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Research Article

On Certain Sufficient Condition Involving Gaussian Hypergeometric Functions

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The authors define a new subclass of $\mathcal{A}$ of functions involving complex order in the open unit disk $\mathbb{U}$. For this new class, we obtain certain inclusion properties involving the Gaussian hypergeometric functions.

1. Introduction and Motivation

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C},\ |z| < 1\}. \tag{1.2}$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ $(0 \le \alpha < 1)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \le \alpha < 1). \tag{1.3}$$

This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that are starlike in $\mathbb{U}$ with respect to the origin.

A function $f \in \mathcal{A}$ is said to be convex of order α in $\mathbb{U}$ $(0 \le \alpha < 1)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \le \alpha < 1). \tag{1.4}$$

The class of convex functions is denoted by the class $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, the well-known standard class of convex functions. It is an established fact that

$$f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha). \tag{1.5}$$

A function $f \in \mathcal{A}$ is said to be in the class UCV of uniformly convex functions in $\mathbb{U}$ if f is a normalized convex function in $\mathbb{U}$ and has the property that, for every circular arc δ contained in the unit disk $\mathbb{U}$, with center ζ also in U, the image curve $f(\delta)$ is a convex arc. The function class UCV was introduced by Goodman [1].

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{1.6}$$

Furthermore, we denote by $k - UCV$ and $k - ST$ two interesting subclasses of $\mathcal{S}$ consisting, respectively, of functions which are k-uniformly convex and k-starlike in $\mathbb{U}$. Thus, we have

$$\begin{aligned} k - UCV &:= \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \ (z \in \mathbb{U};\ 0 \le k < \infty) \right\}, \\ k - ST &:= \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \ (z \in \mathbb{U};\ 0 \le k < \infty) \right\}. \end{aligned} \tag{1.7}$$

The class $k - UCV$ was introduced by Kanas and Wiśniowska [2], where its geometric definition and connections with the conic domains were considered. The class $k - ST$ was investigated in [3]. In fact, it is related to the class $k - UCV$ by means of the well-known Alexander equivalence between the usual classes of convex and starlike functions; see also the work of Kanas and Srivastava [4] for further developments involving each of the classes $k - UCV$ and $k - ST$. In particular, when $k = 1$, we obtain

$$1 - UCV \equiv UCV, \qquad 1 - ST = SP, \tag{1.8}$$

where UCV and SP are the familiar classes of uniformly convex functions and parabolic starlike functions in $\mathbb{U}$, respectively (see for details, [1, 5]). In fact, by making use of a certain fractional calculus operator, Srivastava and Mishra [6] presented a systematic and unified study of the classes UCV and SP.

A function $f \in \mathcal{A}$ is said to be in the class $P_\gamma^\tau(A,B) \subset \mathcal{A}$ if it satisfies the inequality

$$\left|\frac{f'(z)+\gamma z f''(z)-1}{(A-B)\tau - B\left[f'(z)+\gamma z f''(z)-1\right]}\right| < 1 \quad \left(z \in \mathbb{U};\ \tau \in \mathbb{C}\setminus\{0\},\ -1 \le B < A \le 1,\ 0 \le \gamma < 1\right). \tag{1.9}$$

The class $P_0^\tau(A,B)$ was introduced earlier by Dixit and Pal [7]. Two of the many interesting subclasses of the class $P_\gamma^\tau(A,B)$ are worthy of mention here. First of all, by setting

$$\gamma = 0, \quad \tau = e^{i\eta}\cos\eta\left(-\frac{\pi}{2} < \eta < \frac{\pi}{2}\right), \quad A = 1-2\beta \quad (0 \le \beta < 1), \quad B = -1, \tag{1.10}$$

the class $P_\gamma^\tau(A,B)$ reduces essentially to the class $R_\eta(\beta)$ introduced and studied by Ponnusamy and Rønning [8], where

$$R_\eta(\beta) = \left\{ f \in \mathcal{A} : \Re\left(e^{i\eta}\left(f'(z)-\beta\right)\right) > 0 \ \left(z \in \mathbb{U};\ -\frac{\pi}{2} < \eta < \frac{\pi}{2},\ 0 \le \beta < 1\right)\right\}. \tag{1.11}$$

Secondly, if we put

$$\gamma = 0, \quad \tau = 1, \quad A = \beta, \quad B = -\beta \quad (0 < \beta \le 1), \tag{1.12}$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \quad (z \in \mathbb{U};\ 0 < \beta \le 1) \tag{1.13}$$

which was studied by (among others) Padmanabhan [9] and Caplinger and Causey [10].

