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Research Article

k-Kernel Symmetric Matrices

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In this paper we present equivalent characterizations of k-Kernel symmetric Matrices. Necessary and sufficient conditions are determined for a matrix to be k-Kernel Symmetric. We give some basic results of kernel symmetric matrices. It is shown that k-symmetric implies k-Kernel symmetric but the converse need not be true. We derive some basic properties of k-Kernel symmetric fuzzy matrices. We obtain k-similar and scalar product of a fuzzy matrix.

1. Introduction

Throughout we deal with fuzzy matrices that is, matrices over a fuzzy algebra $\mathcal{F}$ with support $[0,1]$ under max-min operations. For $a, b \in \mathcal{F}$, $a + b = \max\{a, b\}$, $a \cdot b = \min\{a, b\}$, let $\mathcal{F}_{mn}$ be the set of all $m \times n$ matrices over $\mathcal{F}$, in short $\mathcal{F}_{nn}$ is denoted as $\mathcal{F}_n$. For $A \in \mathcal{F}_n$, let A^T, A^+, $R(A)$, $C(A)$, $N(A)$, and $\rho(A)$ denote the transpose, Moore-Penrose inverse, Row space, Column space, Null space, and rank of A, respectively. A is said to be regular if $AXA = A$ has a solution. We denote a solution X of the equation $AXA = A$ by A^- and is called a generalized inverse, in short, g-inverse of A. However $A\{1\}$ denotes the set of all g-inverses of a regular fuzzy matrix A. For a fuzzy matrix A, if A^+ exists, then it coincides with A^T [1, Theorem 3.16]. A fuzzy matrix A is range symmetric if $R(A) = R(A^T)$ and Kernel symmetric if $N(A) = N(A^T) = \{x : xA = 0\}$. It is well known that for complex matrices, the concept of range symmetric and kernel symmetric is identical. For fuzzy matrix $A \in \mathcal{F}_n$, A is range symmetric, that is, $R(A) = R(A^T)$ implies $N(A) = N(A^T)$ but converse needs not be true [2, page 217]. Throughout, let k-be a fixed product of disjoint transpositions in $S_n = 1, 2, \ldots, n$ and, K be the associated permutation matrix. A matrix $A = (a_{ij}) \in \mathcal{F}_n$ is k-Symmetric if $a_{ij} = a_{k(j)k(i)}$ for $i, j = 1$ to n. A theory for k-hermitian matrices over the complex field is developed in [3] and the concept of k-EP matrices as a generalization of k-hermitian and EP (or) equivalently kernel symmetric matrices over the complex field is studied in [4–6].

Further, many of the basic results on k-hermitian and EP matrices are obtained for the k-EP matrices. In this paper we extend the concept of k-Kernel symmetric matrices for fuzzy matrices and characterizations of a k-Kernel symmetric matrix is obtained which includes the result found in [2] as a particular case analogous to that of the results on complex matrices found in [5].

2. Preliminaries

For $x = (x_1, x_2, \ldots, x_n) \in \mathcal{F}_{1\times n}$, let us define the function $\kappa(x) = (x_{k(1)}, x_{k(2)}, \ldots, x_{k(n)})^T \in \mathcal{F}_{n\times 1}$. Since K is involutory, it can be verified that the associated permutation matrix satisfy the following properties.

Since K is a permutation matrix, $KK^T = K^TK = I_n$ and K is an involution, that is, $K^2 = I$, we have $K^T = K$.

(P.1) $K = K^T$, $K^2 = I$, and $\kappa(x) = Kx$ for $A \in \mathcal{F}_n$,

(P.2) $N(A) = N(AK)$,

(P.3) if A^+ exists, then $(KA)^+ = A^+K$ and $(AK)^+ = KA^+$

(P.4) A^+ exist if and only if A^T is a g-inverse of A.

Definition 2.1 (see [2, page 119]). For $A \in \mathcal{F}_n$ is kernel symmetric if $N(A) = N(A^T)$, where $N(A) = \{x/xA = 0 \text{ and } x \in \mathcal{F}_{1\times n}\}$, we will make use of the following results.

