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*Research Article*

# Convolution Operators and Bochner-Riesz Means on Herz-Type Hardy Spaces in the Dunkl Setting

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We study the Dunkl convolution operators on Herz-type Hardy spaces $\mathscr{H}^p_{\alpha,2}$ and we establish a version of multiplier theorem for the maximal Bochner-Riesz operators on the Herz-type Hardy spaces $\mathscr{H}^p_{\alpha,\infty}$.

## 1. Introduction

The classical theory of Hardy spaces on $\mathbb{R}^n$ has received an important impetus from the work of Fefferman and Stein, Lu and Yang [1, 2]. Their work resulted in many applications involving sharp estimates for convolution and multiplier operators.

By using the technique of Herz-type Hardy spaces for the Dunkl operator $\Lambda_\alpha$, we are attempting in this paper to study the Dunkl convolution operators, and we establish a version of multiplier theorem for the maximal Bochner-Riesz operators on these spaces.

The Dunkl operator $\Lambda_\alpha$, $\alpha > -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$:

$$\Lambda_\alpha f(x) := \frac{d}{dx}f(x) + \frac{2\alpha+1}{x}\left[\frac{f(x)-f(-x)}{2}\right], \tag{1.1}$$

is the operator devised by Dunkl [3] in connection with a generalization of the classical theory of spherical harmonics. The Dunkl analysis with respect to $\alpha \geq -1/2$ concerns the Dunkl operator $\Lambda_\alpha$, the Dunkl transform $\mathcal{F}_\alpha$, the Dunkl convolution $*_\alpha$, and a certain measure $\mu_\alpha$ on $\mathbb{R}$.

In this paper we define a Herz-type Hardy spaces $\mathscr{H}^p_{\alpha,q}$, $0 < p \leq 1 < q \leq \infty$, in the Dunkl setting. Next, we consider the Dunkl convolution operators $T_k f := k *_\alpha f$, where $k$ is a locally integrable function on $\mathbb{R}$. We use the atomic decomposition of the Herz-type Hardy spaces $\mathcal{H}^{p}_{\alpha,q}$ to study the $\mathcal{H}^{p}_{\alpha,2}-\mathcal{H}^{q}_{\alpha,2}$-bounded and the $\mathcal{H}^{1}_{\alpha,2}-L^1$-bounded of the operators $T_k$. Finally, we establish a version of multiplier theorem for the maximal Bochner-Riesz operators $\sigma^{\eta}_{\alpha}$, $t>0$, $\eta>\alpha+1/2$:

$$\sigma^{\eta}_{\alpha}(f) := \sup_{t>0}\left|\Phi^{\eta}_{\alpha,t} *_{\alpha} f\right|, \tag{1.2}$$

where

$$\Phi^{\eta}_{\alpha,t}(x) := \frac{\Gamma(\eta+1)}{2^{\alpha+1}} t^{2\alpha+2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n! \Gamma(n+\alpha+\eta+2)} (tx)^{2n}, \tag{1.3}$$

on the Herz-type Hardy spaces $\mathcal{H}^{p}_{\alpha,\infty}$. In this version we prove the $\mathcal{H}^{p}_{\alpha,\infty}-L^p$-bounded of the operators $\sigma^{\eta}_{\alpha}$, for $\alpha+1/2<\eta<\alpha+3/2$.

The content of this work is the following. In Section 2, we recall some results about harmonic analysis and we define a Herz-type Hardy spaces $\mathcal{H}^{p}_{\alpha,q}$, $0<p\leq 1<q\leq\infty$ for the Dunkl operator $\Lambda_\alpha$. In Section 3, we study the $\mathcal{H}^{p}_{\alpha,2}-\mathcal{H}^{q}_{\alpha,2}$-bounded and the $\mathcal{H}^{1}_{\alpha,2}-L^1$-bounded of the convolution operators $T_k$. In Section 4, we prove the $\mathcal{H}^{p}_{\alpha,\infty}-L^p$-bounded of the maximal Bochner-Riesz operators $\sigma^{\eta}_{\alpha}$.

Throughout the paper we use the classic notation. Thus $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ are the Schwartz space on $\mathbb{R}$ and the space of tempered distributions on $\mathbb{R}$, respectively. Finally, $C$ will denote a positive constant not necessary the same in each occurrence.

## 2. The Dunkl Harmonic Analysis on $\mathbb{R}$

For $\alpha\geq -1/2$ and $\lambda\in\mathbb{C}$, the initial problem

$$\Lambda_\alpha f(x) = \lambda f(x), \quad f(0)=1, \tag{2.1}$$

has a unique analytic solution $E_\alpha(\lambda x)$ called Dunkl kernel [4–6] given by

$$E_\alpha(\lambda x) = \mathfrak{I}_\alpha(\lambda x) + \frac{\lambda x}{2(\alpha+1)} \mathfrak{I}_{\alpha+1}(\lambda x), \quad x\in\mathbb{R}, \tag{2.2}$$

where

$$\mathfrak{I}_\alpha(\lambda x) := \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(\lambda x)^{2n}}{2^{2n} n!\, \Gamma(n+\alpha+1)}, \tag{2.3}$$

is the modified spherical Bessel function of order $\alpha$.

We notice that, the Dunkl kernel $E_\alpha(\lambda x)$ can be expanded in a power series [7] in the form

$$E_\alpha(\lambda x) = \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{b_n(\alpha)}, \tag{2.4}$$

where

$$b_{2n}(\alpha) = \frac{2^{2n} n!}{\Gamma(\alpha+1)}\Gamma(n+\alpha+1), \qquad b_{2n+1}(\alpha) = 2(\alpha+1)b_{2n}(\alpha+1). \tag{2.5}$$

*Note 1.* Let $\mu_\alpha$ be the measure on $\mathbb{R}$ given by

$$d\mu_\alpha(x) := \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}|x|^{2\alpha+1}dx. \tag{2.6}$$

We denote by $L^p(\mathbb{R},\mu_\alpha)$, $p \in ]0,\infty]$, the space of measurable functions $f$ on $\mathbb{R}$, such that

$$\begin{gathered} \|f\|_{L^p_\alpha} := \left[\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right]^{1/p} < \infty, \quad p \in ]0,\infty[, \\ \|f\|_{L^\infty_\alpha} := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty. \end{gathered} \tag{2.7}$$

The Dunkl kernel gives rise to an integral transform, called Dunkl transform on $\mathbb{R}$, which was introduced by Dunkl in [8], where already many basic properties were established. Dunkl's results were completed and extended later on by de Jeu in [5].