Finally, many of the authors have also studied the class $P_\gamma^1(A,B)$. For details of these works one can refer to the works of Ding Gong [11], R. Singh and S. Singh [12], Owa and Wu [13], and also the references cited by them. Although, many mapping properties of the class $P_\gamma^1(A,B)$ have been studied by these authors, they did not study any mapping properties involving the hypergeometric functions.

The Gaussian hypergeometric function $F(a,b;c,z)$, $z \in \mathbb{U}$ is given by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{1.14}$$

is the solution of the homogeneous hypergeometric differential equation

$$z(1-z)w''(z) + [c-(a+b+1)z]w'(z) - abw(z) = 0 \tag{1.15}$$

and has rich applications in various fields such as conformal mappings, quasiconformal theory, and continued fractions.

Here, a, b, c are complex numbers such that $c \neq 0, -1, -2, -3, \ldots$, $(a)_0 = 1$ for $a \neq 0$, and for each positive integer n, $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer symbol. In the case of $c = -k$, $k = 0, 1, 2, \ldots$, $F(a, b; c; z)$ is defined if $a = -j$ or $b = -j$, where $j \leq k$. In this situation, $F(a, b; c; z)$ becomes a polynomial of degree j in z. Results regarding $F(a, b; c; z)$ when $\Re\ (c - a - b)$ is positive, zero, or negative are abundant in the literature. In particular when $\Re\ (c - a - b) > 0$, the function is bounded. This and the zero balanced case $\Re\ (c - a - b) = 0$ are discussed in detail by many authors (see [14, 15]). The hypergeometric function $F(a, b; c; z)$ has been studied extensively by various authors and it plays an important role in Geometric Function Theory. It is useful in unifying various functions by giving appropriate values to the parameters a, b, and c. We refer to [8, 16–19] and references therein for some important results.

In particular, the close-to-convexity (in turn the univalency), convexity, starlikeness, (for details on these technical terms we refer to [5]), and various other properties of these hypergeometric functions were examined based on the conditions on a, b, and c in [8]. For more interesting properties of hypergeometric functions, one can also refer to [20, 21].

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$ and $g(z)$ univalent. Then we say that $f(z)$ is subordinate to $g(z)$ written as $f(z) \prec g(z)$ if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For $f \in \mathcal{A}$, we recall that the operator $I_{a,b,c}(f)$ of Hohlov [22] which maps $\mathcal{A}$ into itself defined by

$$I_{a,b,c}(f)(z) = zF(a, b; c; z) * f(z), \tag{1.16}$$

where $*$ denotes usual Hadamard product of power series. Therefore, for a function f defined by (1.1), we have

$$I_{a,b,c}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n. \tag{1.17}$$

Using the integral representation,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{dt}{(1-tz)^a}, \quad \Re(c) > R(b) > 0, \tag{1.18}$$

we can write

$$[I_{a,b,c}(f)](z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t} dt * \frac{z}{(1-tz)^a}. \tag{1.19}$$

When $f(z)$ equals the convex function $z/(1-z)$, then the operator $I_{a,b,c}(f)$ in this case becomes $zF(a, b; c; z)$. For $a = 1$, $b = 1 + \delta$, $c = 2 + \delta$ with $\Re(\delta) > -1$ then the convolution operator $I_{a,b,c}(f)$ turns into Bernardi operator

$$B_f(z) = [I_{a,b,c}(f)](z) = \frac{1+\delta}{z^\delta} \int_0^1 t^{\delta-1} f(t) dt. \tag{1.20}$$

Indeed, $I_{1,1,2}(f)$ and $I_{1,2,3}(f)$ are known as Alexander and Libera operators, respectively.

