}^{J_1}\left(q^{2\#I_2}z\right)\xi_{I_2}^{J_2}(z)\right) \otimes v_J(z). \end{aligned} \tag{6.66}$$

On the other hand,

$$\begin{aligned} \Delta_\Lambda\left(v_{I_1}\left(q^{2\#I_2}z\right)\right)\Delta_\Lambda(v_{I_2}(z)) &= \Delta_\Lambda\left(v_{I_1}\left(q^{2\#I_2}z\right)v_{I_2}(z)\right) \\ &= \Delta_\Lambda(S_l(I_1, I_2; \zeta)v_I(z)) \\ &= \sum_J S_l(I_1, I_2; \lambda)\xi_I^J(z) \otimes v_J(z). \end{aligned} \tag{6.67}$$

Equating these expressions proves (6.64) since, by Proposition 5.5, the set $\{v_J(z) : J \subseteq [1,n]\}$ is linearly independent over $M_{\mathfrak{h}^*}$. The second part is completely analogous, using the right comodule algebra Λ' in place of Λ. □

In Section 7.4 we will need the following lemma, relating the left and right signums $S_l(I, J; \zeta)$ and $S_r(I, J; \zeta)$, defined in (6.63). In the nonspectral trigonometric case the corresponding identity was proved in [15, proof of Proposition 4.1.22].

Lemma 6.12. *Let* I, J *be two disjoint subsets of* $[1,n]$*. Then*

$$S_l(I, J; \zeta + \omega(I)) = S_r(J, I; \zeta)^{-1}, \tag{6.68}$$

where $\omega(I) = \sum_{i \in I} \omega(i)$*.*

Proof. First we claim that, we have the following explicit formulas:

$$S_l(I,J;\zeta) = \prod_{i\in I, j\in J} E(\zeta_{ji}+1), \tag{6.69}$$

$$S_r(I,J;\zeta) = \prod_{i\in I, j\in J} E(\zeta_{ij})^{-1}. \tag{6.70}$$

Recall the definition (6.7) of $v_I(z)$. Since E is odd, relation (5.8b) implies that

$$v_i\left(q^2 z\right)v_j(z) = \frac{E(\zeta_{ji}+1)}{E(\zeta_{ij}+1)} v_j\left(q^2 z\right)v_i(z). \tag{6.71}$$

Also, $F_J(\zeta)$ only involves ζ_{ij} with $i,j \in J$ so it commutes with any $v_k(z)$ with $k \in I$ (since $I \cap J = \emptyset$). From these facts we obtain

$$\begin{aligned} v_I\left(q^{2\#J}z\right)v_J(z) &= \frac{F_I(\zeta)^{-1}F_J(\zeta)^{-1}}{F_{I\cup J}(\zeta)^{-1}} \prod_{\substack{i\in I, j\in J\\ i<j}} \frac{E(\zeta_{ji}+1)}{E(\zeta_{ij}+1)} v_{I\cup J}(z) \\ &= \prod_{\substack{(i,j)\in K\\ i<j}} E(\zeta_{ij}+1) \prod_{\substack{i\in I, j\in J\\ i<j}} \frac{E(\zeta_{ji}+1)}{E(\zeta_{ij}+1)} v_{I\cup J}(z) \\ &= \prod_{i\in I, j\in J} E(\zeta_{ji}+1) v_{I\cup J}(z), \end{aligned} \tag{6.72}$$

where $K = (I\times J)\cup(J\times I)$. This proves (6.69). Similarly one proves (6.70). Now we have

$$S_l(J,I;\zeta+\omega(J))^{-1} = \prod_{i\in I, j\in J} E\left((\zeta+\omega(J))_{ij}+1\right)^{-1} = \prod_{i\in I, j\in J} E(\zeta_{ij})^{-1} = S_r(I,J;\zeta). \tag{6.73}$$

Here we used that for any $i \in I$, $j \in J$ we have $\omega(J)(E_{ii}) = 0$, $\omega(J)(E_{jj}) = 1$, and hence $(\omega(J))_{ij} = -1$. □

7. The Cobraiding and the Elliptic Determinant

7.1. Cobraidings for $\mathfrak{h}$-Bialgebroids

The following definition of a cobraiding was given in [14] and studied further in [10]. When $\mathfrak{h} = 0$ the notion reduces to ordinary cobraidings for bialgebras.

Definition 7.1. A *cobraiding* on an $\mathfrak{h}$-bialgebroid A is a $\mathbb{C}$-bilinear map $\langle\cdot,\cdot\rangle : A \times A \to D_{\mathfrak{h}}$ such that, for any $a, b, c \in A$ and $f \in M_{\mathfrak{h}^*}$,

$$\langle A_{\alpha\beta}, A_{\gamma\delta}\rangle \subseteq (D_{\mathfrak{h}})_{\alpha+\gamma,\beta+\delta}, \tag{7.1a}$$

$$\langle \mu_r(f)a, b\rangle = \langle a, \mu_l(f)b\rangle = fT_0 \circ \langle a, b\rangle, \tag{7.1b}$$

$$\langle a\mu_l(f), b\rangle = \langle a, b\mu_r(f)\rangle = \langle a, b\rangle \circ fT_0, \tag{7.1c}$$

$$\langle ab, c\rangle = \sum_i \langle a, c_i'\rangle T_{\beta_i}\langle b, c_i''\rangle, \quad \Delta(c) = \sum_i c_i' \otimes c_i'', \quad c_i'' \in A_{\beta_i\gamma}, \tag{7.1d}$$

$$\langle a, bc\rangle = \sum_i \langle a_i'', b\rangle T_{\beta_i}\langle a_i', c\rangle, \quad \Delta(a) = \sum_i a_i' \otimes a_i'', \quad a_i'' \in A_{\beta_i\gamma}, \tag{7.1e}$$

$$\langle a, 1\rangle = \langle 1, a\rangle = \varepsilon(a), \tag{7.1f}$$

$$\sum_{ij} \mu_l\Big(\Big\langle a_i', b_j'\Big\rangle 1\Big) a_i'' b_j'' = \sum_{ij} \mu_r\Big(\Big\langle a_i'', b_j''\Big\rangle 1\Big) b_j' a_i'. \tag{7.1g}$$

The following definition was given in unpublished notes by Rosengren [16]. The terminology is motivated by Proposition 7.6 concerning FRST algebras A_R, but it makes sense for arbitrary $\mathfrak{h}$-bialgebroids.

