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Research Article

Dependent Elements of Derivations on Semiprime Rings

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We characterize dependent elements of a commuting derivation d on a semiprime ring R and investigate a decomposition of R using dependent elements of d. We show that there exist ideals U and V of R such that $U \oplus V$ is an essential ideal of R, $U \cap V = \{0\}$, $d = 0$ on U, $d(V) \subseteq V$, and d acts freely on V.

1. Introduction and Preliminaries

Murray and von Neumann [1] and von Neumann [2] introduced the notion of free action on abelian von Neumann algebras and used it for construction of certain factors. Kallman [3] generalized the notion of free action of automorphisms to von Neumann algebras, not necessarily abelian, by using implicitly the dependent elements of an automorphism. Dependent elements of automorphisms were later studied by Choda et al. [4] in the context of C^*-algebras. Several other authors have studied dependent elements of automorphisms in the context of operator algebras (see [5, 6] and references therein). A brief account of dependent elements in W^*-algebras has also appeared in the book of Strătilă [7].

It is well known that all C^* and von Neumann algebras are semiprime rings; in particular a von Neumann algebra is prime if and only if its centre consists of the scalar multiples of identity [8]. Thus a natural extension of the notion of a dependent element of mappings on a C^*-algebra or von Neumann algebras is the study of this notion in the context of semiprime rings and prime rings.

Laradji and Thaheem [9] initiated the study of dependent elements of endomorphisms of semiprime rings and generalized a number of results of [4] for semiprime rings. Recently, Vukman and Kosi-Ulbl [10] and Vukman [11, 12] have made further study of dependent elements of some mappings on prime and semiprime rings.

On one hand, motivated by the work of Laradji and Thaheem [9], Vukman and Kosi-Ulbl [10], and Vukman [11, 12] on dependent elements of mappings of semiprime rings and on the other hand by the work done by various researchers on commuting derivations on prime and semiprime rings, we investigate some properties, not already investigated, of dependent elements of commuting derivations on semiprime rings. We show that the dependent elements of a commuting derivation of a semiprime ring are central and form a commutative semiprime subring of R. We also show that for a commuting derivation d on a semiprime ring R, there exist ideals U and V of R such that $U \oplus V$ is an essential ideal of R, $U \cap V = \{0\}$, $d = 0$ on U, $d(V) \subseteq V$, and zero is the only dependent element of $d \setminus V$, the restriction of d on V; that is, d acts freely on V.

Throughout, R will represent an associative ring with centre $Z(R)$. The commutator $xy - yx$ will be denoted by $[x, y]$. We will use the basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = y[x, z] + [x, y]z$. Recall that a ring R is semiprime if $aRa = 0$ implies $a = 0$ and is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \to R$ is called a derivation on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. It is called commuting if $[d(x), x] = 0$ for all $x \in R$. Let $a \in R$, then the mapping $d : R \to R$ given by $d(x) = [a, x]$ is a derivation on R. It is called inner derivation on R.

We call an element element $a \in R$ a dependent element of a derivation $d : R \to R$ if $d(x)a = [a, x]a$ holds for all $x \in R$. Following [3], a derivation $d : R \to R$ is said to act freely on R (or a free action) in case zero is the only dependent element of d. It is known that a semiprime ring R has no central nilpotent elements. We will use this fact without any specific reference. For a derivation $d : R \to R$, $D(d)$ denotes the collection of all dependent elements of d.

It is known that the left and right annihilators of an ideal U of a semiprime ring R coincide. It will be denoted by $\text{Ann}(U)$. It is also known that $U \cap \text{Ann}(U) = \{0\}$ and $U \oplus \text{Ann}(U)$ is an essential ideal of R. We will use these facts without any further reference.

We will use the following result in the sequel.

Theorem 1.1 (see [8, Corollary 3.2]). *If d is a commuting inner derivation on a semiprime ring R, then $d = 0$.*

2. Results

We now prove our results.

Theorem 2.1. *Let d be a commuting derivation of a semiprimering R. Then $a \in D(d)$ if and only if $a \in Z(R)$ and $d(x)a = 0$ for all $x \in R$.*

Proof. Let $a \in D(d)$. Then

$$d(x)a = [a, x]a \quad \forall x \in R. \tag{2.1}$$

Replacing x by xy in (2.1), we get $d(xy)a = [a, xy]a$. That is,

$$d(x)ya + xd(y)a = x[a, y]a + [a, x]ya \quad \forall x, y \in R. \tag{2.2}$$

From (2.1) and (2.2), we get

$$d(x)ya = [a, x]ya \quad \forall x, y \in R. \tag{2.3}$$

Multiplying (2.3) by z on the right, we get

$$d(x)yaz = [a, x]yaz. \tag{2.4}$$

Replacing y by yz in (2.3), we get

$$d(x)yza = [a, x]yza. \tag{2.5}$$

Subtracting (2.5) from (2.4), we get $d(x)y(az - za) = [a, x]y(az - za)$, which implies