Let $0 \le k < \infty$, and let $f \in \mathcal{A}$ be of the form (1.1). If $f \in k - UCV$, then the following coefficient inequalities hold true (cf. [2]):

$$|a_n| \le \frac{(P_1)_{n-1}}{(1)_n}, \quad n \in \mathbb{N} \setminus \{1\}, \tag{1.21}$$

where $P_1 = P_1(k)$ is the coefficient of z in the function

$$p_k(z) = 1 + \sum_{n=1}^{\infty} P_n(k) z^n, \tag{1.22}$$

which is the extremal function for the class $P(p_k)$ related to the class $k - UCV$ by the range of the expression

$$1 + \frac{zf''(z)}{f'(z)} \quad (z \in \mathbb{U}), \tag{1.23}$$

where $P_1 = P_1(k)$ is given, as above, by (1.22).

Similarly, if f of the form (1.1) belong to the class $k - ST$, then (cf. [3])

$$|a_n| \le \frac{(P_1)_{n-1}}{(1)_{n-1}}, \quad n \in \mathbb{N} \setminus \{1\}, \tag{1.24}$$

where $P_1 = P_1(k)$ is given, as above by (1.22).

2. Properties of $P_\gamma^\tau(A, B)$

Theorem 2.1. *Let $f \in \mathcal{S}$ and be of the form* (1.1). *If $f \in P_\gamma^\tau(A, B)$, then*

$$|a_n| \le \frac{(A - B)|\tau|}{n(1 + \gamma(n - 1))}. \tag{2.1}$$

The estimate is sharp.

Proof. Since $f \in P_\gamma^\tau(A, B)$, we have

$$1 + \frac{1}{\tau}\left[f'(z) + \gamma z f''(z) - 1\right] = \frac{1 + Aw(z)}{1 + Bw(z)}, \tag{2.2}$$

where $w(z)$ is analytic in $\mathbb{U}$ and satisfies the condition $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. Hence, we have

$$\frac{1}{\tau}\left[f'(z) + \gamma z f''(z) - 1\right] = w(z)\left[(A - B) - \frac{B}{\tau}\left(f'(z) + \gamma z f''(z) - 1\right)\right]. \tag{2.3}$$

Using $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $w(z) = \sum_{n=1}^{\infty} b_n z^n$, we have

$$\left\{(A-B) - \frac{B}{\tau}\sum_{n=2}^{\infty}(1+\gamma(n-1))na_n z^{n-1}\right\}\left[\sum_{n=1}^{\infty} b_n z^n\right] = \frac{1}{\tau}\left[\sum_{n=2}^{\infty}(1+\gamma(n-1))na_n z^{n-1}\right]. \tag{2.4}$$

By equating the coefficients, we observe that the coefficient a_n in the right-hand side depends only on $a_2, a_3, \ldots, a_{n-1}$ on the left-hand side of the above expression. This gives

$$\begin{aligned}\left[(A-B) - \frac{B}{\tau}\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]na_n z^{n-1}\right)\right]w(z)\\ = \frac{1}{\tau}\left(\sum_{n=2}^{k}[1+\gamma(n-1)]na_n z^{n-1}\right) + \sum_{n=k+1}^{\infty} d_n z^{n-1}.\end{aligned} \tag{2.5}$$

By using $|w(z)| < 1$, we get

$$\begin{aligned}\left|(A-B) - \frac{B}{\tau}\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]na_n z^{n-1}\right)\right|\\ \geq \left|\frac{1}{\tau}\left(\sum_{n=2}^{k}[1+\gamma(n-1)]na_n z^{n-1}\right) + \sum_{n=k+1}^{\infty} d_n z^{n-1}\right|.\end{aligned} \tag{2.6}$$

