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*Research Article*

# Universal Verma Modules and the Misra-Miwa Fock Space

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The Misra-Miwa $v$-deformed Fock space is a representation of the quantized affine algebra $U_v(\widehat{\mathfrak{sl}}_\ell)$. It has a standard basis indexed by partitions, and the nonzero matrix entries of the action of the Chevalley generators with respect to this basis are powers of $v$. Partitions also index the polynomial Weyl modules for $U_q(\mathfrak{gl}_N)$ as $N$ tends to infinity. We explain how the powers of $v$ which appear in the Misra-Miwa Fock space also appear naturally in the context of Weyl modules. The main tool we use is the Shapovalov determinant for a universal Verma module.

## 1. Introduction

Fock space is an infinite dimensional vector space which is a representation of several important algebras, as described in, for example, [1, Chapter 14]. Here we consider the charge zero part of Fock space, which we denote by $\mathbf{F}$, and its $v$-deformation $\mathbf{F}_v$. The space $\mathbf{F}$ has a standard $\mathbb{Q}$-basis $\{|\mu\rangle \mid \lambda \text{ is a partition}\}$ and $\mathbf{F}_v := \mathbf{F}\otimes_{\mathbb{Q}}\mathbb{Q}(v)$. Following Hayashi [2], Misra and Miwa [3] define an action of the quantized universal enveloping algebra $U_v(\widehat{\mathfrak{sl}}_\ell)$ on $\mathbf{F}_v$. The only nonzero matrix elements $\langle\mu|F_{\bar{i}}|\lambda\rangle$ of the Chevalley generators $F_{\bar{i}}$ in terms of the standard basis occur when $\mu$ is obtained by adding a single $\bar{i}$-colored box to $\lambda$, and these are powers of $v$.

We show that these powers of $v$ also appear naturally in the following context: partitions with at most $N$ parts index polynomial Weyl modules $\Delta(\lambda)$ for the integral quantum group $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$. Let $V$ be the standard $N$ dimensional representation of $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$. If the matrix element $\langle\mu|F_{\bar{i}}|\lambda\rangle$ is nonzero then, for sufficiently large $N$, $(\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}V)\otimes_{\mathcal{A}}\mathbb{Q}(q)$ contains the highest weight vector of weight $\mu$. There is a unique such highest weight vector $v_\mu$ which satisfies a certain triangularity condition with respect to an integral basis of $\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}V$. We show that the matrix element $\langle\mu|F_{\bar{i}}|\lambda\rangle$ is equal to $v^{\mathrm{val}_{\phi_{2\ell}}(v_\mu,v_\mu)}$, where $(\cdot,\cdot)$ is the Shapovalov form and $\mathrm{val}_{\phi_{2\ell}}$ is the valuation at the cyclotomic polynomial $\phi_{2\ell}$.

Our proof is computational, making use of the Shapovalov determinant [4–6]. This is a formula for the determinant of the Shapovalov form on a weight space of a Verma module. The necessary computation is most easily done in terms of the universal Verma modules introduced in the classical case by Kashiwara [7] and studied in the quantum case by Kamita [8]. The statement for Weyl modules is then a straightforward consequence.

Before beginning, let us discuss some related work. In [9], Kleshchev carefully analyzed the $\mathfrak{gl}_{N-1}$ highest weight vectors in a Weyl module for $\mathfrak{gl}_N$ and used this information to give modular branching rules for symmetric group representations. Brundan and Kleshchev [10] have explained that highest weight vectors in the restriction of a Weyl module to $\mathfrak{gl}_{N-1}$ give information about highest weight vectors in a tensor product $\Delta(\lambda)\otimes V$ of a Weyl module with the standard $N$-dimensional representation of $\mathfrak{gl}_N$. Our computations put a new twist on the analysis of the highest weight vectors in $\Delta(\lambda)\otimes V$, as we study them in their "universal" versions and by the use of the Shapovalov determinant. Our techniques can be viewed as an application of the theory of Jantzen [11] as extended to the quantum case by Wiesner [12].