$$d(x)y[a, z] = [a, x]y[a, z]. \tag{2.6}$$

Multiplying (2.6) by x on the left, we get

$$xd(x)y[a, z] = x[a, x]y[a, z]. \tag{2.7}$$

Replacing y by xy in (2.6), we get

$$d(x)xy[a, z] = [a, x]xy[a, z]. \tag{2.8}$$

Subtracting (2.7) from (2.8), we get

$$[d(x), x]y[a, z] = [[a, x], x]y[a, z]. \tag{2.9}$$

Since d is commuting, therefore from (2.9) we get

$$[[a, x], x]y[a, z] = 0. \tag{2.10}$$

From (2.10), we get

$$[[a, x], x]y[a, z]z = 0. \tag{2.11}$$

Replacing y by yz in (2.10) and then subtracting the result from (2.11), we get $[[a, x], x]y[[a, z], z] = 0$, which implies $[[a, x], x]y[[a, x], x] = 0$. Using semiprimeness of R, from the last relation we get

$$[[a, x], x] = 0 \quad \forall x \in R. \tag{2.12}$$

Thus inner derivation $\psi : R \to R$ defined by $\psi(x) = [a, x]$ is commuting. Hence $\psi(x) = 0$ by Theorem 1.1, which implies $[a, x] = 0$. Thus $a \in Z(R)$. Further from (2.1), we get $d(x)a = 0$.

Conversely, let $a \in Z(R)$ and let $d(x)a = 0$. Then $d(x)a = 0 = [a, x]a$. So $a \in D(d)$. This completes the proof. □

Corollary 2.2. *Let R be a semiprime ring and let d be a commuting derivation of R. Let $a \in D(d)$, then $d(a) = 0$.*

Proof. Since $a \in D(d)$, therefore

$$d(x)a = 0 \quad \forall x \in R. \tag{2.13}$$

Replacing x by $d(x)$ in (2.13), we get

$$d^2(x)a = 0 \quad \forall x \in R. \tag{2.14}$$

From (2.13), we get $d(d(x)a) = d(0) = 0$, which implies $d^2(x)a + d(x)d(a) = 0$. Using (2.14), from the last relation we get

$$d(x)d(a) = 0. \tag{2.15}$$

Replacing x by ax in (2.15) and using (2.15), we get $0 = d(ax)d(a) = d(a)xd(a) + ad(x)d(a) = d(a)xd(a)$. Thus $d(a)xd(a) = 0$ for all $x \in R$. Using semiprimeness of R, from the last equation we get $d(a) = 0$. □

Corollary 2.3. *Let R be a semiprime ring and let d be a commuting derivation of R. Then $D(d)$ is a commutative semiprime subring of R.*

Proof. Let $a, b \in D(d)$. Then by Theorem 2.1$a, b \in Z(R)$, $d(x)a = 0$, and $d(x)b = 0$ for all $x \in R$. Obviously $a - b \in Z(R)$ and $d(x)ab = 0$. So $a - b$ and $ab \in D(d)$. Since $a, b \in Z(R)$, so $ab = ba$. Thus $D(d)$ is a commutative subring of R. To show semiprimeness of $D(d)$, we consider $aD(d)a = 0$, $a \in D(d)$. Then $axa = 0$ for all $x \in D(d)$. In particular $a^3 = 0$, which implies $a = 0$ because R has no central nilpotents. Thus $D(d)$ is a commutative semiprime subring of ring. □

Corollary 2.4. *Let R be a commutative semiprime ring and let d be a derivation of R. Then $D(d)$ is an ideal of R.*

Proof. Since R is commutative, so d is commuting. Let $a, b \in D(d)$. Then by Corollary 2.3, $a - b \in D(d)$. Let $a \in D(d)$ and let $r \in R$. Then $d(x)a = 0$ for all $x \in R$. Thus $d(x)ar = 0$. Since $ar = ra$, so $d(x)ar = d(x)ra = 0$ for all $x \in R$. Hence $ar = ra \in D(d)$. Thus $D(d)$ is an ideal of R. □

Remark 2.5. (i) If R is a semiprime ring and U an ideal of R, then it is easy to verify that U is a semiprime subring of R and $Z(U) \subseteq Z(R)$.

(ii) If d is a commuting derivation on R and $a \in D(d)$, then by Theorem 2.1, $d(x)a = 0$ for all $x \in R$. This implies $0 = d(xy)a = d(x)ya + xd(y)a = d(x)ya$, which gives $d(x)ya = 0$. Thus $ad(x)yad(x) = 0$ for all $x, y \in R$, which by semiprimeness of R implies $ad(x) = 0$.