Squaring both sides of (2.6) and integrating around $|z| = r$, $0 < r < 1$, we obtain

$$\begin{aligned}(A-B)^2 + \frac{B^2}{|\tau|^2}\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]^2 n^2|a_n|^2 r^{2n-2}\right)\\ \geq \frac{1}{|\tau|^2}\left(\sum_{n=2}^{k}[1+\gamma(n-1)]^2 n^2|a_n|^2 r^{2n-2}\right) + \sum_{n=k+1}^{\infty}|d_n|^2 r^{2n-2}.\end{aligned} \tag{2.7}$$

By letting $r \to 1$, we conclude that

$$(A-B)^2 + \frac{B^2}{|\tau|^2}\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]^2 n^2|a_n|^2\right) \geq \frac{1}{|\tau|^2}\left(\sum_{n=2}^{k}[1+\gamma(n-1)]^2 n^2|a_n|^2\right) \tag{2.8}$$

or

$$\left(\sum_{n=2}^{k}[1+\gamma(n-1)]^2 n^2|a_n|^2\right) \leq (A-B)^2|\tau|^2 + B^2\left(\sum_{n=2}^{k-1}[1+\gamma(n-1)]^2 n^2|a_n|^2\right). \tag{2.9}$$

By making use of the fact that $-1 \leq B < 1$, we get

$$[1+\gamma(n-1)]^2 n^2|a_n|^2 \leq (A-B)^2|\tau|^2. \tag{2.10}$$

This gives

$$|a_n| \leq \frac{(A-B)|\tau|}{n\big(1+\gamma(n-1)\big)}, \quad n = 2, 3, \ldots. \tag{2.11}$$

The result is sharp for the function

$$f(z) = \begin{cases} \dfrac{z}{\tau}\displaystyle\iint_0^1 u^{1/\gamma-1}\frac{1+A\tau(uvz)^{n-1}+B(1-\tau)(uvz)^{n-1}}{1+B(uvz)^{n-1}}\,du\,dv, & \gamma > 0, \\ z\displaystyle\int_0^1 \frac{1+A\tau(uz)^{n-1}+B(1-\tau)(uz)^{n-1}}{1+B(uz)^{n-1}}\,du, & \gamma = 0. \end{cases} \tag{2.12}$$

□

Theorem 2.2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then a sufficient condition for $f \in P_\gamma^\tau(A,B)$ is*

$$\sum_{n=2}^{\infty} n(1+|B|)\big[1+\gamma(n-1)\big]|a_n| \leq (A-B)|\tau|. \tag{2.13}$$

The result is sharp for the function

$$f(z) = z + \frac{(A-B)|\tau|}{n(1+|B|)\big[1+\gamma(n-1)\big]}z^n, \quad n \geq 2. \tag{2.14}$$

Proof. In view of (2.13),

$$\begin{aligned} &\left|\frac{1}{\tau}\left(f'(z)+\gamma z f''(z)-1\right)\right| - \left|(A-B)-\frac{B}{\tau}\left(f'(z)+\gamma z f''(z)-1\right)\right| \\ &\quad= \left|\frac{1}{\tau}\left(1+\sum_{n=2}^{\infty} n a_n z^{n-1} + \gamma\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1} - 1\right)\right| \\ &\qquad - \left|(A-B) - \frac{B}{\tau}\left(1+\sum_{n=2}^{\infty} n a_n z^{n-1} + \gamma\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1} - 1\right)\right| \\ &\quad\leq \frac{1}{|\tau|}\sum_{n=2}^{\infty} n\big(1+\gamma(n-1)\big)|a_n||z|^{n-1} \\ &\qquad - \left\{(A-B) - \frac{|B|}{|\tau|}\left(\sum_{n=2}^{\infty} n\big(1+\gamma(n-1)\big)\right)|a_n||z|^{n-1}\right\} \end{aligned} \tag{2.15}$$

which is clearly less than or equal to zero for all $|z| = r$, $0 < r < 1$. Letting $r \to 1$, we get

$$\left|\frac{f'(z)+\gamma z f''(z)-1}{(A-B)\tau - B\big[f'(z)+\gamma z f''(z)-1\big]}\right| < 1. \tag{2.16}$$