The Dunkl transform of a function $f \in L^1(\mathbb{R},\mu_\alpha)$, is given by

$$\mathcal{F}_\alpha(f)(\lambda) := \int_{\mathbb{R}} E_\alpha(-i\lambda x)f(x)d\mu_\alpha(x), \quad \lambda \in \mathbb{R}. \tag{2.8}$$

For $T \in \mathcal{S}'(\mathbb{R})$, we define the Dunkl transform $\mathcal{F}_\alpha(T)$ of $T$, by

$$\langle \mathcal{F}_\alpha(T), \varphi\rangle := \langle T, \mathcal{F}_\alpha(\varphi)\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}). \tag{2.9}$$

*Note 2.* For all $x, y, z \in \mathbb{R}$, we put

$$W_\alpha(x,y,z) := \{1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}\}\Delta_\alpha(|x|,|y|,|z|), \tag{2.10}$$

where

$$\sigma_{x,y,z} := \begin{cases} \dfrac{x^2+y^2-z^2}{2xy}, & \text{if } x,y \in \mathbb{R}\setminus\{0\} \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta_\alpha(|x|,|y|,|z|) := \begin{cases} d_\alpha \dfrac{\left[\left((|x|+|y|)^2-z^2\right)\left(z^2-(|x|-|y|)^2\right)\right]^{\alpha-1/2}}{|xyz|^{2\alpha}} & \text{if } |z| \in A_{x,y}, \\ 0, & \text{otherwise,} \end{cases} \tag{2.11}$$

$$d_\alpha = \frac{2^{1-\alpha}(\Gamma(\alpha+1))^2}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}, \qquad A_{x,y} = \left[\left||x|-|y|\right|, |x|+|y|\right].$$

We denote by $\nu_{x,y}$ the following signed measure:

$$d\nu_{x,y}(z) := \begin{cases} W_\alpha(x,y,z)\,d\mu_\alpha(z) & \text{if } x,y \in \mathbb{R}\setminus\{0\}, \\ d\delta_x(z) & \text{if } y=0, \\ d\delta_y(z) & \text{if } x=0. \end{cases} \tag{2.12}$$

The Dunkl translation operators $\tau_x$, $x \in \mathbb{R}$ (see [6]) are defined for $f \in \mathcal{C}(\mathbb{R})$ (the space of continuous functions on $\mathbb{R}$), by

$$\tau_x f(y) := \int_{||x|-|y||}^{|x|+|y|} f(z)\,d\nu_{x,y}(z) + \int_{-(|x|+|y|)}^{-||x|-|y||} f(z)\,d\nu_{x,y}(z). \tag{2.13}$$

Let $f$ and $g$ be two functions in $\mathcal{S}(\mathbb{R})$. We define the Dunkl convolution product $*_\alpha$ of $f$ and $g$ by

$$f *_\alpha g(x) := \int_{\mathbb{R}} \tau_x f(-y) g(y)\,d\mu_\alpha(y), \quad x \in \mathbb{R}. \tag{2.14}$$

For $T \in \mathcal{S}'(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$, we define the Dunkl convolution product $T *_\alpha f$ by

$$T *_\alpha f(x) := \langle T(y), \tau_x f(-y)\rangle, \quad x \in \mathbb{R}. \tag{2.15}$$

We begin by recalling the definition of the Herz-type Hardy space in the Dunkl setting. Firstly we introduce a class of fundamental functions that we will call atoms.

Let $0 < p \le 1 < q \le \infty$. A measurable function $a$ on $\mathbb{R}$ is called a $(p,q)$ atom, if $a$ satisfies the following conditions:

(i) there exists $r > 0$ such that $\operatorname{supp}(a) \subset [-r, r]$;

(ii) $\|a\|_{L^q_\alpha} \le r^{-2(\alpha+1)(1/p-1/q)}$, where $r$ is given in (i);

(iii) $\int_{\mathbb{R}} a(x)x^j d\mu_\alpha(x) = 0$, for all $j = 0, 1, \ldots, 2s+1$,

where $s = [(\alpha+1)(1/p-1)]$, (the integer part of $(\alpha+1)(1/p-1)$).

Let $0 < p \le 1 < q \le \infty$. Our Herz-type Hardy space $\mathcal{H}^p_{\alpha,q}$ is constituted by all those $f \in \mathcal{S}'(\mathbb{R})$ that can be represented by

$$f = \sum_{j=0}^{\infty} \lambda_j a_j, \tag{2.16}$$

where $\lambda_j \in \mathbb{C}$ and $a_j$ is a $(p,q)$ atom, for all $j \in \mathbb{N}$, such that $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$ and the series in (2.16) converges in $\mathcal{S}'(\mathbb{R})$.

We define on $\mathcal{H}^p_{\alpha,q}$ the norm $\|\cdot\|_{\mathcal{H}^p_{\alpha,q}}$ by

$$\|f\|_{\mathcal{H}^p_{\alpha,q}} := \inf \left( \sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p}, \tag{2.17}$$

where the infimum is taken over all those sequences $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ such that $f$ is given by (2.16) for certain $(p,q)$ atoms $a_j$, $j \in \mathbb{N}$.

As the same in [7], we prove the following theorem.

**Theorem 2.1.** *Let $0 < p \le 1 < q \le \infty$ and $f \in \mathcal{H}^p_{\alpha,q}$. Then*

$$\left|\mathcal{F}_\alpha(f)(y)\right| \le C\left|y\right|^{2(\alpha+1)(1/p-1)} \|f\|_{\mathcal{H}^p_{\alpha,q}}, \quad y \in \mathbb{R}. \tag{2.18}$$

## 3. Dunkl Convolution Operators on $\mathcal{H}^p_{\alpha,2}$

In the following, we study on $\mathcal{H}^p_{\alpha,2}$, $0 < p \le 1$ the Dunkl convolution operators defined by $T_k f := k *_\alpha f$, where $k$ is a locally integrable function on $\mathbb{R}$.

**Theorem 3.1.** *Let $0 < p \le q \le 1$. Assume that for every $n \in \mathbb{N}$ we are given $\xi_n > 0$ and a function $g_n$ such that*

(i) $g_n(x) = 0, \quad |x| \ge 2^{-n}$,

(ii) $\|g_n\|_{L^1_\alpha} \le \xi_n 2^{2(\alpha+1)(1/q-1/p)n}$,

(iii) $\|x^{2s+2}\mathcal{F}_\alpha(g_n)\|_{L^2_\alpha} \le \xi_n 2^{2(\alpha+1)(1/q-1/2)n}, \quad s = [(\alpha+1)(1/p-1)]$.