Brundan [13] generalized Kleshchev's [9] techniques and used this information to give modular branching rules for Hecke algebras. As discussed in [14, 15], these branching rules are reflected in the fundamental representation of $\widehat{\mathfrak{sl}}_p$ and its crystal graph, recovering much of the structure of the Misra-Miwa Fock space. Using Hecke algebras at a root of unity, Ryom-Hansen [16] recovered the full $U_v(\widehat{\mathfrak{sl}}_\ell)$ action on Fock space. To complete the picture, one should construct a graded category, where multiplication by $v$ in the $\widehat{\mathfrak{sl}}_\ell$ representation corresponds to a grading shift. Recent work of Brundan-Kleshchev [17] and Ariki [18] explains that one solution to this problem is through the representation theory of Khovanov-Lauda-Rouquier algebras [19, 20]. It would be interesting to explicitly describe the relationship between their category and the present work. Another related construction due to Brundan-Stroppel considers the case when the Fock space is replaced by $\wedge^m V\otimes\wedge^n V$, where $V$ is the natural $\mathfrak{gl}_\infty$ module and $m, n$ are fixed natural numbers.

We would also like to mention very recent work of Peng Shan [21] which independently develops a similar story to the one presented here, but using representations of a quantum Schur algebra where we use representations of $U_\varepsilon(\mathfrak{gl}_N)$. The approach taken there is somewhat different and in particular relies on localization techniques of Beĭlinson and Bernstein [22].

This paper is arranged as follows. Sections 2 and 3 are background on the quantum group $U_q(\mathfrak{gl}_N)$ and the Fock space $\mathbf{F}_v$. Sections 4 and 5 explain universal Verma modules and the Shapovalov determinant. Section 6 contains the statement and proof of our main result relating Fock space and Weyl modules.

## 2. The Quantum Group $U_q(\mathfrak{gl}_N)$ and Its Integral Form $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$

This is a very brief review, intended mainly to fix notation. With slight modifications, the construction in this section works in the generality of symmetrizable Kac-Moody algebras. See [23, Chapters 6 and 9] for details.

### *2.1. The Rational Quantum Group*

$U_q(\mathfrak{gl}_N)$ is the associative algebra over the field of rational functions $\mathbb{Q}(q)$ generated by

$$X_1,\ldots,X_{N-1},\qquad Y_1,\ldots,Y_{N-1},\qquad L_1^{\pm 1},\ldots,L_N^{\pm 1}, \tag{2.1}$$

with relations

$$
\begin{gathered}
L_iL_j = L_jL_i, \qquad L_iL_i^{-1} = L_i^{-1}L_i = 1, \qquad X_iY_j - Y_jX_i = \delta_{i,j}\frac{L_iL_{i+1}^{-1} - L_{i+1}L_i^{-1}}{q-q^{-1}}, \\
L_iX_jL_i^{-1} = \begin{cases} qX_j, & \text{if } i = j, \\ q^{-1}X_j, & \text{if } i = j+1, \\ X_j, & \text{otherwise}, \end{cases} \qquad L_iY_jL_i^{-1} = \begin{cases} q^{-1}Y_j, & \text{if } i = j, \\ qY_j, & \text{if } i = j+1, \\ Y_j, & \text{otherwise}, \end{cases} \\
X_iX_j = X_jX_i, \qquad Y_iY_j = Y_jY_i, \quad \text{if } |i-j| \geq 2, \\
X_i^2X_j - \left(q+q^{-1}\right)X_iX_jX_i + X_jX_i^2 = Y_i^2Y_j - \left(q+q^{-1}\right)Y_iY_jY_i + Y_jY_i^2 = 0, \quad \text{if } |i-j| = 1.
\end{gathered} \tag{2.2}
$$