Thus, $f \in P_\gamma^\tau(A,B)$. □

3. Results Involving Gaussian Hypergeometric Function

Theorem 3.1. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + |b| + 2$. Then a sufficient condition for the function $zF(a,b;c;z)$ to be in the class $P_{\gamma}^{\tau}(A,B)$ is that*

$$S \leq \frac{(A-B)|\tau|}{1+|B|} + 1, \tag{3.1}$$

where

$$S = \frac{\Gamma(c)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)} \times \left[\gamma|ab|(1+|a|)(1+|b|) + (1+2\gamma)|ab|(c-|a|-|b|-2) + (c-|a|-|b|-2)(c-|a|-|b|-1)\right]. \tag{3.2}$$

Proof. $zF(a,b;c;z)$ has the series representation given by

$$zF(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n. \tag{3.3}$$

In view of Theorem 2.2, it suffices to show that

$$S(a,b,c,\gamma) := \sum_{n=2}^{\infty} n(1+|B|)(1+\gamma(n-1)) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| \leq (A-B)|\tau|. \tag{3.4}$$

From the fact that $|(a)_n| \leq (|a|)_n$, we observe that c is real and positive, under the hypothesis

$$S(a,b,c,\gamma) \leq \sum_{n=2}^{\infty} n[1+\gamma(n-1)] \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \tag{3.5}$$

By writing $n[1+\gamma(n-1)]$ as, $\gamma(n-1)(n-2) + (n-1)(1+2\gamma) + 1$, we get

$$\begin{aligned} S(a,b,c,\gamma) &\leq \gamma \sum_{n=2}^{\infty} (n-1)(n-2) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &\quad + (1+2\gamma) \sum_{n=2}^{\infty} (n-1) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} + \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= \gamma \sum_{n=3}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-3}} + (1+2\gamma) \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-2}} + \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \end{aligned} \tag{3.6}$$

Using the fact that

$$(a)_n = a(a+1)_{n-1}, \tag{3.7}$$

it is easy to see that

$$
\begin{aligned}
S(a,b,c,\gamma) \le{}& \gamma\frac{|ab|(1+|a|)(1+|b|)}{c(1+c)}\sum_{n=3}^{\infty}\frac{(2+|a|)_{n-3}(2+|b|)_{n-3}}{(2+c)_{n-3}(1)_{n-3}} \\
&+(1+2\gamma)\frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(1+|a|)_{n-2}(1+|b|)_{n-2}}{(1+c)_{n-2}(1)_{n-2}}+\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}.
\end{aligned}
\tag{3.8}
$$

From (1.14),

$$
\begin{aligned}
S(a,b,c,\gamma) \le{}& \gamma\frac{|ab|(1+|a|)(1+|b|)}{c(1+c)}F(2+|a|,2+|b|;2+c;1) \\
&+(1+2\gamma)\frac{|ab|}{c}F(1+|a|,1+|b|;1+c;1)+F(|a|,|b|;c;1)-1.
\end{aligned}
\tag{3.9}
$$

By using the Gauss summation theorem

$$
F(a,b;c;1)=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)},
\tag{3.10}
$$

we get

$$
\begin{aligned}
S(a,b,c,\gamma) \le{}& \gamma\frac{|ab|(1+|a|)(1+|b|)}{c(1+c)}\frac{\Gamma(c-|a|-|b|-2)\Gamma(c+2)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\
&+(1+2\gamma)\frac{|ab|}{c}\frac{\Gamma(c-|a|-|b|-1)\Gamma(c+1)}{\Gamma(c-|a|)\Gamma(c-|b|)}+\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1.
\end{aligned}
\tag{3.11}
$$

Equation (3.4) now follows by an application of (3.1) and (3.2). □

Theorem 3.2. *Let $a,b\in\mathbb{C}\setminus\{0\}$. Also, let c be a real number such that $c>|a|+|b|$. If $f\in P_{\gamma}^{\tau}(A,B)$, and if the inequality*

$$
\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\le\frac{1}{1+|B|}+1
\tag{3.12}
$$

is satisfied, then $zF(a,b;c;z^k)*f(z)\in P_{\gamma}^{\tau}(A,B)$, where $k\in\mathbb{N}$.