Definition 7.2. A cobraiding $\langle\cdot,\cdot\rangle$ on an $\mathfrak{h}$-bialgebroid A is called *unitary* if

$$\varepsilon(ab) = \sum_{(a),(b)} \langle a', b'\rangle T_{\omega_{12}(a)+\omega_{12}(b)}\langle a'', b''\rangle \tag{7.2}$$

for all $a, b \in A$. In such sums we always assume, without loss of generality, that all a', a'', b', b'' are homogenous and use the notation $\omega_{12}(a) = \gamma$ if $a' \in A_{\alpha\gamma}$ for some α (or equivalently, if $a'' \in A_{\gamma\beta}$ for some β).

7.2. Cobraidings for the FRST Algebras A_R

Now let $R : \mathfrak{h}^* \times \mathbb{C}^\times \to \operatorname{End}_{\mathfrak{h}}(V \otimes V)$ be a meromorphic function, and let A_R be the $\mathfrak{h}$-bialgebroid associated to R as in Section 3.2.

Proposition 7.3. *Assume that $\varphi : \mathbb{C}^\times \to \mathbb{C}$ is a holomorphic function, not vanishing identically, such that, for each $x, y, a, b \in X$, $z \in \mathbb{C}^\times$, the limit $\lim_{w\to z}(\varphi(w)R_{xy}^{ab}(\zeta, w))$ exists and defines a meromorphic function in $M_{\mathfrak{h}^*}$. Then the following statements are equivalent:*

(i) *there exists a cobraiding $\langle\cdot,\cdot\rangle : A_R \times A_R \to D_{\mathfrak{h}}$ satisfying*

$$\langle L_{ij}(z_1), L_{kl}(z_2)\rangle = \lim_{w\to z_1/z_2}\Big(\varphi(w)R_{ik}^{jl}(\zeta, w)\Big)T_{-\omega(i)-\omega(k)}, \tag{7.3}$$

(ii) *R satisfies the QDYBE* (2.8).

Remark 7.4. (a) The identity (7.1g) is not necessary when proving that (i) implies (ii). Without assuming (7.1g), $\langle\cdot,\cdot\rangle$ is a *pairing on* $A^{\mathrm{cop}} \times A$. See [14].

(b) Without the factor $\varphi(w)$, the cobraiding is not well defined if $R(\zeta, z)$ has poles in the z variable. We also remark that the residual relations (3.8) are necessary for (ii) to imply (i).

Proof. The proof is straightforward and is carried out in [15, Lemma 2.2.5], under the assumption that the R-matrix is regular in the spectral variable. □

We will now generalize slightly the notion of a unitary cobraiding on A_R to account for spectral singularities in the R-matrix as follows.

Call $a \in A_R$ *spectrally homogenous* if there exist $k \in \mathbb{Z}_{\geq 0}$ and $z_1, \ldots, z_k \in \mathbb{C}^\times$ such that

$$a \in \sum_{\sigma \in S_k} \sum_{i_l, j_l \in X} M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} L_{i_1 j_1}\left(z_{\sigma(1)}\right) \cdots L_{i_k j_k}\left(z_{\sigma(k)}\right). \tag{7.4}$$

The multiset $\{z_i\}_i$ is called the *spectral degree* of a and is denoted by $\mathrm{sdeg}(a)$. Note that the spectral degree of a nonzero spectrally homogenous element is uniquely defined, since the RLL relations (3.8) are spectrally homogenous.

Let $\varphi : \mathbb{C}^\times \to \mathbb{C}$ be holomorphic. For spectrally homogenous elements $a, b \in A_R$, define the *regularizing factor* $\widehat{\varphi}(a, b)$ by

$$\widehat{\varphi}(a, b) = \prod_{1 \leq i \leq k, 1 \leq j \leq l} \varphi\left(\frac{z_i}{w_j}\right), \tag{7.5}$$

where $\{z_i\}_i = \mathrm{sdeg}(a)$, $\{w_j\}_j = \mathrm{sdeg}(b)$.

Definition 7.5. Let $\varphi : \mathbb{C}^\times \to \mathbb{C}$ be holomorphic. A cobraiding $\langle\cdot,\cdot\rangle$ on A_R is *unitary with respect to* φ if

$$\widehat{\varphi}(a, b)\widehat{\varphi}(b, a)\varepsilon(ab) = \sum_{(a),(b)} \langle a', b'\rangle T_{\omega_{12}(a)+\omega_{12}(b)} \langle a'', b''\rangle \tag{7.6}$$

for all spectrally homogenous $a, b \in A_R$.

The following proposition was proved in [16] if the spectral variables are taken to be generic so that no regularizing factors are needed.

Proposition 7.6. *Suppose that* $R : \mathfrak{h}^* \times \mathbb{C}^\times \to \mathrm{End}_{\mathbb{C}}(V \otimes V)$ *satisfies the QDYBE and is unitary:* $R(\zeta, z) R(\zeta, z^{-1})^{(21)} = \mathrm{Id}_{V \otimes V}$. *Suppose that* $\varphi : \mathbb{C}^\times \to \mathbb{C}$ *is nonzero holomorphic such that* $\lim_{w \to z}(\varphi(w) R_{xy}^{ab}(\zeta, w))$ *exists and is a holomorphic function in* $M_{\mathfrak{h}^*}$. *Then the cobraiding* $\langle\cdot,\cdot\rangle$ *on* A_R *given in Proposition 7.3 is unitary with respect to* φ.

Proof. Since both sides are holomorphic in the spectral variables, it is enough to prove it for generic values. We claim that for such values, $\widehat{\varphi}(a, b)\langle a, b\rangle_R = \langle a, b\rangle$ where $\langle\cdot,\cdot\rangle_R$ is the cobraiding, defined only for generic spectral values, determined by $\langle L_{ij}(z), L_{kl}(w)\rangle_R = R_{ik}^{jl}(\zeta, z/w) T_{-\omega(i)-\omega(k)}$. Indeed, this claim follows by induction from the identities (7.1d) and

(7.1e) using that $\widehat{\varphi}(a_1, b)\widehat{\varphi}(a_2, c) = \widehat{\varphi}(a_3, bc)$ and $\widehat{\varphi}(c, a_1)\widehat{\varphi}(b, a_2) = \widehat{\varphi}(cb, a_3)$ for spectrally homogenous $a_i, b, c \in A_R$, the a_i having the same spectral degree.