Lemma 2.2 (see [2, page 125]). *For $A, B \in \mathcal{F}_n$ and P being a permutation matrix, $N(A) = N(B) \Leftrightarrow N(PAP^T) = N(PBP^T)$*

Theorem 2.3 (see [2, page 127]). *For $A \in \mathcal{F}_n$, the following statements are equivalent:*

(1) *A is Kernel symmetric,*

(2) *PAP^T is Kernel symmetric for some permutation matrix P,*

(3) *there exists a permutation matrix P such that $PAP^T = \left[\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with* $\det D > 0$.

3. k-Kernel Symmetric Matrices

Definition 3.1. A matrix $A \in \mathcal{F}_n$ is said to be k-Kernel symmetric if $N(A) = N(KA^TK)$

Remark 3.2. In particular, when $\kappa(i) = i$ for each $i = 1$ to n, the associated permutation matrix K reduces to the identity matrix and Definition 3.1 reduces to $N(A) = N(A^T)$, that is, A is Kernel symmetric. If A is symmetric, then A is k-Kernel symmetric for all transpositions k in S_n.

Further, A is k-Symmetric implies it is k-kernel symmetric, for $A = KA^TK$ automatically implies $N(A) = N(KA^TK)$. However, converse needs not be true. This is, illustrated in the following example.

Example 3.3. Let

$$A = \begin{bmatrix} 0 & 0 & 0.6 \\ 0.5 & 1 & 0 \\ 0.5 & 0.3 & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$KA^TK = \begin{bmatrix} 0 & 0 & 0.6 \\ 0.3 & 1 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}. \tag{3.1}$$

Therefore, A is not k-symmetric.

For this A, $N(A) = \{0\}$, since A has no zero rows and no zero columns.

$N(KA^TK) = \{0\}$. Hence A is k-Kernel symmetric, but A is not k-symmetric.

Lemma 3.4. *For $A \in \mathcal{F}_n$, A^+ exists if and only if $(KA)^+$ exists.*

Proof. By [1, Theorem 3.16], For $A \in \mathcal{F}_{mn}$ if A^+ exists then $A^+ = A^T$ which implies A^T is a g-inverse of A. Conversely if A^T is a g-inverse of A, then $AA^TA = A \Rightarrow A^TAA^T = A^T$. Hence A^T is a 2 inverse of A. Both AA^T and A^TA are symmetric. Hence $A^T = A^+$:

$$\begin{aligned} A^+\text{exists} &\Longleftrightarrow AA^TA = A \\ &\Longleftrightarrow KAA^TA = KA \\ &\Longleftrightarrow (KA)(KA)^T(KA) = KA \\ &\Longleftrightarrow (KA)^T \in (KA)\{1\} \\ &\Longleftrightarrow (KA)^+, \text{ exists} \quad (\text{By, P.4}). \end{aligned} \tag{3.2}$$

□

For sake of completeness we will state the characterization of k-kernel symmetric fuzzy matrices in the following. The proof directly follows from Definition 3.1 and by (P.2).

Theorem 3.5. *For $A \in \mathcal{F}_n$, the following statements are equivalent:*

(1) *A is k-Kernel symmetric,*

(2) *KA is Kernel symmetric,*

(3) *AK is Kernel symmetric,*

(4) *$N(A^T) = N(KA)$,*

(5) *$N(A) = N((AK)^T)$,*

Lemma 3.6. *Let $A \in \mathcal{F}_n$, then any two of the following conditions imply the other one,*

(1) *A is Kernel symmetric,*

(2) *A is k-Kernel symmetric,*

(3) *$N(A^T) = N((AK)^T)$.*

Proof. However, (1) and (2) ⇒ (3):

$$\begin{aligned} A \text{ is } k\text{-Kernel symmetric} &\Longrightarrow N(A) = N\left(KA^TK\right) \\ &\Longrightarrow N(A) = N\left(KA^T\right) \quad (\text{By, P.2}) \\ \text{Hence, (1) and (2)} &\Longrightarrow N\left(A^T\right) = N(A) = N\left((AK)^T\right). \end{aligned} \tag{3.3}$$

Thus (3) holds.