*Suppose also that $\sum_{n=0}^{\infty}(\xi_n)^q < \infty$ and define $k = \sum_{n=0}^{\infty} g_n$. Then $T_k$ defines a bounded linear mapping from $\mathcal{H}^p_{\alpha,2}$ into $\mathcal{H}^q_{\alpha,2}$.*

*To prove this theorem the following lemma is needed.*

**Lemma 3.2.** *For $y, z \in \mathbb{R}$ and $j \in \mathbb{N}$, there are a constants $A_{j,\alpha}$, such that*

$$\tau_y\left(x^j\right)(z) = A_{j,\alpha} \sum_{i=0}^{j} \frac{j!}{b_i(\alpha) b_{j-i}(\alpha)} y^i z^{j-i}, \tag{3.1}$$

*where $b_j(\alpha)$ are the constants given by* (2.4).

*Proof.* Let $y, z \in \mathbb{R}$. By dominated convergence theorem, we can write

$$\tau_y\left(x^j\right)(z) = \lim_{t \to 0} \int_{\mathbb{R}} x^j E_\alpha(tx) d\nu_{y,z}(x), \tag{3.2}$$

and by derivation under the integral sign, we get

$$\tau_y\left(x^j\right)(z) = \lim_{t \to 0} \Lambda_{\alpha,\, t}^{j} \int_{\mathbb{R}} E_\alpha(tx) d\nu_{y,z}(x). \tag{3.3}$$

Then, from Theorem 2.4, [6] we obtain

$$\tau_y\left(x^j\right)(z) = \lim_{t \to 0} \Lambda_{\alpha,\, t}^{j} (E_\alpha(ty) E_\alpha(tz)). \tag{3.4}$$

Let $F(t) = E_\alpha(ty) E_\alpha(tz)$ for $|t| < \xi$, where $\xi > 0$. According to (2.11), [9],

$$\Lambda_{\alpha,\, t}^{j} F(t) = F^{(j)}(t) + \sum_{i=1}^{j-1} \left\{ 2^i P_{j-1}\left(t_1^0, \dots, t_i^0\right) F^{(j)}\left(t \cdot t_1^0 \dots t_i^0\right) + 2^j Q_{j-1}\left(t_1^1, \dots, t_{j-1}^1\right) F^{(j)}\left(t \cdot t_1^1 \dots t_j^1\right) \right\}, \tag{3.5}$$

where $P_{j-1}(t_1^0, \dots, t_i^0)$, $i = 1, 2, \dots, j-1$, and $Q_{j-1}(t_1^1, \dots, t_{j-1}^1)$ are polynomials of degree at most $j-1$ with respect to each variable. Moreover,

$$F^{(j)}(t) = \sum_{i=0}^{j} \binom{j}{i} y^i z^{j-i} E_\alpha^{(i)}(ty) E_\alpha^{(j-i)}(tz). \tag{3.6}$$

Then, from (2.4) we deduce

$$\lim_{t \to 0} F^{(j)}(t) = \sum_{i=0}^{j} \binom{j}{i} \frac{i!(j-i)!}{b_i(\alpha) b_{j-i}(\alpha)} y^i z^{j-i} = \sum_{i=0}^{j} \frac{j!}{b_i(\alpha) b_{j-i}(\alpha)} y^i z^{j-i}. \tag{3.7}$$

Therefore,

$$\tau_y\left(x^j\right)(z) = \lim_{t \to 0} \Lambda_{\alpha,\, t}^{j} F(t) = A_{j,\alpha} \sum_{i=0}^{j} \frac{j!}{b_i(\alpha) b_{j-i}(\alpha)} y^i z^{j-i}, \tag{3.8}$$

where

$$A_{j,\alpha} = 1 + \sum_{i=1}^{j-1} \left\{ 2^i P_{j-1}\left(t_1^0, \dots, t_i^0\right) + 2^j Q_{j-1}\left(t_1^1, \dots, t_{j-1}^1\right) \right\}. \tag{3.9}$$

This finishes the proof of the lemma. □

*Proof of Theorem 3.1.* Firstly, notice that $\|g_n\|_{L^1_\alpha} \leq \xi_n$, $n \in \mathbb{N}$. Hence, the series defining $k$ converges in $L^1(\mathbb{R}, \mu_\alpha)$ and $k \in L^1(\mathbb{R}, \mu_\alpha)$.

Let $a$ be a $(p, 2)$ atom. Suppose that $a(x) = 0$, $|x| > r$ and that $\|a\|_{L^2_\alpha} \leq r^{-2(\alpha+1)(1/p-1/2)}$, where $r > 0$. We can write

$$T_k a = \sum_{n=0}^{\infty} g_n *_\alpha a. \tag{3.10}$$

*Step 1.* Let $n \in \mathbb{N}$. From (2.13), for $\left||y| - |z|\right| \geq 2^{-n}$, we have

$$\tau_y g_n(z) = \int_{\left||y|-|z|\right|}^{|y|+|z|} g_n(x) d\nu_{y,z}(x) + \int_{-\left(\left||y|+|z|\right|\right)}^{-\left||y|-|z|\right|} g_n(x) d\nu_{y,z}(x) = 0. \tag{3.11}$$

Hence, for $|y| > r + 2^{-n}$, we deduce

$$g_n *_\alpha a(y) = \int_{-r}^{r} a(-z) \tau_y g_n(z) d\mu_\alpha(z) = 0. \tag{3.12}$$

*Step 2.* Firstly, let us consider that $r \geq 2^{-n}$. From (Proposition 3(i), [10]) and condition (ii) of the theorem, we have

$$\left\|g_n *_\alpha a\right\|_{L^2_\alpha} \leq 4 \left\|g_n\right\|_{L^1_\alpha} \|a\|_{L^2_\alpha} \leq 4\xi_n \left(r + 2^{-n}\right)^{-2(\alpha+1)(1/q-1/2)}. \tag{3.13}$$

Assume now that $r < 2^{-n}$. Since $\int_{\mathbb{R}} a(x) x^j d\mu_\alpha(x) = 0$, $j = 0, 1, \dots, 2s + 1$, with $s = [(\alpha + 1)(1/p - 1)]$, we have

$$g_n *_\alpha a(x) = \int_{\mathbb{R}} a(-y) \left[ \tau_y g_n(x) - \sum_{j=0}^{2s+1} \frac{y^j}{b_j(\alpha)} \Lambda_\alpha^j g_n(x) \right] d\mu_\alpha(y), \quad x \in \mathbb{R}, \tag{3.14}$$

where $b_j(\alpha)$ are the constants given by (2.4).