The algebra $U_q(\mathfrak{gl}_N)$ is a Hopf algebra with coproduct and antipode given by

$$
\begin{gathered}
\Delta(L_i) = L_i \otimes L_i, \qquad S(L_i) = L_i^{-1}, \\
\Delta(X_i) = X_i \otimes L_iL_{i+1}^{-1} + 1 \otimes X_i, \qquad S(X_i) = -X_iL_i^{-1}L_{i+1}, \\
\Delta(Y_i) = Y_i \otimes 1 + L_i^{-1}L_{i+1} \otimes Y_i, \qquad S(Y_i) = -L_iL_{i+1}^{-1}Y_i,
\end{gathered} \tag{2.3}
$$

respectively, (see [23, Section 9.1]).

As a $\mathbb{Q}(q)$-vector space, $U_q(\mathfrak{gl}_N)$ has a triangular decomposition

$$
U_q(g\mathfrak{l}_N) \cong U_q(g\mathfrak{l}_N)^{<0} \otimes U_q(g\mathfrak{l}_N)^0 \otimes U_q(g\mathfrak{l}_N)^{>0}, \tag{2.4}
$$

where the inverse isomorphism is given by multiplication (see [23, Proposition 9.1.3]). Here $U_q(\mathfrak{gl}_N)^{<0}$ is the subalgebra generated by the $Y_i$ for $i = 1, \dots, N-1$, $U_q(\mathfrak{gl}_N)^{>0}$ is the subalgebra generated by the $X_i$ for $i = 1, \dots, N-1$, and $U_q(\mathfrak{gl}_N)^0$ is the subalgebra generated by the $L_i^{\pm 1}$ for $i = 1, \dots, N$.

### *2.2. The Integral Quantum Group*

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. For $n, k \in \mathbb{Z}_{>0}$ and $c \in \mathbb{Z}$, let

$$
[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad x^{(k)} := \frac{x^k}{[k][k-1]\cdots[2][1]}, \qquad \begin{bmatrix} x; c \\ k \end{bmatrix} := \prod_{s=1}^{k} \frac{xq^{c+1-s} - x^{-1}q^{s-1-c}}{q^s - q^{-s}}, \tag{2.5}
$$

in $\mathbb{Q}(q,x)$. The *restricted integral form* $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ is the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{gl}_N)$ generated by $X_i^{(k)}, Y_i^{(k)}, L_i^{\pm 1}$ and $\left[\begin{smallmatrix} L_i;c \\ k \end{smallmatrix}\right]$ for $1 \le i \le N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0}$. As discussed in [24, Section 6], this is an integral form in the sense that

$$U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathbb{Q}(q) = U_q(\mathfrak{gl}_N). \tag{2.6}$$

As with $U_q(\mathfrak{gl}_N)$, the algebra $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ has a triangular decomposition

$$U_q^{\mathcal{A}}(\mathfrak{gl}_N) \cong U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \otimes U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{0} \otimes U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0}, \tag{2.7}$$

where the isomorphism is given by multiplication (see [23, Proposition 9.3.3]). In this case, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0}$ is the subalgebra generated by the $Y_i^{(k)}$, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0}$ is the subalgebra generated by the $X_i^{(k)}$, and $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{0}$ is generated by $L_i^{\pm 1}$ and $\left[\begin{smallmatrix} L_i;c \\ k \end{smallmatrix}\right]$ for $1 \le i \le N$, $c \in \mathbb{Z}$, and $k \in \mathbb{Z}_{>0}$.