Since the R-matrix R is unitary, the statement of the lemma now follows from the identity

$$\varepsilon(ab) = \sum_{(a),(b)} \langle a', b' \rangle_R T_{\omega_{12}(a)+\omega_{12}(b)} \langle a'', b'' \rangle_R \tag{7.7}$$

holding for generic spectral values which was proved by Rosengren [16]. □

7.2.1. The Case of $\mathcal{F}_{\mathrm{ell}}(M(n))$

Specializing further to the algebra of interest, $\mathcal{F}_{\mathrm{ell}}(M(n))$, we obtain the following corollary.

Corollary 7.7. *The $\mathfrak{h}$-bialgebroid $\mathcal{F}_{\mathrm{ell}}(M(n))$ carries a cobraiding $\langle \cdot, \cdot \rangle$ satisfying*

$$\langle e_{ij}(z), e_{kl}(w) \rangle = \widetilde{R}^{jl}_{ik}\left(\zeta, \frac{z}{w}\right) T_{-\omega(i)-\omega(k)} \quad \forall z, w \in \mathbb{C}^{\times},\ i, j \in [1, n], \tag{7.8}$$

where

$$\widetilde{R}^{jl}_{ik}(\zeta, z) = \lim_{w \to z} \left(\theta\left(q^2 w\right) R^{jl}_{ik}(\zeta, w) \right). \tag{7.9}$$

Moreover, this cobraiding is unitary with respect to $\varphi : \mathbb{C}^{\times} \to \mathbb{C}$, $\varphi(z) = \theta(q^2 z)$.

Proof. It suffices to notice that, by (4.3), (2.15), and (2.16), $\widetilde{R}$ is regular in z and apply Propositions 7.3 and 7.6. □

7.3. Properties of the Elliptic Determinant

A common method used to study quantum minors and prove that quantum determinants are central is the fusion procedure, going back to work by Kulish and Sklyanin [17]. Another approach, using representation theory, was developed by Noumi et al. [18]. In this section we show how to prove that the elliptic determinant is central using the properties of the cobraiding on $\mathcal{F}_{\mathrm{ell}}(M(n))$ and how to resolve technical issues connected with the spectral singularities of the elliptic R-matrix.

Let $A = \mathcal{F}_{\mathrm{ell}}(M(n))$. When $I = J = [1, n]$ we set

$$\det(z) = \xi^J_I(z) \tag{7.10}$$

for $z \in \mathbb{C}^{\times}$, where $\xi^J_I(z)$ is the elliptic minor given in Theorem 6.10. Thus one possible expression for $\det(z)$ is

$$\det(z) = \sum_{\sigma \in S_n} \frac{F^{[1,n]}(\rho)}{\sigma\left(F^{[1,n]}(\lambda)\right)} e_{\sigma(1)1}(z) e_{\sigma(2)2}\left(q^2 z\right) \cdots e_{\sigma(n)n}\left(q^{2(n-1)} z\right). \tag{7.11}$$

Theorem 7.8. (a) $\det(z)$ *is a grouplike element of A for each $z \in \mathbb{C}^\times$, that is,*

$$\Delta(\det(z)) = \det(z) \otimes \det(z), \qquad \varepsilon(\det(z)) = 1. \tag{7.12}$$

(b) $\det(z)$ *is a central element in $\mathcal{F}_{ell}(M(n))$:*

$$[e_{ij}(z), \det(w)] = [f(\lambda), \det(w)] = [f(\rho), \det(w)] = 0 \tag{7.13}$$

for all $f \in M_{\mathfrak{h}^*}$, $i, j \in [1, n]$ and all $z, w \in \mathbb{C}^\times$.

Proof. Let $\Lambda^n(z) = M_{\mathfrak{h}^*} v_I(z)$, where $I = [1, n]$. It is a one-dimensional subcorepresentation of the left exterior corepresentation Λ. Its matrix element is $\det(z)$, that is,

$$\Delta(v_I(z)) = \det(z) \otimes v_I(z). \tag{7.14}$$

From the coassociativiy and counity axioms for a corepresentation, it follows that $\det(z)$ is grouplike, proving part (a).

The rest of this section is devoted to the proof of part (b). It follows from the definition that $\det(z) \in A_{00}$ and thus it commutes with $f(\rho)$ and $f(\lambda)$ for any $f \in M_{\mathfrak{h}^*}$. To prove that it commutes with the generators $e_{ij}(z)$ we need several lemmas which we now state and prove.

Lemma 7.9. *For $i, j \in [1, n]$, $I, J \subseteq [1, n]$, $\#I = \#J = 2$, we have*

$$\left\langle \xi_I^J(w), e_{ij}(z) \right\rangle = 0, \quad \text{if } \frac{w}{z} \in p^{\mathbb{Z}}, \tag{7.15}$$

$$\left\langle e_{ij}(z), \xi_I^J(w) \right\rangle = 0, \quad \text{if } \frac{q^2 w}{z} \in p^{\mathbb{Z}}. \tag{7.16}$$

Proof. Let $I = \{i_1, i_2\}$, $i_1 < i_2$, $J = \{j_1, j_2\}$, $j_1 < j_2$. Using the left expansion formula (6.13) and (7.1b), (7.1c) we have

$$\begin{aligned} \left\langle \xi_I^J(w), e_{ij}(z) \right\rangle &= \left\langle \frac{E(\rho_{j_1 j_2} + 1)}{E(\lambda_{i_1 i_2} + 1)} e_{i_2 j_2}\left(q^2 w\right) e_{i_1 j_1}(w), e_{ij}(z) \right\rangle + [j_1 \longleftrightarrow j_2] \\ &= E(\zeta_{j_1 j_2 + 1} + 1) \left\langle e_{i_2 j_2}\left(q^2 w\right) e_{i_1 j_1}(w), e_{ij}(z) \right\rangle \frac{1}{E(\zeta_{i_1 i_2} + 1)} + [j_1 \longleftrightarrow j_2]. \end{aligned} \tag{7.17}$$