Using the properties of the Dunkl transform established by de Jeu [5] (see also [7, 10]), we deduce

$$\begin{aligned}\|g_n *_\alpha a\|_{L^2_\alpha} &\leq \int_{\mathbb{R}} |a(-y)| \left\| \tau_y g_n - \sum_{j=0}^{2s+1} \frac{y^j}{b_j(\alpha)} \Lambda_\alpha^j g_n \right\|_{L^2_\alpha} d\mu_\alpha(y) \\ &= \int_{\mathbb{R}} |a(-y)| \left\| \mathcal{F}_\alpha(\tau_y g_n) - \sum_{j=0}^{2s+1} \frac{y^j}{b_j(\alpha)} \mathcal{F}_\alpha\left(\Lambda_\alpha^j g_n\right) \right\|_{L^2_\alpha} d\mu_\alpha(y) \\ &= \int_{\mathbb{R}} |a(-y)| \left\| \left[ E_\alpha(ixy) - \sum_{j=0}^{2s+1} \frac{(ixy)^j}{b_j(\alpha)} \right] \mathcal{F}_\alpha(g_n) \right\|_{L^2_\alpha} d\mu_\alpha(y).\end{aligned} \tag{3.15}$$

According to page 302, [7]

$$\left| E_\alpha(ixy) - \sum_{j=0}^{2s+1} \frac{(ixy)^j}{b_j(\alpha)} \right| \leq \frac{1}{2^{2\alpha}\Gamma(\alpha+1)} |xy|^{2s+2}. \tag{3.16}$$

Using condition (iii) of the theorem and Hölder's inequality, we get

$$\begin{aligned}\|g_n *_\alpha a\|_{L^2_\alpha} &\leq \frac{1}{2^{2\alpha}\Gamma(\alpha+1)} \left\| x^{2s+2} \mathcal{F}_\alpha(g_n) \right\|_{L^2_\alpha} \int_{-r}^{r} |a(y)| y^{2s+2}\, d\mu_\alpha(y) \\ &\leq \frac{1}{2^{2\alpha}\Gamma(\alpha+1)} \left\| x^{2s+2} \mathcal{F}_\alpha(g_n) \right\|_{L^2_\alpha} \|a\|_{L^2_\alpha} \left[ 2 \int_0^r y^{4s+4} d\mu_\alpha(y) \right]^{1/2} \\ &\leq \gamma_n 2^{2(\alpha+1)(1/q-1/2)n} r^{2s-2(\alpha+1)(1/p-1)+2},\end{aligned} \tag{3.17}$$

where

$$\gamma_n := \frac{\xi_n}{2^{2\alpha}\Gamma(\alpha+1)\sqrt{(2s+\alpha+3)2^{\alpha+1}\Gamma(\alpha+1)}} = c\xi_n. \tag{3.18}$$

Using the fact that $s - (\alpha+1)(1/p-1) + 1 > 0$ and $r < 2^{-n}$, we obtain

$$\|g_n *_\alpha a\|_{L^2_\alpha} \leq \gamma_n 2^{2(\alpha+1)(1/q-1/2)n} \leq \gamma_n \left(r + 2^{-n}\right)^{-2(\alpha+1)(1/q-1/2)}. \tag{3.19}$$

*Step 3.* We now prove for all $j = 0, \ldots, 2s+1$; $s = [(\alpha+1)(1/p-1)]$, that

$$\int_{\mathbb{R}} x^j (g_n *_\alpha a)(x) d\mu_\alpha(x) = 0. \tag{3.20}$$

Fubini's theorem and [6] (see also page 20, [10]) lead to

$$\begin{aligned}
&\int_{\mathbb{R}} x^j (g_n *_\alpha a)(x) d\mu_\alpha(x) \\
&\quad = \int_{\mathbb{R}} x^j \int_{\mathbb{R}} a(y) \left[\int_{\mathbb{R}} g_n(z) d\nu_{x,-y}(z)\right] d\mu_\alpha(y) d\mu_\alpha(x) \\
&\quad = \int_{\mathbb{R}} a(y) \int_{\mathbb{R}} g_n(z) \left[\int_{\mathbb{R}} x^j d\nu_{y,z}(x)\right] d\mu_\alpha(z) d\mu_\alpha(y) \\
&\quad = \int_{\mathbb{R}} a(y) \int_{\mathbb{R}} g_n(z) \tau_y\left(x^j\right)(z) d\mu_\alpha(z) d\mu_\alpha(y).
\end{aligned} \tag{3.21}$$

Hence, by Lemma 3.2 and by taking into account that $\int_{\mathbb{R}} a(x) x^j d\mu_\alpha(x) = 0$, $j = 0, 1, \ldots, 2s+1$, with $s = [(\alpha+1)(1/p-1)]$, we get

$$\int_{\mathbb{R}} x^j (g_n *_\alpha a)(x) d\mu_\alpha(x) = A_{j,\alpha} \sum_{i=0}^{j} \frac{j!}{b_i(\alpha) b_{j-i}(\alpha)} \left[\int_{\mathbb{R}} y^i a(y) d\mu_\alpha(y)\right] \left[\int_{\mathbb{R}} g_n(z) z^{j-i} d\mu_\alpha(z)\right] = 0. \tag{3.22}$$

According to the previous three steps, we conclude that $(1/\gamma_n)(g_n *_\alpha a)$ is a $(q, 2)$ atom. Then, $T_k a \in \mathscr{H}^q_{\alpha,2}$, and

$$\|T_k a\|_{\mathscr{H}^q_{\alpha,2}} \leq c\left(\sum_{n=0}^{\infty} (\xi_n)^q\right)^{1/q}. \tag{3.23}$$