### 2.3. Rational Representations

The Lie algebra $\mathfrak{gl}_N = M_N(\mathbb{C})$ of $N \times N$ matrices has standard basis $\{E_{ij} \mid 1 \le i,j \le N\}$, where $E_{ij}$ is the matrix with 1 in position $(i,j)$ and 0 everywhere else. Let $\mathfrak{h} = \operatorname{span}\{E_{11}, E_{22}, \dots, E_{NN}\}$. Let $\varepsilon_i \in \mathfrak{h}^*$ be the weight of $\mathfrak{gl}_N$ given by $\varepsilon_i(E_{jj}) = \delta_{i,j}$. Define

$$\begin{aligned}
\mathfrak{h}_{\mathbb{Z}}^* &: = \{\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_N\varepsilon_N \in \mathfrak{h}^* \mid \lambda_1, \dots, \lambda_N \in Z\}, \\
\left(\mathfrak{h}_{\mathbb{Z}}^*\right)^+ &: = \left\{\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_N\varepsilon_N \in \mathfrak{h}_{\mathbb{Z}}^* \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N\right\}, \\
P^+ &: = \left\{\lambda = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \cdots + \lambda_N\varepsilon_N \in \left(\mathfrak{h}_{\mathbb{Z}}^*\right)^+ \mid \lambda_N \ge 0\right\}, \\
R^+ &: = \left\{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le N\right\}, \\
Q &: = \operatorname{span}_{\mathbb{Z}}(R^+), \qquad Q^+ := \operatorname{span}_{\mathbb{Z}_{\ge 0}}(R^+), \qquad Q^- := \operatorname{span}_{\mathbb{Z}_{\le 0}}(R^+)
\end{aligned} \tag{2.8}$$

to be the set of *integral weights*, the set of *dominant integral weights*, the set of *dominant polynomial weights*, the set of *positive roots*, the *root lattice*, the *positive part of the root lattice*, and the *negative part of the root lattice*, respectively.

For an integral weight $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_N\varepsilon_N$, the *Verma module* $M(\lambda)$ for $U_q(\mathfrak{gl}_N)$ of the highest weight $\lambda$ is

$$M(\lambda) := U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^{\ge 0}} \mathbb{Q}(q)_\lambda, \tag{2.9}$$

where $\mathbb{Q}(q)_\lambda = \operatorname{span}_{\mathbb{Q}(q)}\{v_\lambda\}$ is the one dimensional vector space over $\mathbb{Q}(q)$ with $U_q(\mathfrak{gl}_N)^{\ge 0}$ action given by

$$X_i \cdot v_\lambda = 0, \qquad L_j \cdot v_\lambda = q^{\lambda_j} v_\lambda, \quad \text{for } 1 \le i \le N-1,\ 1 \le j \le N. \tag{2.10}$$

**Theorem 2.1** (see [23, Chapter 10.1]). *If $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)^+$ then $M(\lambda)$ has a unique finite dimensional quotient $\Delta(\lambda)$ and the map $\lambda \mapsto \Delta(\lambda)$ is a bijection between $(\mathfrak{h}_{\mathbb{Z}}^*)^+$, and the set of irreducible finite dimensional $U_q(\mathfrak{gl}_N)$-modules.*

A *singular vector* in a representation of $U_q(\mathfrak{gl}_N)$ is a vector $v$ such that $X_i \cdot v = 0$ for all $i$.

### *2.4. Integral Representations*

The *integral Verma module* $M^{\mathcal{A}}(\lambda)$ is the $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-submodule of $M(\lambda)$ generated by $v_\lambda$. The *integral Weyl module* $\Delta^{\mathcal{A}}(\lambda)$ is the $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-submodule of $\Delta(\lambda)$ generated by $v_\lambda$. Using (2.6) and (2.4),

$$M^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{Q}(q) = M(\lambda), \qquad \Delta^{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} \mathbb{Q}(q) = \Delta(\lambda). \tag{2.11}$$

In general, $\Delta^{\mathcal{A}}(\lambda)$ is not irreducible as a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ module.

## 3. Partitions and Fock Space

We now describe the $v$-deformed Fock space representation of $U_v(\widehat{\mathfrak{sl}}_\ell)$ constructed by Misra and Miwa [3] following work of Hayashi [2]. Our presentation largely follows [25, Chapter 10].