Proof. Let f be of the form (1.1) belong to the class $P_{\gamma}^{\tau}(A,B)$. By virtue of Theorem 2.2, it suffices to show that

$$
S_0:=\sum_{n=2}^{\infty}(k(n-1)+1)(1+|B|)(1+\gamma k(n-1))\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_{k(n-1)+1}\right|\le(A-B)|\tau|.
\tag{3.13}
$$

Taking into account inequality (2.1) and the relation $|(a)_{n-1}| \leq (|a|)_{n-1}$, we deduce that

$$\begin{aligned} S_0 &\leq (1+|B|)\sum_{n=2}^{\infty}(k(n-1)+1)(1+\gamma k(n-1))\frac{(A-B)|\tau|}{(k(n-1)+1)(1+\gamma k(n-1))}\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \\ &\leq (1+|B|)(A-B)|\tau|\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= (1+|B|)(A-B)|\tau|(F(|a|,|b|;c;1)-1) \end{aligned} \tag{3.14}$$

which is bounded previously by $(A-B)|\tau|$, in view of inequality (3.12). □

Repeating the previous reasoning for $b = \overline{a}$, we can improve the assertion of Theorem 3.2 as follows.

Theorem 3.3. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > \max\{0, 2\Re(a)\}$. If $f \in P_{\gamma}^{\tau}(A,B)$, and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-2\Re(a))}{\Gamma(c-|a|)\Gamma(c-|\overline{a}|)} \leq \frac{1}{1+|B|}+1 \tag{3.15}$$

is satisfied, then $zF(a,\overline{a};c;z^k) * f(z) \in P_{\gamma}^{\tau}(A,B)$, where $k \in \mathbb{N}$.

In the special case when $b = 1$, Theorem 3.2 immediately yields the following new result.

Theorem 3.4. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a|+1$. If $f \in P_{\gamma}^{\tau}(A,B)$, and if the inequality*

$$\frac{c-1}{c-|a|-1} \leq \frac{1}{1+|B|}+1 \tag{3.16}$$

is satisfied, then $zF(a,1;c;z^k) * f(z) \in P_{\gamma}^{\tau}(A,B)$, where $k \in \mathbb{N}$.

Theorem 3.5. *Let $a,b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a|+|b|+3$. If $f \in \mathcal{S}$, and if the inequality*

$$\begin{aligned} &\frac{|a||b|\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[4+\frac{(1+|a|)(1+|b|)}{c-|a|-|b|-2}\left(5+\frac{(2+|a|)(2+|b|)}{c-|a|-|b|-3}\right)\right] \\ &\quad+\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+3\frac{|a||b|}{c-|a|-|b|-1}+\frac{|a||b|(1+|a|)(1+|b|)}{c(1+c)}\right] \\ &\leq \frac{(A-B)|\tau|}{1+|B|}+1 \end{aligned} \tag{3.17}$$

is satisfied, then $I_{a,b,c}(f) \in P_{\gamma}^{\tau}(A,B)$.

Proof. Let $f \in \mathcal{S}$. Applying the well-known estimate for the coefficients of the functions $f \in \mathcal{S}$, due to de Branges [23], we need to show that

$$\sum_{n=2}^{\infty} n^2 [1+\gamma(n-1)] \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| \le \frac{(A-B)|\tau|}{1+|B|}. \tag{3.18}$$

The left-hand side of (3.18) can be written as

$$\sum_{n=2}^{\infty} n^2 \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| + \gamma \sum_{n=2}^{\infty} n^2(n-1) \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right|. \tag{3.19}$$