Let now $f$ be in $\mathscr{H}^p_{\alpha,2}$. Assume that $f = \sum_{j=0}^{\infty} \lambda_j a_j$, where $\lambda_j \in \mathbb{C}$ and $a_j$ is a $(p, 2)$ atom, for every $j \in \mathbb{N}$, and such that $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. The series defining $f$ converges in $L^1(\mathbb{R}, \mu_\alpha)$. In fact, it is sufficient to note that $\|a\|_{L^1_\alpha} \leq (1/\sqrt{2^\alpha \Gamma(\alpha+1)}) r^{2(\alpha+1)(1-1/p)}$. Hence $f \in L^1(\mathbb{R}, \mu_\alpha)$. Moreover, $k \in L^1(\mathbb{R}, \mu_\alpha)$. Then by (Proposition 3(i), [10]), the operator $T_k$ is bounded from $L^1(\mathbb{R}, \mu_\alpha)$ into itself, and from this, we deduce that $T_k f = \sum_{j=0}^{\infty} \lambda_j T_k a_j$. Using the fact that $\sum_{j=0}^{\infty} |\lambda_j| \leq (\sum_{j=0}^{\infty} |\lambda_j|^p)^{1/p}$, we obtain

$$\|T_k f\|_{\mathscr{H}^q_{\alpha,2}} \leq c\left(\sum_{n=0}^{\infty} (\xi_n)^q\right)^{1/q} \|f\|_{\mathscr{H}^p_{\alpha,2}}. \tag{3.24}$$

This completes the proof of the Theorem 3.1. □

We now study the Dunkl convolution operators $T_k$ on the Herz-type Hardy spaces $\mathscr{H}^1_{\alpha,2}$.

**Theorem 3.3.** *Let $k$ be a locally integrable function on $\mathbb{R}$. Assume that the following three conditions are satisfied:*

(i) *$T_k$ defines a bounded linear operator from $L^2(\mathbb{R}, \mu_\alpha)$ into itself.*

(ii) *$T_k$ defines a bounded linear operator from $L^1(\mathbb{R}, \mu_\alpha)$ into $\mathcal{S}'(\mathbb{R})$.*

(iii) *There exist $A$ and $c > 1$ such that*

$$\int_{|z|>cR} |\tau_x k(z) - k(z)| d\mu_\alpha(z) \leq A, \quad |x| \in (0, R) \ , \ R > 0. \tag{3.25}$$

*Then $T_k$ defines a bounded linear mapping from $\mathcal{H}^1_{\alpha,2}$ into $L^1(\mathbb{R}, \mu_\alpha)$.*

*Proof.* Let $a$ be a $(1,2)$ atom. We choose $r > 0$ such that $\operatorname{supp}(a) \subset [-r, r]$ and $\|a\|_{L^2_\alpha} \leq r^{-(\alpha+1)}$. We can write

$$\int_{\mathbb{R}} |T_k a(x)| d\mu_\alpha(x) = \left[\int_{|x|<cr} + \int_{|x|\geq cr}\right] |T_k a(x)| d\mu_\alpha(x) := I_1 + I_2. \tag{3.26}$$

Here $c > 1$ is the one given in (iii).

From condition (i) of the theorem and Hölder's inequality, we deduce that

$$I_1 \leq \left[\int_{\mathbb{R}} |T_k a(x)|^2 d\mu_\alpha(x)\right]^{1/2} \left[2\int_0^{cr} d\mu_\alpha(x)\right]^{1/2} \leq C\|a\|_{L^2_\alpha} \, r^{\alpha+1} \leq C. \tag{3.27}$$

Also, by taking into account that $\int_{\mathbb{R}} a(y) d\mu_\alpha(y) = 0$, the condition (iii) of the theorem allows us to write

$$\begin{aligned}
I_2 &= \int_{|x|\geq cr} \left| \int_{\mathbb{R}} \tau_x k(y) a(-y) d\mu_\alpha(y) \right| d\mu_\alpha(x) \\
&= \int_{|x|\geq cr} \left| \int_{\mathbb{R}} \{\tau_x k(y) - k(x)\} a(-y) d\mu_\alpha(y) \right| d\mu_\alpha(x) \\
&\leq \int_{-r}^{r} |a(-y)| \int_{|x|\geq cr} |\tau_y k(x) - k(x)| d\mu_\alpha(x) d\mu_\alpha(y) \\
&\leq C \int_{-r}^{r} |a(y)| d\mu_\alpha(y) \leq C\|a\|_{L^2_\alpha} \left[2\int_0^r d\mu_\alpha(y)\right]^{1/2} \leq C.
\end{aligned} \tag{3.28}$$

Hence, it concludes that

$$\|T_k a\|_{L^1_\alpha} \leq C. \tag{3.29}$$

Note that the positive constant $C$ is not depending on the $(1,2)$ atom $a$.

Let now $f$ be in $\mathcal{H}^1_{\alpha,2}$. Then $f \in \mathcal{S}'(\mathbb{R})$ and $f = \sum_{j=0}^{\infty} \lambda_j a_j$, where $\lambda_j \in \mathbb{C}$ and $a_j$ is a $(1,2)$ atom, for every $j \in \mathbb{N}$ and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$.

The series defining $f$ converges in $L^1(\mathbb{R}, \mu_\alpha)$. In fact, it is sufficient to note that $\|a\|_{L^1_\alpha} \leq 1/\sqrt{2^\alpha \Gamma(\alpha+1)}$, for every $(1,2)$ atom $a$. Hence $f \in L^1(\mathbb{R}, \mu_\alpha)$. Then the condition (ii) of the theorem implies that

$$T_k f = \sum_{j=0}^{\infty} \lambda_j T_k a_j. \tag{3.30}$$

By (3.29) the series in (3.30) converges in $L^1(\mathbb{R}, \mu_\alpha)$ and $\|T_k f\|_{L^1_\alpha} \leq C \sum_{j=0}^{\infty} |\lambda_j|$. Hence $\|T_k f\|_{L^1_\alpha} \leq C \|f\|_{\mathcal{H}^1_{\alpha,2}}$. □

## 4. Maximal Bochner-Riesz Operators on $\mathcal{H}^p_{\alpha,\infty}$

The Bochner-Riesz mean $\sigma^\eta_{\alpha,t}$, for $t > 0$ and $\eta > \alpha + 1/2$ associated to the Dunkl transform $\mathcal{F}_\alpha$ is defined by

$$\sigma^\eta_{\alpha,t}(f)(x) := \int_{-t}^{t} \left(1 - \frac{y^2}{t^2}\right)^\eta E_\alpha(ixy) \mathcal{F}_\alpha(f)(y) d\mu_\alpha(y), \quad f \in \mathcal{S}(\mathbb{R}). \tag{4.1}$$