Theorem 2.6. *Let R be a semiprime ring and let d be a commuting derivation on R. Then there exist ideals U and V of R such that*

(a) *$U \oplus V$ is an essential ideal of R and $U \cap V = \{0\}$,*

(b) *$d = 0$ on U and $d(V) \subseteq V$,*

(c) *$D(d \setminus V) = \{0\}$, where $d \setminus V$ is restriction of d on V. That is, d acts freely on V.*

Proof. (a) Let U be the ideal of R generated by $D(d)$. Let $V = \text{Ann}(U)$. Then V is an ideal of R, $U \oplus V$ is an essential ideal of R, and $U \cap V = \{0\}$.

(b) By Corollary 2.2 and Theorem 2.1, $d(a) = 0$ and $d(x)a = 0$ for all $x \in R$ and $a \in D(d)$. By Remark 2.5(ii) $ad(x) = 0$. Thus $d(ax) = d(a)x + ad(x) = 0$, $d(xa) = d(x)a + xd(a) = 0$, and $d(xay) = d(x)ay + xd(a)y + xad(y) = 0$ for $a \in D(d)$ and $x, y \in R$. Hence $d = 0$ on U.

Let $v \in V = \text{Ann}(U)$. Thus $va = 0$ for all $a \in U$. So $d(va) = d(0) = 0$, which implies $d(v)a + vd(a) = 0$. Thus $d(v)a = 0$ because $d = 0$ on U. So, $d(v) \in \text{Ann}(U) = V$. Hence $d(V) \subseteq V$.

(c) Since V is an ideal of R, so by Remark 2.5(i) V is a semiprime subring of R and $Z(V) \subseteq Z(R)$. Since $d(V) \subseteq V$, so $d \setminus V$ is a derivation on V. Let $c \in V$ be a dependent element of $d \setminus V$, so by Theorem 2.1 and Corollary 2.2 $c \in Z(V) \subseteq Z(R)$, $d \setminus V(c) = 0$, and $d \setminus V(v)c = cd \setminus V(v) = 0$. Let $x \in R$, so $xv \in V$. Thus $d \setminus V(xv)c = 0$, which implies $d(xv)c = 0$. That is, $d(x)vc + xd(v)c = 0$, which implies $d(x)vc = 0$. Since V is an ideal of R, so $cvd(x) \in V$ and $d(x)c \in V$ for all $x \in R$. Replacing v by $cvd(x)$ in $d(x)vc = 0$, we get $d(x)cvd(x)c = 0$. Using semiprimeness of V, we get $d(x)c = 0$. Since $c \in Z(R)$ and $d(x)c = 0$ for all $x \in R$, therefore $c \in D(d) \subseteq U$. So $c \in U$ and $c \in V = \text{Ann}(U)$. Thus $c = 0$. Hence $D(d \setminus V) = \{0\}$. That is, d acts freely on V. □

Since every derivation d on a commutative ring is a commuting derivation and $D(d)$ is an ideal of R by Corollary 2.4, therefore in case of a commutative ring $U = D(d)$ and $V = \text{Ann}(D(d))$. Thus we have the following corollary.

Corollary 2.7. *Let R be a commutative semiprime ring and let d be a derivation on R. Then there exist ideals $U = D(d)$ and $V = Ann(D(d))$ of R such that*

(a) *$U \oplus V$ is an essential ideal of R and $U \cap V = \{0\}$,*

(b) *$d = 0$ on U, $d(V) \subseteq V$,*

(c) *$D(d \setminus V) = \{0\}$, where $d \setminus V$ is restriction of d on V. That is, d acts freely on V.*

The authors are thankful to the referee for suggesting another proof of Theorem 2.6 with different ideals V and U. The ideal V is generated by $d(R)$ and $U = \text{Ann}_R(V)$. The proof suggested by the referee is based on a theorem of Chuang and Lee [13]. The statement of the said theorem and the proof of Theorem 2.6 as suggested by the referee are given below.

Theorem 2.8 (see [13]). *Let R be a semiprime ring with a derivation d and let λ be a left ideal of R. Suppose that $[d(x), x] \in Z(R)$ for all $x \in \lambda$, where $Z(R)$ denotes the center of R. Then $[\lambda, R]d(R) = 0$.*

Now we give the proof of Theorem 2.6 as suggested by the referee.

Proof. By Theorem 2.8, $[R, R]d(R) = 0$. Let V be the ideal of R generated by $d(R)$ and let $U = \text{Ann}_R(V)$. Clearly, $[R, R] \subseteq U$, $U \cap V = 0$, $U \oplus V$ is essential in R, and $d(V) \subseteq V \subseteq Z(R)$.

Since $d(U) \subseteq U \cap V$, we have $d(U) = 0$. Let c be a dependent element of $d \setminus V$. Let $v \in V$. Then $d(v)c = [c, v]c \in UV = 0$. Thus $d(U \oplus V)c = d(V)c = 0$. Since $U \oplus V$ is an essential ideal of the semiprime ring R, we have $d(R)c = 0$, implying that $c \in U$. So $c \in U \cap V = 0$, as asserted. This proves the theorem. □

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