### *3.1. Partitions*

A partition $\lambda$ is a finite length nonincreasing sequence of positive integers. Associated to a partition is its Ferrers diagram. We draw these diagrams as in Figure 1 so that, if $\lambda = (\lambda_1, \ldots, \lambda_N)$, then $\lambda_i$ is the number of boxes in row $i$ (rows run southeast to northwest $\nwarrow$). Say that $\lambda$ is contained in $\mu$ if the diagram for $\lambda$ fits inside the diagram for $\mu$ and let $\mu/\lambda$ be the collection of boxes of $\mu$ that are not in $\lambda$. For each box $b \in \lambda$, the *content* $c(b)$ is the horizontal position of $b$ and the *color* $\overline{c}(b)$ is the residue of $c(b)$ modulo $\ell$. In Figure 1, the numbers $c(b)$ are listed below the diagram. The *size* $|\lambda|$ of a partition $\lambda$ is the total number of boxes in its Ferrers diagram.

The set $P^+$ of dominant polynomial weights from Section 2.3 is naturally identified with partitions with at most $N$ parts. If $\lambda \in P^+$, then

$$\Delta(\lambda) \otimes \Delta(\varepsilon_1) \cong \bigoplus_{\substack{1 \le k \le N \\ \lambda + \varepsilon_k \in P^+}} \Delta(\lambda + \varepsilon_k) \tag{3.1}$$

as $U_q(\mathfrak{gl}_N)$-modules. The diagram of $\lambda + \varepsilon_k$ is obtained from the diagram of $\lambda$ by adding a box on row $k$, and $\Delta(\lambda + \varepsilon_k)$ appears in the sum on the right side of (3.1) if and only if $\lambda + \varepsilon_k$ is a partition. See, for example, [26, Section 6.1, Formula 6.8] for the classical statement and [23, Proposition 10.1.16] for the quantum case.

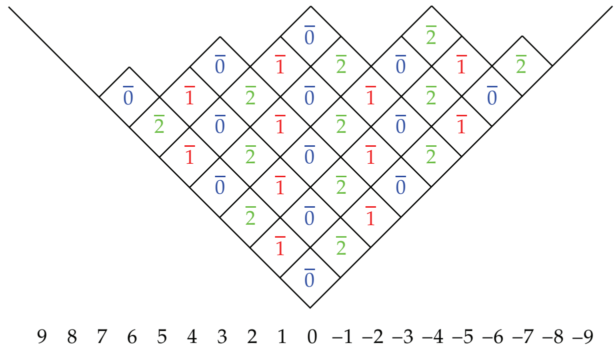
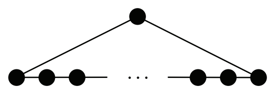



**Figure 1:** The partition $(7,6,6,5,5,3,3,1)$ with each box containing its color for $\ell = 3$. The content $c(b)$ of a box $b$ is the horizontal position of $b$ reading right to left. The contents of boxes are listed beneath the diagram so that $c(b)$ is aligned with all boxes $b$ of that content.

### 3.2. *The Quantum Affine Algebra*

Let $U'_v(\widehat{\mathfrak{sl}}_\ell)$ be the quantized universal enveloping algebra corresponding to the $\ell$-node Dynkin diagram



More precisely, $U'_v(\widehat{\mathfrak{sl}}_\ell)$ is the algebra generated by $E_{\bar{i}}, F_{\bar{i}}, K_{\bar{i}}^{\pm 1}$, for $\bar{i} \in \mathbb{Z}/\ell\mathbb{Z}$, with relations