The maximal operator $\sigma^\eta_\alpha$, $\eta > \alpha + 1/2$ associated to the Bochner-Riesz means $\sigma^\eta_{\alpha,t}$, $t > 0$, is defined by

$$\sigma^\eta_\alpha(f) := \sup_{t>0} \left| \sigma^\eta_{\alpha,t}(f) \right|. \tag{4.2}$$

**Lemma 4.1.** *For $t > 0$ and $\eta > \alpha + 1/2$, one has*

(i) $\sigma^\eta_{\alpha,t}(f) = \Phi^\eta_{\alpha,t} *_\alpha f$, *where*

$$\Phi^\eta_{\alpha,t}(x) := \frac{\Gamma(\eta+1)}{2^{\alpha+1}\Gamma(\alpha+\eta+2)} t^{2\alpha+2}\, \mathfrak{I}_{\alpha+\eta+1}(itx). \tag{4.3}$$

*Here $\mathfrak{I}_\alpha$ is the modified spherical Bessel function given by* (2.3).

(ii) *The operator $\sigma^\eta_\alpha$ is bounded from $L^p(\mathbb{R}, \mu_\alpha)$, $1 \leq p \leq \infty$ into itself.*

*Proof.* Let $t > 0$ and $\eta > \alpha + 1/2$.

(i) By taking into account that the functions $|z|^{\alpha+1/2} \mathfrak{I}_\alpha(iz)$ and $\mathfrak{I}_\alpha(iz)$ are bounded on $\mathbb{R}$ it is not hard to see that

$$\left\| \Phi^\eta_{\alpha,t} \right\|_{L^1_\alpha} = \left\| \Phi^\eta_{\alpha,1} \right\|_{L^1_\alpha} = \frac{\Gamma(\eta+1)}{2^{\alpha+1}\Gamma(\alpha+\eta+2)} \int_{\mathbb{R}} \left| \mathfrak{I}_{\alpha+\eta+1}(ix) \right| d\mu_\alpha(x) < \infty. \tag{4.4}$$

On the other hand, from [11], we have

$$\int_0^1 \left(1-y^2\right)^{\eta} \mathfrak{I}_{\alpha}(itxy) y^{2\alpha+1} dy = \frac{\Gamma(\alpha+1)\Gamma(\eta+1)}{2\Gamma(\alpha+\eta+2)} \mathfrak{I}_{\alpha+\eta+1}(itx). \tag{4.5}$$

Thus,

$$\begin{aligned} \Phi^{\eta}_{\alpha,t}(x) &= \frac{t^{2\alpha+2}}{2^{\alpha}\Gamma(\alpha+1)} \int_0^1 \left(1-y^2\right)^{\eta} \mathfrak{I}_{\alpha}(itxy) y^{2\alpha+1} dy \\ &= \int_{-t}^{t} \left(1-\frac{y^2}{t^2}\right)^{\eta} E_{\alpha}(ixy) d\mu_{\alpha}(y). \end{aligned} \tag{4.6}$$

Applying Inversion Theorem[5], we obtain

$$\mathcal{F}_{\alpha}\left(\Phi^{\eta}_{\alpha,t}\right)(y) = \left(1-\frac{y^2}{t^2}\right)^{\eta} \chi_{(-t,t)}(y), \tag{4.7}$$

where $\chi_{(-t,t)}$ is the characteristic function of the set $(-t, t)$. Thus,

$$\sigma^{\eta}_{\alpha,t}(f)(x) = \int_{\mathbb{R}} E_{\alpha}(ixy) \mathcal{F}_{\alpha}\left(\Phi^{\eta}_{\alpha,t}\right)(y) \mathcal{F}_{\alpha}(f)(y) d\mu_{\alpha}(y), \tag{4.8}$$

and from (Proposition 3 (ii), [10]), we deduce that

$$\sigma^{\eta}_{\alpha,t}(f)(x) = \Phi^{\eta}_{\alpha,t} *_{\alpha} f(x). \tag{4.9}$$

(ii) Using (i) and (Proposition 3 (i), [10]), we obtain

$$\left\|\sigma^{\eta}_{\alpha}(f)\right\|_{L^p_{\alpha}} \le 4\left\|\Phi^{\eta}_{\alpha,1}\right\|_{L^1_{\alpha}} \|f\|_{L^p_{\alpha}}. \tag{4.10}$$

This clearly yields the result. □

**Theorem 4.2.** *Let $0 < p \le 1 < q \le \infty$ and $f \in \mathscr{H}^p_{\alpha,q}$. For $t > 0$ and $\eta > \alpha + 1/2$, the operator $\sigma^{\eta}_{\alpha,t}$ extended to a bounded operator from $\mathscr{H}^p_{\alpha,q}$ into $\mathcal{S}'(\mathbb{R})$.*

*Proof.* According to Theorem 2.1, if $f \in \mathscr{H}^p_{\alpha,q}$, then $\sigma^{\eta}_{\alpha,t}(f)$ is in $\mathcal{S}'(\mathbb{R})$ and it is defined by

$$\left\langle \sigma^{\eta}_{\alpha,t}(f), \varphi \right\rangle = \int_{-t}^{t} \left(1-\frac{y^2}{t^2}\right)^{\eta} \mathcal{F}_{\alpha}(f)(y) \mathcal{F}_{\alpha}(\varphi)(y) d\mu_{\alpha}(y), \quad \varphi \in \mathcal{S}(\mathbb{R}). \tag{4.11}$$

Moreover,

$$\left|\left\langle \sigma_{\alpha,t}^{\eta}(f),\varphi\right\rangle\right| \leq C\|f\|_{\mathscr{H}_{\alpha,q}^{p}}\int_{\mathbb{R}}|y|^{2(\alpha+1)(1/p-1)}|\mathcal{F}_{\alpha}(\varphi)(y)|d\mu_{\alpha}(y),\quad \varphi\in\mathcal{S}(\mathbb{R}). \tag{4.12}$$

Hence $\sigma_{\alpha,t}^{\eta}$ is a bounded operator from $\mathscr{H}_{\alpha,q}^{p}$ into $\mathcal{S}'(\mathbb{R})$. □

We now study the behavior of the maximal Bochner-Riesz operator $\sigma_{\alpha}^{\eta}$ on $\mathscr{H}_{\alpha,\infty}^{p}$.