The second expression of (3.19), by virtue of the triangle inequality for the pochhammer symbol $|(a)_{n-1}| \le (|a|)_{n-1}$, is less than or equal to

$$\gamma \sum_{n=2}^{\infty} n^2(n-1) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \gamma \sum_{n=1}^{\infty} (n+1)^2 \frac{(|a|)_n(|b|)_n}{(c)_n(1)_{n-1}} =: S_1. \tag{3.20}$$

Now, making use of the relation (3.7), we get

$$\begin{aligned} S_1 &= \gamma \frac{|ab|}{c} \sum_{n=1}^{\infty} (n+1)^2 \frac{(|a|+1)_{n-1}(|b|+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} \\ &= \gamma \frac{|ab|}{c} \sum_{n=0}^{\infty} (n+2)^2 \frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n} \\ &= \gamma \frac{|ab|}{c} \sum_{n=0}^{\infty} n(n-1) \frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n} \\ &\quad + 5\gamma \frac{|ab|}{c} \sum_{n=0}^{\infty} n \frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n} \\ &\quad + 4\gamma \frac{|ab|}{c} \sum_{n=0}^{\infty} \frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n}, \end{aligned} \tag{3.21}$$

where we are writing $(n+2)^2 = n(n-1) + 5n + 4$. By repeating the use of (3.7) and the Gauss summation formula, we have

$$\begin{aligned} S_1 &\le \frac{|ab|(|a|+1)(|b|+1)(|a|+2)(|b|+2)\Gamma(c)\Gamma(c-|a|-|b|-3)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\ &\quad + \frac{5|ab|(|a|+1)(|b|+1)\Gamma(c)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)} + \frac{4|ab|\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)} \\ &= \frac{|ab|\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[4 + \frac{(1+|a|)(1+|b|)}{c-|a|-|b|-2} \left(5 + \frac{(2+|a|)(2+|b|)}{c-|a|-|b|-3} \right) \right]. \end{aligned} \tag{3.22}$$

As a next step, we consider the first expression of equation. By making use of the triangle inequality for the pochhammer symbol as stated in evaluating S_1, we get

$$\sum_{n=2}^{\infty} n^2 \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right| = \sum_{n=0}^{\infty} (n+2)^2 \left| \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+-1}} \right| \le \sum_{n=0}^{\infty} (n+2)^2 \frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} := S_2. \tag{3.23}$$

Now making use of relation (3.7), we obtain

$$S_2 = \sum_{n=0}^{\infty} (n+1)^2 \frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + 2\sum_{n=0}^{\infty} (n+1) \frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \sum_{n=0}^{\infty} \frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}}, \tag{3.24}$$

where we write $(n+2)^2 = (n+1)^2 + 2(n+1) + 1$. By repeating the use of (3.7) and the Gauss summation formula, we have

$$S_2 \le \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)} \left[\frac{|ab|(|a|+1)(|b|+1)}{c(1+c)} + \frac{3|ab|}{c-|a|-|b|} + 1 \right]. \tag{3.25}$$

The proof of Theorem 3.5 now follows by an application of the inequalities of the terms dealing with S_1, S_2 and inequality (3.17). □

Repeating the previous reasoning for $b = \overline{a}$, we can improve the assertion of Theorem 3.5 as follows.

Theorem 3.6. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > \max\{0, 2\Re(a) + 3)\}$. If $f \in \mathcal{S}$, and if the inequality*

$$\frac{|a|^2\Gamma(c)\Gamma(c-2\Re(a)-1)}{\Gamma(c-|a|)\Gamma(c-|\overline{a}|)} \left[4 + \frac{(1+|a|)(1+|\overline{a}|)}{c-2\Re(a)-2} \left(5 + \frac{(2+|a|)(2+|\overline{a}|)}{c-2\Re(a)-3} \right) \right] + \frac{\Gamma(c)\Gamma(c-2\Re(a))}{\Gamma(c-|a|)\Gamma(c-|\overline{a}|)} \left[1 + 3\frac{|a|^2}{c-2\Re(a)-1} + \frac{|a|^2(1+|a|)(1+|\overline{a}|)}{c(1+c)} \right] \le \frac{(A-B)|\tau|}{1+|B|} + 1 \tag{3.26}$$

is satisfied, then $I_{a,\overline{a},c}(f) \in P_\gamma^\tau(A,B)$.