Thus we need to prove that for $w/z \in p^{\mathbb{Z}}$, the first term is antisymmetric in j_1, j_2. By (7.1d),

$$\begin{aligned} &E(\zeta_{j_1 j_2} + 1) \left\langle e_{i_2 j_2}\left(q^2 w\right) e_{i_1 j_1}(w), e_{ij}(z) \right\rangle \\ &\quad = E(\zeta_{j_1 j_2} + 1) \sum_x \left\langle e_{i_2 j_2}\left(q^2 w\right), e_{ix}(z) \right\rangle T_{\omega(x)} \langle e_{i_1 j_1}(w), e_{xj}(z) \rangle \\ &\quad = E(\zeta_{j_1 j_2} + 1) \sum_x \widetilde{R}_{i_2 i}^{j_2 x}\left(\zeta, \frac{q^2 w}{z}\right) \widetilde{R}_{i_1 x}^{j_1 j}\left(\zeta - \omega(j_2), \frac{w}{z}\right) T_{-\omega(j_1) - \omega(j_2) - \omega(j)}. \end{aligned} \tag{7.18}$$

Take now $w = p^k z$ where $k \in \mathbb{Z}$. One checks that

$$\widetilde{R}^{ab}_{xy}\left(\zeta, p^k\right) = \theta\left(q^2 p^k\right) q^{2k(\zeta_{ba}+1-\delta_{ab})} \delta_{ay}\delta_{bx}, \tag{7.19}$$

where δ_{xy} is the Kronecker delta. In particular, only the $x = j_1$ term is nonzero. Now the antisymmetry of (7.18) in j_1, j_2 follows by applying the identities

$$\widetilde{R}^{j_1 j}_{j j_1}\left(\zeta - \omega(j_2), p^k\right) = q^{2k\zeta_{j_2 j_1}} \cdot \widetilde{R}^{j_2 j}_{j j_2}\left(\zeta - \omega(j_1), p^k\right), \tag{7.20}$$

$$E(\zeta_{j_1 j_2} + 1) \widetilde{R}^{j_2 j_1}_{i_2 i}\left(\zeta, q^2 p^k\right) = -q^{2k\zeta_{j_1 j_2}} \cdot E(\zeta_{j_2 j_1} + 1) \widetilde{R}^{j_1 j_2}_{i_2 i}\left(\zeta, q^2 p^k\right). \tag{7.21}$$

Relation (7.20) can be proved directly from (7.19) while for (7.21) one can use that

$$\frac{\widetilde{R}^{ab}_{xy}(\zeta, p^k z)}{\widetilde{R}^{ba}_{xy}(\zeta, p^k z)} = q^{2k\zeta_{ba}} \frac{\widetilde{R}^{ab}_{xy}(\zeta, z)}{\widetilde{R}^{ba}_{xy}(\zeta, z)} \tag{7.22}$$

together with the relation

$$E(\zeta_{j_1 j_2} + 1)\widetilde{\alpha}\left(\zeta_{j_2 j_1}, q^2\right) = -E(\zeta_{j_2 j_1} + 1)\widetilde{\beta}\left(\zeta_{j_2 j_1}, q^2\right) \tag{7.23}$$

which holds for any $j_1 \neq j_2$ which is easily proved by applying $\theta(z^{-1}) = -z^{-1}\theta(z)$ three times.

Relation (7.16) can be proved analogously, using the right expansion formula (6.14) for $\xi^J_I(w)$ instead. □

Since the cobraiding depends holomorphically on the spectral variables, and all zeros of θ are simple and of the form p^k, we conclude that the following limits exist for all $z, w \in \mathbb{C}^\times$, i, j, I, J, $\#I = \#J = 2$:

$$\begin{aligned} \left\langle \xi^J_I(w), e_{ij}(z) \right\rangle' &:= \lim_{(z_1,w_1)\to(z,w)} \frac{\left\langle \xi^J_I(w_1), e_{ij}(z_1) \right\rangle}{\theta(z_1/w_1)}, \\ \left\langle e_{ij}(z), \xi^J_I(w) \right\rangle' &:= \lim_{(z_1,w_1)\to(z,w)} \frac{\left\langle e_{ij}(z_1), \xi^J_I(w_1) \right\rangle}{\theta(q^2 w_1/z_1)}. \end{aligned} \tag{7.24}$$

Taking $a = e_{ij}(z_1)$, $b = \xi^J_I(w_1)$ in (7.6), dividing both sides by $\theta(z_1/w_1)\theta(q^2 w_1/z_1)$ and taking the limits $(z_1, w_1) \to (z, w)$, where $z, w \in \mathbb{C}^\times$ are arbitrary, we get

$$\psi(z,w)\varepsilon\left(e_{ij}(z)\xi^J_I(w)\right) = \sum_{x,X} \left\langle \xi^X_I(w), e_{ix}(z) \right\rangle' T_{\omega(x)+\omega(x_1)+\omega(x_2)} \left\langle e_{xj}(z), \xi^J_X(w) \right\rangle', \tag{7.25}$$

and interchanging a and b,

$$\psi(z,w)\varepsilon\left(\xi_I^J(w)e_{ij}(z)\right)=\sum_{x,X}\left\langle e_{ix}(z),\xi_I^X(w)\right\rangle' T_{\omega(x)+\omega(x_1)+\omega(x_2)}\left\langle \xi_X^J(w),e_{xj}(z)\right\rangle', \tag{7.26}$$

for all $z,w\in\mathbb{C}^\times$, where $\psi:(\mathbb{C}^\times)^2\to\mathbb{C}$ is given by

$$\psi(z,w)=\theta\left(\frac{q^2z}{w}\right)\theta\left(\frac{q^4w}{z}\right). \tag{7.27}$$

We are now ready to prove the key identity.