$$K_{\bar{i}}K_{\bar{j}} = K_{\bar{j}}K_{\bar{i}}, \qquad K_{\bar{i}}K_{\bar{i}}^{-1} = K_{\bar{i}}^{-1}K_{\bar{i}} = 1, \qquad E_{\bar{i}}F_{\bar{j}} - F_{\bar{j}}E_{\bar{i}} = \delta_{\bar{i},\bar{j}}\frac{K_{\bar{i}} - K_{\bar{i}}^{-1}}{v - v^{-1}},$$
$$K_{\bar{i}}E_{\bar{j}}K_{\bar{i}}^{-1} = \begin{cases} v^2 E_{\bar{j}}, & \text{if } \bar{i} = \bar{j}, \\ v^{-1}E_{\bar{j}}, & \text{if } \bar{i} = \bar{j} \pm 1, \\ E_{\bar{j}}, & \text{otherwise}, \end{cases} \qquad K_{\bar{i}}F_{\bar{j}}K_{\bar{i}}^{-1} = \begin{cases} v^{-2} F_{\bar{j}}, & \text{if } \bar{i} = \bar{j}, \\ vF_{\bar{j}}, & \text{if } \bar{i} = \bar{j} \pm 1, \\ F_{\bar{j}}, & \text{otherwise}, \end{cases} \tag{3.2}$$
$$E_{\bar{i}}E_{\bar{j}} = E_{\bar{j}}E_{\bar{i}}, \qquad F_{\bar{i}}F_{\bar{j}} = F_{\bar{j}}F_{\bar{i}}, \quad \text{if } \left|\bar{i} - \bar{j}\right| \geq 2,$$
$$E_{\bar{i}}^2E_{\bar{j}} - \left(v + v^{-1}\right)E_{\bar{i}}E_{\bar{j}}E_{\bar{i}} + E_{\bar{j}}E_{\bar{i}}^2 = F_{\bar{i}}^2F_{\bar{j}} - \left(v + v^{-1}\right)F_{\bar{i}}F_{\bar{j}}F_{\bar{i}} + F_{\bar{j}}F_{\bar{i}}^2 = 0, \quad \text{if } \left|\bar{i} - \bar{j}\right| = 1.$$

See [23, Definition Proposition 9.1.1]. The algebra $U'_v(\widehat{\mathfrak{sl}}_\ell)$ is the quantum group corresponding to the nontrivial central extension $\widehat{\mathfrak{sl}}'_\ell = \mathfrak{sl}_\ell[t, t^{-1}] \oplus \mathbb{C}c$ of the algebra of polynomial loops in $\mathfrak{sl}_\ell$.

### 3.3. Fock Space

Define *v-deformed Fock space* to be the $\mathbb{Q}(v)$ vector space $\mathbf{F}_v$ with basis $\{|\mu\rangle \mid \lambda \text{ is a partition}\}$. Our $\mathbf{F}_v$ is only the charge 0 part of Fock space described in [27]. Fix $\bar{i} \in \mathbb{Z}/\ell\mathbb{Z}$ and partitions $\lambda \subseteq \mu$ such that $\mu/\lambda$ is a single box. Define

$$\begin{aligned}
A_{\bar{i}}(\lambda) &:= \left\{\text{boxes } b \mid b \notin \lambda,\ b \text{ has color } \bar{i} \text{ and } \lambda \cup b \text{ is a partition}\right\},\\
R_{\bar{i}}(\lambda) &:= \left\{\text{boxes } b \mid b \in \lambda,\ b \text{ has color } \bar{i} \text{ and } \lambda \setminus b \text{ is a partition}\right\},\\
N^l_{\bar{i}}(\mu/\lambda) &:= \left|\left\{b \in R_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\right\}\right| - \left|\left\{b \in A_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\right\}\right|,\\
N^r_{\bar{i}}(\mu/\lambda) &:= \left|\left\{b \in R_{\bar{i}}(\lambda) \mid b \text{ is to the right of } \mu/\lambda\right\}\right| - \left|\left\{b \in A_{\bar{i}}(\lambda) \mid b \text{ is to the right of } \mu/\lambda\right\}\right|,
\end{aligned} \tag{3.3}$$

to be the set of *addable boxes of color* $\bar{i}$, the set of *removable boxes of color* $\bar{i}$, the *left removable-addable difference*, and the *right removable-addable difference*, respectively.