Also (1) and (3) ⇒ (2):

$$\begin{aligned} A \text{ is Kernel symmetric} &\Longrightarrow N(A) = N\left(A^T\right) \\ \text{Hence, (1) and (3)} &\Longrightarrow N(A) = N\left((AK)^T\right) \\ &\Longrightarrow N(AK) = N\left((AK)^T\right) \quad (\text{By, P.2}) \\ &\Longrightarrow AK \text{ is Kernel symmetric} \\ &\Longrightarrow A \text{ is } k\text{-Kernel symmetric} \quad (\text{by Theorem (3.5)}). \end{aligned} \tag{3.4}$$

Thus (2) holds.

However, (2) and (3) ⇒ (1):

$$\begin{aligned} A \text{ is } k\text{-Kernel symmetric} &\Longrightarrow N(A) = N\left(KA^TK\right) \\ &\Longrightarrow N(A) = N\left((AK)^T\right) \quad (\text{by, P.2}) \\ \text{Hence (2) and (3)} &\Longrightarrow N(A) = N\left(A^T\right). \end{aligned} \tag{3.5}$$

Thus, (1) holds.

Hence, Theorem. □

Toward characterizing a matrix being k-Kernel symmetric, we first prove the following lemma.

Lemma 3.7. *Let* $B = \left[\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right]$, *where* D *is* $r \times r$ *fuzzy matrix with no zero rows and no zero columns, then the following equivalent conditions hold:*

(1) *B is k-Kernel symmetric,*

(2) $N(B^T) = N((BK)^T)$,

(3) $K = \left[\begin{smallmatrix} K_1 & 0 \\ 0 & K_2 \end{smallmatrix}\right]$ *where K_1 and K_2 are permutation matrices of order r and n-r, respectively,*

(4) *$k = k_1k_2$ where k_1 is the product of disjoint transpositions on $S_n = \{1, 2, \ldots, n\}$ leaving $(r+1, r+2, \ldots, n)$ fixed and k_2 is the product of disjoint transposition leaving $(1, 2, \ldots, r)$ fixed.*

Proof. Since D has no zero rows and no zero columns $N(D) = N(D^T) = \{0\}$. Therefore $N(B) = N(B^T) \neq \{0\}$ and B is Kernel symmetric.

Now we will prove the equivalence of (1),(2), and (3). B is k-Kernel symmetric $\Leftrightarrow$ $N(B^T) = N((BK)^T)$ follows from By Lemma (3.6).

Choose $z = [0\ y]$ with each component of $y \neq 0$ and partitioned in conformity with that of $B = \left[\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Clearly, $z \in N(B) = N((B^T)) = N((BK)^T)$. Let us partition K as $K = \left[\begin{smallmatrix} K_1 & K_3 \\ K_3^T & K_2 \end{smallmatrix}\right]$, Then

$$KB^T = \begin{bmatrix} K_1 & K_3 \\ K_3^T & K_2 \end{bmatrix}\begin{bmatrix} D^T & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} K_1D^T & 0 \\ K_3^TD^T & 0 \end{bmatrix}. \tag{3.6}$$

Now

$$\begin{aligned} z = [0\ y] &\in N(B) = N\left(KB^T\right) \\ &\Longrightarrow [0\ \mathrm{y}]\begin{bmatrix} K_1D^T & 0 \\ K_3^TD^T & 0 \end{bmatrix} = 0 \\ &\Longrightarrow yK_3^TD^T = 0 \end{aligned} \tag{3.7}$$

Since $N(D^T) = 0$, it follows that $yK_3^T = 0$.