Lemma 7.10. *For any $i,j\in[1,n]$, $I,J\subseteq[1,n]$, $\#I=\#J=2$ and any $z,w\in\mathbb{C}^\times$, $q^2w/z\notin p^{\mathbb{Z}}$ we have*

$$\begin{aligned}\psi(z,w)&\sum_{x,X}\mu_l\left(\left\langle \xi_I^X(w),e_{ix}(z)\right\rangle' 1\right)\xi_X^J(w)e_{xj}(z)\\&=\psi(z,w)\sum_{x,X}\mu_r\left(\left\langle \xi_X^J(w),e_{xj}(z)\right\rangle' 1\right)e_{ix}(z)\xi_I^X(w).\end{aligned} \tag{7.28}$$

Proof. Using the counit axiom followed by (7.25) we have

$$\begin{aligned}\psi(z,w)e_{ij}(z)\xi_I^J(w)&=\psi(z,w)\sum_{x,X}\mu_l\left(\varepsilon\left(e_{ix}(z)\xi_I^X(w)\right)1\right)e_{xj}(z)\xi_X^J(w)\\&=\sum_{x,y,X,Y}\mu_l\left(\left\langle \xi_I^Y(w),e_{iy}(z)\right\rangle' 1\right)\mu_l\left(\left\langle e_{yx}(z),\xi_Y^X(w)\right\rangle' 1\right)e_{xj}(z)\xi_X^J(w).\end{aligned} \tag{7.29}$$

Applying the identity obtained by dividing by $\theta(q^2w/z)$ in both sides of the cobraiding identity (7.1g) with $a=e_{yj}(z)$, $b=\xi_Y^J(w)$ in the right hand side of (7.29) gives

$$\psi(z,w)e_{ij}(z)\xi_I^J(w)=\sum_{x,y,X,Y}\mu_l\left(\left\langle \xi_I^Y(w),e_{iy}(z)\right\rangle' 1\right)\mu_r\left(\left\langle e_{xj}(z),\xi_X^J(w)\right\rangle' 1\right)\xi_Y^X(w)e_{yx}(z). \tag{7.30}$$

Now multiply both sides by $\mu_r(\langle \xi_J^K(w),e_{jk}(z)\rangle' 1)$, and sum over j, J. After applying (7.26) in the right hand side we get

$$\begin{aligned}\psi(z,w)&\sum_{j,J}\mu_r\left(\left\langle \xi_J^K(w),e_{jk}(z)\right\rangle' 1\right)e_{ij}(z)\xi_I^J(w)\\&=\psi(z,w)\sum_{x,y,X,Y}\mu_l\left(\left\langle \xi_I^Y(w),e_{iy}(z)\right\rangle' 1\right)\mu_r\left(\varepsilon\left(\xi_X^K(w)e_{xk}(z)\right)1\right)\xi_Y^X(w)e_{yx}(z).\end{aligned} \tag{7.31}$$

By the counit axiom the last expression equals

$$\varphi(z,w)\sum_{y,Y}\mu_l\left(\left\langle \xi_I^Y(w), e_{iy}(z)\right\rangle' 1\right)\xi_Y^K(w)e_{yk}(z). \tag{7.32}$$

□

Lemma 7.11. (a) *The limit*

$$\langle \det(w), e_{ij}(z)\rangle' := \lim_{(z_1,w_1)\to(z,w)} \frac{\langle \det(w_1), e_{ij}(z_1)\rangle}{\theta(w_1/z_1)\theta(q^2w_1/z_1)\cdots\theta(q^{2(n-2)}w_1/z_1)} \tag{7.33}$$

exists for any $z, w \in \mathbb{C}^\times$.

(b) *We have*

$$\mu_l(\langle \det(w), e_{11}(z)\rangle' 1)\det(w)e_{11}(z) = \mu_r(\langle \det(w), e_{11}(z)\rangle' 1)e_{11}(z)\det(w) \tag{7.34}$$

for any $z, w \in \mathbb{C}^\times$.

Proof. (a) We must show that $\langle \det(w), e_{ij}(z)\rangle$ vanishes for $q^{2k}w/z \in p^{\mathbb{Z}}$, where $k \in \{0,1,\dots,n-2\}$. Applying the Laplace expansion (6.65) twice we get

$$\det(w) = \sum_{I_1\cup I_2\cup I_3=[1,n]} S_r(I_1,I_2,I_3;\lambda)\xi_{I_1}^{J_1}(w)\xi_{I_2}^{J_2}\left(q^{2\#J_1}w\right)\xi_{I_3}^{J_3}\left(q^{2(\#J_1+2)}w\right), \tag{7.35}$$

where $J_1 = \{1,\dots,k\}$, $J_2 = \{k+1,k+2\}$, $J_3 = \{k+3,\dots,n\}$. Substituting this in the pairing and applying the multiplication-comultiplication relation (7.1d) we see that each term contains a factor of the form $\langle \xi_X^Y(q^{2k}w), e_{xy}(z)\rangle$, where $\#X = \#Y = 2$, which indeed vanishes for $q^{2k}w/z \in p^{\mathbb{Z}}$ by Lemma 7.9.

(b) By (7.1a), $\langle \det(w), e_{xy}(z)\rangle = 0$, if $x \neq y$. Thus (7.34) can be written

$$\sum_x \mu_l(\langle \det(w), e_{1x}(z)\rangle' 1)\det(w)e_{x1}(z) = \sum_x \mu_r(\langle \det(w), e_{x1}(z)\rangle' 1)e_{1x}(z)\det(w). \tag{7.36}$$

If $q^{2k}w/z \notin p^{\mathbb{Z}}$ for any $k \in \{0,1,\dots,n-2\}$, this follows from the cobraiding identity (7.1g) with $a = \det(w)$, $b = e_{11}(z)$ by dividing by the nonzero number $\prod_{k=0}^{n-2}\theta(q^k w/z)$.