**Theorem 3.1** (see [25, Theorem 10.6]). *There is an action of $U'_v(\widehat{\mathfrak{sl}}_\ell)$ on $\mathbf{F}_v$ determined by*

$$E_{\bar{i}}|\lambda\rangle := \sum_{\bar{c}(\lambda/\mu)=\bar{i}} v^{-N^r_{\bar{i}}(\lambda/\mu)}|\mu\rangle, \qquad F_{\bar{i}}|\lambda\rangle := \sum_{\bar{c}(\mu/\lambda)=\bar{i}} v^{N^l_{\bar{i}}(\mu/\lambda)}|\mu\rangle, \tag{3.4}$$

*where $\bar{c}(\lambda/\mu)$ denotes the color of $\lambda/\mu$ and the sum is over partitions $\mu$ which differ from $\lambda$ by removing (resp. adding) a single $\bar{i}$-colored box.*

As a $U'_v(\widehat{\mathfrak{sl}}_\ell)$-module, $\mathbf{F}_v$ is isomorphic to an infinite direct sum of copies of the basic representation $V(\Lambda_0)$. Using the grading of $\mathbf{F}_v$ where $|\lambda\rangle$ has degree $|\lambda|$, the highest weight vectors in $\mathbf{F}_v$ occur in degrees divisible by $\ell$, and the number of the highest weight vectors in degree $\ell k$ is the number of partitions of $k$. Then, $\mathbf{F}_v \cong V(\Lambda_0) \otimes \mathbb{C}[x_1, x_2, \ldots]$, where $x_k$ has degree $\ell k$, and $U'_v(\widehat{\mathfrak{sl}}_\ell)$ acts trivially on the second factor (see [27, Proposition 2.3]). Note that we are working with the "derived" quantum group $U'_v(\widehat{\mathfrak{sl}}_\ell)$, not the "full" quantum group $U_v(\widehat{\mathfrak{sl}}_\ell)$, which is why there are no $\delta$-shifts in the summands of $\mathbf{F}_v$.

*Comment 1.* Comparing with [25, Chapter 10], our $N^l_{\bar{i}}(\mu/\lambda)$ is equal to Ariki's $-N^a_{\bar{i}}(\mu/\lambda)$ and our $N^r_{\bar{i}}(\mu/\lambda)$ is equal to Ariki's $-N^b_{\bar{i}}(\mu/\lambda)$. However, these numbers play a slightly different role in Ariki's work, which is explained by a different choice of conventions.

## 4. Universal Verma Modules

The purpose of this section is to construct a family of representations which are universal Verma modules in the sense that each can be "evaluated" to obtain any given Verma module. This notion was defined by Kashiwara [7] in the classical case and was studied in the quantum case by Kamita [8].

### *4.1. Rational Universal Verma Modules*

Let $\mathbb{K} := \mathbb{Q}(q, z_1, z_2, \ldots, z_N)$. This field is isomorphic to the field of fractions of $U_q(\mathfrak{gl}_N)^0$ via the map

$$\psi : U_q(\mathfrak{gl}_N)^0 \longrightarrow \mathbb{K}, \quad \text{defined by } \psi\left(L_i^{\pm 1}\right) = z_i^{\pm 1}. \tag{4.1}$$

For each $\mu \in \mathfrak{h}_{\mathbb{Z}}^*$, define a $\mathbb{Q}(q)$-linear automorphism $\sigma_\mu : \mathbb{K} \to \mathbb{K}$ by

$$\sigma_\mu(z_i) := q^{(\mu, \varepsilon_i)} z_i, \quad \text{for } 1 \le i \le N, \tag{4.2}$$

where $(\cdot, \cdot)$ is the inner product on $\mathfrak{h}_{\mathbb{Z}}^*$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. Let $\mathbb{K}_\mu = \operatorname{span}_{\mathbb{K}}\{v_{\mu+}\}$ be the one-dimensional vector space over $\mathbb{K}$ with basis vector $v_\mu^+$ and $U_q(\mathfrak{gl}_N)^{\ge 0}$ action given by

$$X_i \cdot v_{\mu+} = 0, \quad \text{for } 1 \le i \le N-1, \quad a \cdot v_{\mu+} = \sigma_\mu(\psi(a)) v_{\mu+}, \ \text{ for } a \in U_q(\mathfrak{gl}_N)^0. \tag{4.3}$$