Since each component of $y \neq 0$ under max-min composition $yK_3^T = 0$, this implies $K_3^T = 0 \Rightarrow K_3 = 0$.

Therefore

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}. \tag{3.8}$$

Thus, (3) holds, Conversely, if (3) holds, then

$$KB^T = \begin{bmatrix} K_1D^T & 0 \\ 0 & 0 \end{bmatrix}, \qquad N\left(KB^T\right) = N(B). \tag{3.9}$$

Thus (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3) holds.

However, (3)$\Leftrightarrow$(4): the equivalence of (3) and (4) is clear from the definition of k. □

Definition 3.8. For $A, B \in \mathcal{F}_n$, A is k-similar to B if there exists a permutation matrix P such that $A = (KP^TK)BP$.

Theorem 3.9. *For $A \in \mathcal{F}_n$ and $k = k_1k_2$ (where k_1k_2 as defined in Lemma 3.7). Then the following are equivalent:*

(1) *A is k-Kernel symmetric of rank r,*

(2) *A is k-similar to a diagonal block matrix $\left[\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right]$ with* $\det D > 0$,

(3) *$A = KGLG^T$ and $L \in \mathcal{F}_r$ with* $\det L > 0$ *and $G^TG = I_r$.*

Proof. (1)⇔(2).

By using Theorem 2.3 and Lemma 3.7 the proof runs as follows.

$$\begin{aligned}
A \text{ is } k\text{-Kernel symmetric} &\Longleftrightarrow KA \text{ is Kernel symmetric}: \\
&\Longleftrightarrow PKAP^T = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \text{ with } \det E > 0 \\
&\quad \text{for some permutation matrix } P \quad (\text{By Theorem } (2.3)) \\
&\Longleftrightarrow A = KP^T \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P \\
&\Longleftrightarrow A = \left(KP^T K\right) K \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P \quad (\text{By P.1}) \\
&\Longleftrightarrow A = KP^T K \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P \\
&\Longleftrightarrow A = KP^T K \begin{bmatrix} K_1 E & 0 \\ 0 & 0 \end{bmatrix} P \\
&\Longleftrightarrow A = KP^T K \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P.
\end{aligned} \tag{3.10}$$

Thus A is k-similar to a diagonal block matrix $\left[\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix}\right]$, where $D = K_1 E$ and $\det D > 0$.

However, (2)⇔(3):

$$\begin{aligned}
A &= KP^T K \begin{bmatrix} K_1 E & 0 \\ 0 & 0 \end{bmatrix} P \\
&= K \begin{bmatrix} P_1^T & P_3^T \\ P_2^T & P_4^T \end{bmatrix} \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \\
&= K \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} K_1 D \begin{bmatrix} P_1 & P_2 \end{bmatrix} \\
&= KGLG^T, \quad \text{where } G = \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix},\ G^T = \begin{bmatrix} P_1 & P_2 \end{bmatrix},\ L = K_1 D \in \mathcal{F}_r \\
G^T G &= \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} = P_1 P_1^T + P_2 P_2^T = I_r, \qquad L \in \mathcal{F}_r.
\end{aligned} \tag{3.11}$$

Hence the Proof. □

Let $x, y \in \mathcal{F}_{1\times n}\dot{A}$ scalar product of x and y is defined by $xy^T = \langle x, y\rangle$. For any subset $S \in \mathcal{F}_{1\times n}$, $S^{\perp} = \{y : \langle x, y\rangle = 0, \text{ for all } x \in S\}$.

Remark 3.10. In particular, when $\kappa(i) = i, K$ reduces to the identity matrix, then Theorem 3.9 reduces to Theorem 2.3. For a complex matrix A, it is well known that $N(A)^{\perp} = R(A^*)$, where $N(A)^{\perp}$ is the orthogonal complement of $N(A)$. However, this fails for a fuzzy matrix hence $C^n = N(A) \oplus R(A)$ this decomposition fails for Kernel fuzzy matrix. Here we shall prove the partial inclusion relation in the following.