So assume $q^{2k}w/z \in p^{\mathbb{Z}}$ for some $k \in \{0,1,\dots,n-2\}$. We again use the iterated Laplace expansion (7.35). For simplicity of notation, we write it as $\det(w) = \sum a_1a_2a_3$ where a_2 is the 2×2 minor. Put $b = e_{11}(z)$. Substituting this, and expanding $\langle a_1a_2a_3, b'\rangle$ using (7.1d), we get after simplification

$$\begin{aligned}&\sum_x \mu_l(\langle \det(w), e_{1x}(z)\rangle' 1)\det(w)e_{x1}(z)\\ &\quad= \frac{1}{\prod_{m=0,m\neq k}^{n-2}\theta(q^{2m}w/z)}\sum_{(a),(b)}\mu_l(\langle a_1', b'\rangle 1)a_1''\mu_l\left(\langle a_2', b''\rangle' 1\right)a_2''\mu_l\left(\left\langle a_3', b^{(3)}\right\rangle 1\right)a_3''b^{(4)}.\end{aligned} \tag{7.37}$$

Now using the cobraiding identity (7.1g) and its primed version for quadratic minors (7.28), we can move the b all the way to the left. Doing the steps backwards the claim follows. □

It remains to calculate $\langle \det(w), e_{11}(z)\rangle' 1$.

Lemma 7.12. *We have*

$$\langle \det(w), e_{11}(z)\rangle' 1 = q^{n-1}\theta\left(\frac{q^{2n}w}{z}\right). \tag{7.38}$$

Proof. Expanding $\det(w)$ using the left expansion formula (6.13) with

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}, \tag{7.39}$$

the longest element in S_n, and applying (7.1d) repeatedly we have (putting $I = [1, n]$)

$$\begin{aligned}
&\langle \det(w), e_{11}(z)\rangle \\
&\quad = \sum_{\tau\in S_n} \left\langle \frac{\tau(F_I(\rho))}{\sigma(F_I(\lambda))} e_{1\tau(n)}\left(q^{2(n-1)}w\right)\cdots e_{n\tau(1)}(w), e_{11}(z)\right\rangle \\
&\quad = \sum_{\substack{\tau\in S_n\\ x_1,\dots,x_{n-1}}} \tau(F_I(\zeta))\left\langle e_{1\tau(n)}\left(q^{2(n-1)}w\right), e_{1x_1}(z)\right\rangle T_{\omega(x_1)}\cdots T_{\omega(x_{n-1})}\langle e_{n\tau(1)}(w), e_{x_{n-1}1}(z)\rangle \sigma\left(F_I(\zeta)^{-1}\right).
\end{aligned} \tag{7.40}$$

One proves inductively that in all nonzero terms we have $\tau(j) = n+1-j$ and $x_{n-j} = 1$ for all $1 \le j \le n-1$ by looking from right to left: $\langle e_{n\tau(1)}(w), e_{x_{n-1}1}(z)\rangle = \widetilde{R}^{\tau(1)1}_{nx_{n-1}}(\zeta, w/z)T_{-\omega(n)-\omega(x_{n-1})}$ which, if $1 \ne n$, is nonzero only for $\tau(1) = n$ and $x_{n-1} = 1$ by (4.3). Then looking at the second pairing from the right we see that $\tau(2) = n-1$ and $x_{n-2} = 1$ if it is nonzero, and so on. Thus only the term $\tau = \sigma$ and $x_1 = \cdots = x_{n-1} = 1$ survives and it equals

$$\begin{aligned}
&\sigma(F_I(\zeta))\widetilde{R}^{11}_{11}\left(\zeta, \frac{q^{2(n-1)}w}{z}\right)T_{-\omega(1)}\widetilde{R}^{21}_{21}\left(\zeta, \frac{q^{2(n-2)}w}{z}\right)T_{-\omega(2)} \\
&\qquad \cdots T_{-\omega(n-1)}\widetilde{R}^{n1}_{n1}\left(\zeta, \frac{w}{z}\right)T_{-\omega(n)-\omega(1)}\sigma(F_I(\zeta))^{-1}.
\end{aligned} \tag{7.41}$$

Using that $\sigma(F_I(\zeta)) = \prod_{i<j} E(\zeta_{ji}+1)$ and that $\widetilde{R}_{j1}^{j1}(\zeta - \omega(1), z) = \widetilde{\alpha}(\zeta_{j1}+1, z) = q\theta(z)(E(\zeta_{j1}+2)/E(\zeta_{j1}+1))$ we get

$$q^{n-1}\theta\left(\frac{q^{2n}w}{z}\right)\cdot\theta\left(\frac{q^{2(n-2)}w}{z}\right)\theta\left(\frac{q^{2(n-3)}w}{z}\right)\cdots\theta\left(\frac{w}{z}\right)$$
$$\cdot\prod_{i<j}E(\zeta_{ji}+1)\prod_{1<j}\frac{E(\zeta_{j1}+2)}{E(\zeta_{j1}+1)}T_{-\omega(1)}\prod_{i<j}E(\zeta_{ji}+1)^{-1}. \tag{7.42}$$

The factors involving the dynamical variable ζ cancel and the claim follows. □

By Lemmas 7.11(b) and 7.12 we conclude that $\det(w)$ commutes with $e_{11}(z)$ if $q^{2n}w/z \notin p^{\mathbb{Z}}$. By applying an automorphism from the $S_n \times S_n$-action on A as defined in Section 5.3 and using that $\det(z)$ is fixed by those, by relation (6.15), we conclude that $\det(w)$ commutes with any $e_{ij}(z)$ as long as $q^{2n}w/z \notin p^{\mathbb{Z}}$.

For the remaining case we can note that relations (4.4a)–(4.4d) and (4.5) imply that there is a $\mathbb{C}$-linear map $T : \mathcal{F}_{\mathrm{ell}}(M(n)) \to \mathcal{F}_{\mathrm{ell}}(M(n))$ such that $T(ab) = T(b)T(a)$ for all $a, b \in \mathcal{F}_{\mathrm{ell}}(M(n))$, given by

$$T(e_{ij}(z)) = e_{ij}\left(z^{-1}\right), \qquad T(f(\lambda)) = f(-\lambda), \qquad T(f(\rho)) = f(-\rho), \tag{7.43}$$

for all $f \in M_{\mathfrak{h}^*}$, $i, j \in [1, n]$, and $z \in \mathbb{C}^\times$. One verifies that $T(\det(z)) = \det(q^{-2(n-1)}z^{-1})$.