In the special case when $b = 1$, Theorem 3.2 immediately yields a result concerning the Carlson-Shaffer operator $\mathcal{L}(a, c)$.

Theorem 3.7. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + 4$. If $f \in \mathcal{S}$, and if the inequality*

$$\begin{aligned}
&\frac{|a|(c-1)}{(c-|a|-1)(c-|a|-2)}\left[4+\frac{2(1+|a|)}{c-|a|-3}\left(5+\frac{3(2+|a|)}{c-|a|-4}\right)\right] \\
&\quad+\frac{c}{(c-|a|-1)}\left[1+\frac{3|a|}{c-|a|-2}+\frac{2|a|(1+|a|)}{c(1+c)}\right] \\
&\quad\leq \frac{(A-B)|\tau|}{1+|B|}+1
\end{aligned} \tag{3.27}$$

is satisfied, then $\mathcal{L}(a, c)(f) \in P_{\gamma}^{\tau}(A, B)$.

Theorem 3.8. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + |b| + P_1$, where $P_1 = P_1(k)$ is given with* (1.22). *If, for some k $(0 \leq k < \infty)$, $f \in k-UCV$, and the inequality*

$${}_3F_2(|a|,|b|,P_1;c,1;1)+\frac{P_1\gamma|ab|}{c}{}_3F_2(|a|+1,|b|+1,P_1+1;c+1,2;1) \leq \frac{(A-B)|\tau|}{1+|B|}+1 \tag{3.28}$$

is satisfied, then $I_{a,b,c}(f) \in P_{\gamma}^{\tau}(A, B)$.

Proof. By means of (1.17) and (2.13), the following inequality must be satisfied:

$$\sum_{n=2}^{\infty} n[1+\gamma(n-1)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leq \frac{(A-B)|\tau|}{1+|B|}. \tag{3.29}$$

Applying the estimates for the coefficients given by (1.21), and making use of the relations (3.7) and $|(d)_n| \leq (|d|)_n$, condition (3.29) will be satisfied if

$$\begin{aligned}
&(1+|B|)\left[{}_3F_2(|a|,|b|,P_1;c,1;1)-1\right] \\
&\quad+(1+|B|)\frac{P_1\gamma|ab|}{c}{}_3F_2(|a|+1,|b|+1,P_1+1;c+1,2;1) \leq (A-B)|\tau|
\end{aligned} \tag{3.30}$$

provided $c > |a| + |b| + P_1$. The proof of the Theorem 3.8 is now completed by virtue of hypothesis (3.28). □

Theorem 3.9. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a| + |b| + P_1$, where $P_1 = P_1(k)$ is given with (1.22). If, for some k $(0 \leq k < \infty)$, $f \in k - ST$, and the inequality*

$$\begin{aligned} {}_3F_2(|a|, |b|, P_1; c, 1; 1) &+ \frac{P_1(1+\gamma)|ab|}{c} {}_3F_2(|a|+1, |b|+1, P_1+1; c+1, 2; 1) \\ &+ \frac{P_1\gamma|ab|}{c} {}_3F_2(|a|+1, |b|+1, P_1+1; c+1, 1; 1) \leq \frac{(A-B)|\tau|}{1+|B|} + 1 \end{aligned} \tag{3.31}$$

is satisfied, then $I_{a,b,c}(f) \in P_\gamma^\tau(A, B)$.

Proof. Proceeding as in the proof of Theorem 3.8, and applying the estimates for the coefficients given by (1.24) instead of (1.21), and making use of relations (3.7) and $|(d)_n| \leq (|d|)_n$, the proof of the theorem by virtue of hypothesis (3.31) is complete. □

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