Theorem 3.11. *For $A \in \mathcal{F}_n$, if $N(A) \neq \{0\}$, then $R(A^T) \subseteq N(A)^{\perp}$ and $R(A^T) \neq \mathcal{F}_{1\times n}$.*

Proof. Let $x \neq 0 \in N(A)$, since $x \neq 0$, $x_{io} \neq 0$ for atleast one i_o. Suppose $x_i \neq 0$ (say) then under the max-min composition $xA = 0$ implies, the ith row of $A = 0$, therefore, the ith column of $A^T = 0$. If $x \in R(A^T)$, then there exists $y \in \mathcal{F}_{1\times n}$ such that $yA^T = x$. Since ith column of $A^T = 0$, it follows that, ith component of $x = 0$, that is, $x_i = 0$ which is a contradiction. Hence $x \notin R(A^T)$ and $R(A^T) \neq \mathcal{F}_{1\times n}$.

For any $z \in R(A^T)$, $z = yA^T$ for some $y \in \mathcal{F}_{1\times n}$. For any $x \in N(A)$, $xA = 0$ and

$$\begin{aligned} \langle x, z\rangle &= xz^T \\ &= x\left(yA^T\right)^T \\ &= xAy^T \\ &= 0. \end{aligned} \tag{3.12}$$

Therefore, z $\in N(A)^{\perp}$, $R(A^T) \subseteq N(A)^{\perp}$. □

Remark 3.12. We observe that the converse of Theorem 3.11 needs not be true. That is , if $R(A^T) \neq \mathcal{F}_{1\times n}$, then $N(A) \neq \{0\}$ and $N(A)^{\perp} \subseteq R(A^T)$ need not be true. These are illustrated in the following Examples.

Example 3.13. Let

$$A = \begin{bmatrix} 0 & 0 & 0.6 \\ 0.5 & 1 & 0 \\ 0.5 & 0.3 & 0 \end{bmatrix} \tag{3.13}$$

since A has no zero columns, $N(A) = \{0\}$.

For this $A, R(A^T) = \{(x, y, z) : 0 \leq x \leq 0.6, 0 \leq y \leq 1, 0 \leq z \leq 0.5\}$.

Therefore, $R(A^T) \neq \mathcal{F}_{1\times 3}$.

Example 3.14. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.14}$$

For this A,

$$N(\mathrm{A}) = \{(0,0,z) : z \in \mathcal{F}\},$$
$$N(A)^{\perp} = \{(x,y,0) : x,y \in \mathcal{F}\}, \tag{3.15}$$

Here, $R(A^T) = \{(x,y,0) : 0 \leq y \leq x \leq 1\} \neq \mathcal{F}_{1\times 3}$.

Therefore, for $x > y \in \mathcal{F}$, $(x,y,0) \in N(A)^{\perp}$ but $(x,y,0) \notin R(A^T)$.

Therefore, $N(A)^{\perp}$ is not contained in $R(A^T)$.

References

[1] K. H. Kim and F. W. Roush, "Generalized fuzzy matrices," *Fuzzy Sets and Systems*, vol. 4, no. 3, pp. 293–315, 1980.
[2] A. R. Meenakshi, *Fuzzy Matrix: Theory and Applications*, MJP, Chennai, India, 2008.
[3] R. D. Hill and S. R. Waters, "On κ-real and κ-Hermitian matrices," *Linear Algebra and Its Applications*, vol. 169, pp. 17–29, 1992.
[4] T. S. Baskett and I. J. Katz, "Theorems on products of EP_r matrices," *Linear Algebra and Its Applications*, vol. 2, pp. 87–103, 1969.
[5] A. R. Meenakshi and S. Krishnamoorthy, "On κ-EP matrices," *Linear Algebra and Its Applications*, vol. 269, pp. 219–232, 1998.
[6] H. Schwerdtfeger, *Introduction to Linear Algebra and the Theory of Matrices*, P. Noordhoff, Groningen, The Netherlands, 1962.