We have proved that $[\det(w), e_{ij}(z)] = 0$ if $q^{2n}w/z \notin p^{\mathbb{Z}}$. Assume $q^{2n}w/z \in p^{\mathbb{Z}}$. Then

$$T([\det(w), e_{ij}(z)]) = \left[e_{ij}\left(z^{-1}\right), \det\left(q^{-2(n-1)}w^{-1}\right)\right] = 0 \tag{7.44}$$

since $q^{-2(n-1)}w^{-1}/z^{-1} = q^2(q^{2n}w/z)^{-1} \notin p^{\mathbb{Z}}$. This finishes the proof of Theorem(b). □

7.4. The Antipode

We use the following definition for the antipode, given in [13].

Definition 7.13. An $\mathfrak{h}$-*Hopf algebroid* is an $\mathfrak{h}$-bialgebroid A equipped with a $\mathbb{C}$-linear map $S : A \to A$, called the antipode, such that

$$S(\mu_r(f)a) = S(a)\mu_l(f), \quad S(a\mu_l(f)) = \mu_r(f)S(a), \quad a \in A,\ f \in M_{\mathfrak{h}^*}, \tag{7.45}$$

$$m \circ (\mathrm{Id} \otimes S) \circ \Delta(a) = \mu_l(\varepsilon(a)1), \quad a \in A,$$
$$m \circ (S \otimes \mathrm{Id}) \circ \Delta(a) = \mu_r(T_\alpha(\varepsilon(a)1)), \quad a \in A_{\alpha\beta}, \tag{7.46}$$

where m denotes the multiplication, and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in M_{\mathfrak{h}^*}$.

Let $\mathcal{F}_{\mathrm{ell}}(M(n))[\det(z)^{-1} : z \in \mathbb{C}^{\times}]$ be the polynomial algebra in uncountably many variables $\det(z)^{-1}$, $z \in \mathbb{C}^{\times}$, with coefficients in $\mathcal{F}_{\mathrm{ell}}(M(n))$. We define $\mathcal{F}_{\mathrm{ell}}(GL(n))$ to be

$$\frac{\mathcal{F}_{\mathrm{ell}}(M(n))\left[\det(z)^{-1} : z \in \mathbb{C}^{\times}\right]}{J}, \tag{7.47}$$

where J is the ideal generated by the relations $\det(z)\det(z)^{-1} = 1 = \det(z)^{-1}\det(z)$ for each $z \in \mathbb{C}^{\times}$. We extend the bigrading of $\mathcal{F}_{\mathrm{ell}}(M(n))$ to $\mathcal{F}_{\mathrm{ell}}(M(n))[\det(z)^{-1} : z \in \mathbb{C}^{\times}]$ by requiring that $\det(z)^{-1}$ has bidegree $0,0$ for each $z \in \mathbb{C}^{\times}$. Then J is homogenous and the bigrading descends to $\mathcal{F}_{\mathrm{ell}}(GL(n))$. We extend the comultiplication and counit by requiring that $\det(z)^{-1}$ is grouplike for each $z \in \mathbb{C}^{\times}$, that is,

$$\Delta\left(\det(z)^{-1}\right) = \det(z)^{-1} \otimes \det(z)^{-1}, \qquad \varepsilon\left(\det(z)^{-1}\right) = 1. \tag{7.48}$$

Here 1 denotes the identity operator in $D_{\mathfrak{h}}$. One verifies that J is a coideal and that $\varepsilon(J) = 0$, which induces operations Δ, ε on $\mathcal{F}_{\mathrm{ell}}(GL(n))$. In this way $\mathcal{F}_{\mathrm{ell}}(GL(n))$ becomes an $\mathfrak{h}$-bialgebroid. This algebra is nontrivial since $\varepsilon(J) = 0$ implies that J is a proper ideal.

For $i \in [1, n]$ we set $\hat{\imath} = \{1, \ldots, i-1, i+1, \ldots, n\}$. For a meromorphic function f on $\mathfrak{h}^*$, we denote the images of f under the left and right moment maps in $\mathcal{F}_{\mathrm{ell}}(GL(n))$ also by $f(\lambda)$ and $f(\rho)$ respectively.

Theorem 7.14. *$\mathcal{F}_{\mathrm{ell}}(GL(n))$ is an $\mathfrak{h}$-Hopf algebroid with antipode S given by*

$$S(f(\lambda)) = f(\rho), \qquad S(f(\rho)) = f(\lambda), \tag{7.49}$$

$$S(e_{ij}(z)) = \frac{S_r(\hat{\jmath}, \{j\}; \lambda)}{S_r(\hat{\imath}, \{i\}; \rho)} \det\left(q^{-2(n-1)} z\right)^{-1} \xi_{\hat{\jmath}}^{\hat{\imath}}\left(q^{-2(n-1)} z\right), \tag{7.50}$$

$$S\left(\det(z)^{-1}\right) = \det(z), \tag{7.51}$$

for all $f \in M_{\mathfrak{h}^*}$, $i, j \in [1, n]$ and $z \in \mathbb{C}^{\times}$.

Proof. We proceed in steps.

Step 1. Define S on the generators of $\mathcal{F}_{\mathrm{ell}}(M(n))$ by (7.49), (7.50). We show that the antipode axiom (7.46) holds if a is a generator. Indeed for $a = f(\lambda)$ or $a = f(\rho)$, $f \in M_{\mathfrak{h}^*}$, this is easily checked. Let $a = e_{ij}(z)$. Using the right Laplace expansion (6.65) with $J_1 = \hat{\imath}$, $J_2 = \{j\}$, $I = [1, n]$ and z replaced by $q^{-2(n-1)}z$ we obtain

$$\sum_{x=1}^{n} S(e_{ix}(z)) e_{xj}(z) = \delta_{ij}. \tag{7.52}$$

Similarly, using the left Laplace expansion (6.64) with $I_1 = \{i\}$, $I_2 = \hat{\jmath}$, $J = [1, n]$, and z replaced by $q^{-2(n-1)}z$, together with the identity (6.68), we get

$$\sum_{x=1}^{n} e_{ix}(z) S(e_{xj}(z)) = \delta_{ij}, \tag{7.53}$$

using also the crucial fact that, by Theorem 7.8, $e_{ij}(z)$ commutes in $\mathcal{F}_{\text{ell}}(M(n))$ with $\det(q^{-2(n-1)}z)$ and hence in $\mathcal{F}_{\text{ell}}(GL(n))$ with $\det(q^{-2(n-1)}z)^{-1}$. This proves that the antipode axiom (7.46) is satisfied for $a = e_{ij}(z)$.