**Theorem 4.3.** *Let $2(\alpha+1)/(\alpha+\eta+3/2) < p \leq 1$. Then the maximal Bochner-Riesz operator $\sigma_{\alpha}^{\eta}$, $\alpha+1/2<\eta<\alpha+3/2$ is bounded from $\mathscr{H}_{\alpha,\infty}^{p}$ into $L^{p}(\mathbb{R},\mu_{\alpha})$.*

*To prove this theorem we need the following lemma.*

**Lemma 4.4.** (i) *For $x,y\in\mathbb{R}$ and $\eta>\alpha+1/2$,*

$$\left|\tau_x\Phi_{\alpha,t}^{\eta}(y)\right| \leq Ct^{\alpha-\eta+1/2}\big||x|-|y|\big|^{-(\alpha+\eta+3/2)}. \tag{4.13}$$

(ii) *For $0<|y|<|x|$ and $\eta>\alpha+1/2$,*

$$\left|\tau_x\Phi_{\alpha,t}^{\eta}(y)-\Phi_{\alpha,t}^{\eta}(x)\right| \leq C|y|t^{\alpha-\eta+3/2}\ \big||x|-|y|\big|^{-(\alpha+\eta+3/2)}. \tag{4.14}$$

*Proof.* (i) Since the function $|z|^{\alpha+1/2}\mathfrak{J}_{\alpha}(iz)$ is bounded on $\mathbb{R}$, it follows that

$$\left|\Phi_{\alpha,t}^{\eta}(z)\right| \leq Ct^{\alpha-\eta+1/2}|z|^{-(\alpha+\eta+3/2)};\quad t>0,\ \ z\in\mathbb{R}. \tag{4.15}$$

According to [6] and the fact that $\Phi_{\alpha,t}^{\eta}$ is even, we obtain

$$\tau_x\Phi_{\alpha,t}^{\eta}(y) = \int_0^{\pi}\Phi_{\alpha,t}^{\eta}\left((x,y)_{\theta}\right)d\rho_{x,y}(\theta), \tag{4.16}$$

where

$$\begin{gathered} d\rho_{x,y}(\theta) := \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\{1-\operatorname{sgn}(xy)\cos\theta\}\sin^{2\alpha}\theta d\theta\ , \\ (x,y)_{\theta} := \sqrt{x^2+y^2-2|xy|\cos\theta}. \end{gathered} \tag{4.17}$$

By (4.15) and using the fact that $(x,y)_{\theta} \geq ||x|-|y||$, we deduce that

$$\begin{aligned} \left|\tau_x\Phi_{\alpha,t}^{\eta}(y)\right| &\leq Ct^{\alpha-\eta+1/2}\int_0^{\pi}(x,y)_{\theta}^{-(\alpha+\eta+3/2)}d\rho_{x,y}(\theta) \\ &\leq Ct^{\alpha-\eta+1/2}\big||x|-|y|\big|^{-(\alpha+\eta+3/2)}. \end{aligned} \tag{4.18}$$

(ii) From [11], we have

$$\frac{d}{dz}\Phi^{\eta}_{\alpha,t}(z) = -\frac{\Gamma(\eta+1)}{2^{\alpha+2}\Gamma(\alpha+\eta+3)}t^{2\alpha+4}\; z\Im_{\alpha+\eta+2}(itz). \tag{4.19}$$

Then, similarly to the proof in (i), we have

$$\left|\frac{d}{dz}\Phi^{\eta}_{\alpha,t}(z)\right| \leq Ct^{\alpha-\eta+3/2}|z|^{-(\alpha+\eta+3/2)}; \quad t>0, \;\; z\in\mathbb{R}, \tag{4.20}$$

$$\tau_x\Phi^{\eta}_{\alpha,t}(y) - \Phi^{\eta}_{\alpha,t}(x) = \int_0^{\pi}\left[\Phi^{\eta}_{\alpha,t}\left((x,y)_\theta\right) - \Phi^{\eta}_{\alpha,t}\left((x,0)_\theta\right)\right]d\rho_{x,y}(\theta), \tag{4.21}$$

which can be written as

$$\begin{aligned}\tau_x\Phi^{\eta}_{\alpha,t}(y) - \Phi^{\eta}_{\alpha,t}(x) &= \int_0^{\pi}\int_0^1 \frac{d}{ds}\left[\Phi^{\eta}_{\alpha,t}\left((x,sy)_\theta\right)\right]ds\,d\rho_{x,y}(\theta) \\ &\leq |y|\int_0^1\int_0^{\pi}\frac{d}{ds}\Phi^{\eta}_{\alpha,t}\left((x,sy)_\theta\right)d\rho_{x,y}(\theta)ds.\end{aligned} \tag{4.22}$$

Then from (4.20), it follows

$$\begin{aligned}\left|\tau_x\Phi^{\eta}_{\alpha,t}(y) - \Phi^{\eta}_{\alpha,t}(x)\right| &\leq C|y|t^{\alpha-\eta+3/2}\int_0^1\int_0^{\pi}(x,sy)_\theta^{-(\alpha+\eta+3/2)}d\rho_{x,y}(\theta)ds \\ &\leq C|y|t^{\alpha-\eta+3/2}\int_0^1\left||x|-s|y|\right|^{-(\alpha+\eta+3/2)}ds.\end{aligned} \tag{4.23}$$

Hence, if $0<|y|<|x|$, we obtain

$$\left|\tau_x\Phi^{\eta}_{\alpha,t}(y) - \Phi^{\eta}_{\alpha,t}(x)\right| \leq C|y|t^{\alpha-\eta+3/2}\left||x|-|y|\right|^{-(\alpha+\eta+3/2)}, \tag{4.24}$$

which completes the proof of the lemma. □

*Proof of Theorem 4.3.* Let us first show that $C>0$, exists such that

$$\left\|\sigma^{\eta}_{\alpha}(a)\right\|_{L^p_\alpha} \leq C, \tag{4.25}$$

for every $(p,\infty)$ atom $a$.