Step 2. We show that S extends to a $\mathbb{C}$-linear map $S : \mathcal{F}_{\text{ell}}(M(n)) \to \mathcal{F}_{\text{ell}}(GL(n))$ satisfying $S(ab) = S(b)S(a)$. For this we must verify that S preserves the relations, (4.1), (4.2), (4.5) of $\mathcal{F}_{\text{ell}}(M(n))$. Since $S(e_{ij}(z)) \in \mathcal{F}_{\text{ell}}(GL(n))_{\omega(\hat{\jmath}),\omega(\hat{\imath})}$ and $\omega(i) + \omega(\hat{\imath}) = 0$, we have

$$\begin{aligned} S(e_{ij}(z)) S(f(\lambda)) &= S(e_{ij}(z)) f(\rho) \\ &= f(\rho - \omega(\hat{\imath})) S(e_{ij}(z)) \\ &= f(\rho + \omega(i)) S(e_{ij}(z)) \\ &= S(f(\lambda + \omega(i))) S(e_{ij}(z)). \end{aligned} \tag{7.54}$$

Similarly, $S(e_{ij}(z)) S(f(\rho)) = S(f(\rho + \omega(j))) S(e_{ij}(z))$ so relations (4.1) are preserved. Next, consider the RLL relation

$$\sum_{x,y=1}^{n} R_{ac}^{xy}\left(\lambda, \frac{z_1}{z_2}\right) e_{xb}(z_1) e_{yd}(z_2) = \sum_{x,y=1}^{n} R_{xy}^{bd}\left(\rho, \frac{z_1}{z_2}\right) e_{cy}(z_2) e_{ax}(z_1). \tag{7.55}$$

Multiply (7.55) from the left by $S(e_{ic}(z_2))$ and from the right by $S(e_{dk}(z_2))$, sum over c, d, and use (7.52), (7.53) to obtain

$$\sum_{x,c} R_{ac}^{xk}\left(\lambda - \omega(\hat{c}), \frac{z_1}{z_2}\right) S(e_{ic}(z_2)) e_{xb}(z_1) = \sum_{x,d} R_{xi}^{bd}\left(\rho - \omega(\hat{\imath}), \frac{z_1}{z_2}\right) e_{ax}(z_1) S(e_{dk}(z_2)). \tag{7.56}$$

Then multiply from the left by $S(e_{ja}(z_1))$ and from the right by $S(e_{bl}(z_1))$, sum over a, b, and use (7.52), (7.53) again to get

$$\begin{aligned} &\sum_{a,c} R_{ac}^{lk}\left(\lambda - \omega(\hat{a}) - \omega(\hat{c}), \frac{z_1}{z_2}\right) S(e_{ja}(z_1)) S(e_{ic}(z_2)) \\ &\quad = \sum_{b,d} R_{\text{ji}}^{bd}\left(\rho - \omega(\hat{\jmath}) - \omega(\hat{\imath}), \frac{z_1}{z_2}\right) S(e_{dk}(z_2)) S(e_{bl}(z_1)). \end{aligned} \tag{7.57}$$

Since $S(e_{ij}(z)) \in \mathcal{F}_{\mathrm{ell}}(GL(n))_{\hat{j},\hat{\imath}}$ and $R_{ji}^{bd}(\rho - \omega(\hat{j}) - \omega(\hat{\imath}), z_1/z_2) = R_{ji}^{bd}(\rho - \omega(\widehat{b}) - \omega(\widehat{d}), z_1/z_2)$ by the $\mathfrak{h}$-invariance of R, (7.57) can be rewritten

$$\sum_{a,c} S(e_{ja}(z_1)) S(e_{ic}(z_2)) R_{ac}^{lk}\left(\lambda, \frac{z_1}{z_2}\right) = \sum_{b,d} S(e_{dk}(z_2)) S(e_{bl}(z_1)) R_{ji}^{bd}\left(\rho, \frac{z_1}{z_2}\right). \tag{7.58}$$

This is the result of formally applying S to the RLL relations, proving that S preserves (4.2). Similarly (4.5) is preserved.

Step 3. Since, by the above steps, (7.46) holds on the generators of $\mathcal{F}_{\mathrm{ell}}(M(n))$ and $S(ab) = S(b)S(a)$ for all $a, b \in \mathcal{F}_{\mathrm{ell}}(M(n))$, it follows that (7.46) holds for any $a \in \mathcal{F}_{\mathrm{ell}}(M(n))$. By taking in particular $a = \det(z)$ we get

$$\det(z) S(\det(z)) = 1, \qquad S(\det(z)) \det(z) = 1, \tag{7.59}$$

respectively. Thus, definining S on $\det(z)^{-1}$ by (7.51), the relations $\det(z)\det(z)^{-1} = 1 = \det(z)^{-1}\det(z)$ are preserved by S. Hence S extends to an antimultiplicative $\mathbb{C}$-linear map $S : \mathcal{F}_{\mathrm{ell}}(GL(n)) \to \mathcal{F}_{\mathrm{ell}}(GL(n))$ satisfying the antipode axiom (7.46) on $\mathcal{F}_{\mathrm{ell}}(M(n))$ and on $\det(z)^{-1}$. Hence (7.46) holds for any $a \in \mathcal{F}_{\mathrm{ell}}(GL(n))$. □

8. Concluding Remarks

To define the antipode we only needed that $e_{ij}(z)$ commutes with $\det(q^{-2(n-1)}z)$. This can also be proved using the Laplace expansions.

Perhaps one could avoid problems with spectral poles and zeros of the R-matrix by thinking of the algebra as generated by meromorphic sections of a $M_{\mathfrak{h}^* \oplus \mathfrak{h}^*}$-line bundle over the elliptic curve $\mathbb{C}^\times/\{z \sim pz\}$. In this direction we found that the relation $e_{ij}(pz) = q^{\lambda_i - \rho_j} e_{ij}(z)$ respects the RLL relation (here $\mathfrak{h}$ should be the Cartan subalgebra of $\mathfrak{gl}_n$). This relation should then most likely be added to the algebra.

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