Let $a$ be an $(p,\infty)$ atom. Suppose that $a(x) = 0$, $|x| > r$ and $\|a\|_{L^\infty_\alpha} \le r^{-2(\alpha+1)/p}$. We choose $\ell \in \mathbb{Z}$ such that $2^{\ell-1} < r \le 2^\ell$, we write

$$\int_{|x|\ge 4r} \left|\sigma^\eta_\alpha(a)(x)\right|^p d\mu_\alpha(x) \le I_1 + I_2, \tag{4.26}$$

where

$$I_1 := \sum_{i=1}^{\infty} \int_{(i+1)2^\ell \le |x| \le (i+2)2^\ell} \sup_{t\ge\delta_i} \left|\sigma^\eta_{\alpha,t}(a)(x)\right|^p d\mu_\alpha(x), \tag{4.27}$$

$$I_2 := \sum_{i=1}^{\infty} \int_{(i+1)2^\ell \le |x| \le (i+2)2^\ell} \sup_{t<\delta_i} \left|\sigma^\eta_{\alpha,t}(a)(x)\right|^p d\mu_\alpha(x), \tag{4.28}$$

being $\delta_i = 2^{-\ell}/i^b$, where $b$ will be specified later.

According to Lemma 4.4 (i), for $|x| \in [(i+1)2^\ell, (i+2)2^\ell]$, $i = 1, 2, \ldots$, we get

$$\begin{aligned}\left|\sigma^\eta_{\alpha,t}(a)(x)\right| &\le \int_{-r}^{r} |a(-y)| \left|\tau_x \Phi^\eta_{\alpha,t}(y)\right| d\mu_\alpha(y)\\ &\le C t^{\alpha-\eta+1/2} r^{-2(\alpha+1)/p} \int_0^{2^\ell} \big||x| - |y|\big|^{-(\alpha+\eta+3/2)} d\mu_\alpha(y)\\ &\le C \frac{\left(2^\ell t\right)^{\alpha-\eta+1/2}}{i^{\alpha+\eta+3/2} r^{2(\alpha+1)/p}}.\end{aligned} \tag{4.29}$$

Then, using the fact that $2^{\ell-1} < r \le 2^\ell$, we obtain

$$I_1 \le C \sum_{i=1}^{\infty} \left( \frac{\delta_i^{\alpha-\eta+1/2} 2^{\ell\{\alpha-\eta+1/2-2(\alpha+1)/p\}}}{i^{\alpha+\eta+3/2}} \right)^p i^{2\alpha+1} 2^{2\ell(\alpha+1)}, \tag{4.30}$$

and hence, it concludes that

$$I_1 \le C \sum_{i=1}^{\infty} i^{2\alpha+1-\{\alpha+\eta+3/2+(\alpha-\eta+1/2)b\}p}. \tag{4.31}$$

Note that the last series is convergent provide that $b < (p(\alpha+\eta+3/2) - 2(\alpha+1)/p(\eta-\alpha-1/2))$.

On the other hand, since $\int_{\mathbb{R}} a(y)d\mu_\alpha(y) = 0$, from Lemma 4.4 (ii) we have

$$\begin{aligned}\left|\sigma^{\eta}_{\alpha,t}(a)(x)\right| &\leq \int_{-r}^{r} |a(-y)|\left|\tau_x\Phi^{\eta}_{\alpha,t}(y) - \Phi^{\eta}_{\alpha,t}(x)\right|d\mu_\alpha(y)\\ &\leq Ct^{\alpha-\eta+3/2}r^{-2(\alpha+1)/p}\int_{0}^{2^\ell} \left||x| - |y|\right|^{-(\alpha+\eta+3/2)}|y|d\mu_\alpha(y)\\ &\leq C\frac{(2^\ell t)^{\alpha-\eta+3/2}}{i^{\alpha+\eta+3/2}r^{2(\alpha+1)/p}}.\end{aligned} \tag{4.32}$$

Then, for $|x| \geq 4r$, we obtain

$$I_2 \leq C\sum_{i=1}^{\infty}\left(\frac{\delta_i^{\alpha-\eta+3/2}2^{\ell\{\alpha-\eta+3/2-2(\alpha+1)/p\}}}{i^{\alpha+\eta+3/2}}\right)^p i^{2\alpha+1}2^{2\ell(\alpha+1)}, \tag{4.33}$$

and in fact, we deduce that

$$I_2 \leq C\sum_{i=1}^{\infty} i^{2\alpha+1-\{\alpha+\eta+3/2+(\alpha-\eta+3/2)b\}p}. \tag{4.34}$$

The last series converges provided that $b > \dfrac{p(\alpha+\eta+3/2) - 2(\alpha+1)}{p(\eta-\alpha-3/2)}$.

Note that we can find $b$ such that the series in (4.31) and (4.34) converge if and only if $p > (2(\alpha+1)/\alpha+\eta+3/2)$. By combining (4.31) and (4.34) we show that

$$\int_{|x|\geq 4r}\left|\sigma^{\eta}_{\alpha}(a)(x)\right|^p d\mu_\alpha(x) \leq C, \tag{4.35}$$

for a certain $C > 0$ that is not depending on $a$.

From Lemma 4.1 (ii) and (4.35) we deduce that

$$\begin{aligned}\left\|\sigma^{\eta}_{\alpha}(a)\right\|^p_{L^p_\alpha} &\leq \left[\int_{|x|<4r} + \int_{|x|\geq 4r}\right]\left|\sigma^{\eta}_{\alpha}(a)(x)\right|^p d\mu_\alpha(x)\\ &\leq C\left[\|a\|^p_{L^\infty_k}\int_{|x|<4r} d\mu_\alpha(x) + 1\right] \leq C,\end{aligned} \tag{4.36}$$

that is, (4.25) holds. Let now $f$ be in $\mathscr{H}^p_{\alpha,\infty}$. Assume that $f = \sum_{j=0}^{\infty}\lambda_j a_j$, where the series converges in $\mathcal{S}'(\mathbb{R})$, and for every $j \in \mathbb{N}$, $a_j$ is a $(p,\infty)$ atom and $\lambda_j \in \mathbb{C}$, such that $\sum_{j=0}^{\infty}|\lambda_j|^p < \infty$. From Theorem 4.2, for $0 < p \leq 1$, we can write

$$\sigma^{\eta}_{\alpha,t}(f)(x) = \sum_{j=0}^{\infty}\lambda_j\sigma^{\eta}_{\alpha,t}(a_j)(x); \quad t > 0,\ \ x \in \mathbb{R}. \tag{4.37}$$

Hence, from (4.25) it follows $\|\sigma_\alpha^\eta(f)\|_{L_\alpha^p}^p \leq C\sum_{j=0}^{\infty}|\lambda_j|^p$. Thus we conclude that $\|\sigma_\alpha^\eta(f)\|_{L_\alpha^p} \leq C\|f\|_{\mathscr{H}_{\alpha,\infty}^p}$. □


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