The *$\mu$-shifted rational universal Verma module* ${}^\mu\widetilde{M}$ is the $U_q(\mathfrak{gl}_N)$-module

$${}^\mu\widetilde{M} := U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^{\ge 0}} \mathbb{K}_\mu. \tag{4.4}$$

The universal Verma module ${}^\mu\widetilde{M}$ is actually a module over $U_q(\mathfrak{gl}_N) \otimes_{U_q(\mathfrak{gl}_N)^0} \widetilde{U}_q(\mathfrak{gl}_N)^0$, where $\widetilde{U}_q(\mathfrak{gl}_N)^0$ is the field of fractions of $U_q(\mathfrak{gl}_N)^0$. However, if we identify $\widetilde{U}_q(\mathfrak{gl}_N)^0$ with $\mathbb{K}$ using the map $\psi$, the action of $\widetilde{U}_q(\mathfrak{gl}_N)^0$ on ${}^\mu\widetilde{M}$ is not by multiplication, but rather is twisted by the automorphism $\sigma_\mu$. It is to keep track of the difference between the action of $U_q(\mathfrak{gl}_N)^0$ and multiplication that we use different notation for the generators of $\mathbb{K}$ and $U_q(\mathfrak{gl}_N)^0$ (i.e., $z_i$ versus $L_i$).

### *4.2. Integral Universal Verma Modules*

The field $\mathbb{K}$ contains an $\mathcal{A}$-subalgebra

$$\mathcal{R} \text{ generated by } z_i^{\pm 1}, \qquad \begin{bmatrix} z_i; c \\ k \end{bmatrix}, \quad (1 \le i \le N, c \in \mathbb{Z}, k \in \mathbb{Z}_{>0}), \tag{4.5}$$

which is isomorphic to $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^0$ via the restriction of the map $\psi$ in (4.1). The *integral universal Verma module* ${}^\mu\widetilde{M}^{\mathcal{R}}$ is the $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-submodule of ${}^\mu\widetilde{M}$ generated by $v_{\mu+}$. By restricting (4.4),

$${}^\mu\widetilde{M}^{\mathcal{R}} = U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{\ge 0}} \mathcal{R}_\mu, \tag{4.6}$$

where $\mathcal{R}_\mu$ is the $\mathcal{R}$-submodule of $\mathbb{K}_\mu$ spanned by $v_{\mu+}$. In particular, ${}^\mu\widetilde{M}^{\mathcal{R}}$ is a free $\mathcal{R}$-module.

### *4.3. Evaluation*

Let $\mathrm{ev}_\lambda^{\mathcal{R}} : \mathcal{R} \to \mathcal{A}$ be the map defined by

$$\mathrm{ev}_\lambda^{\mathcal{R}}(z_i) = q^{(\lambda,\varepsilon_i)}, \qquad \mathrm{ev}_\lambda^{\mathcal{R}}\begin{bmatrix} z_i; c \\ n \end{bmatrix} = \begin{bmatrix} q^{(\lambda,\varepsilon_i)}; c \\ n \end{bmatrix}, \tag{4.7}$$

where $(\cdot,\cdot)$ is the inner product on $\mathfrak{h}^*$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$. There is a surjective $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-module homomorphism "evaluation at $\lambda$"

$$\mathrm{ev}_\lambda : {}^{\mu}\widetilde{M}^{\mathcal{R}} \longrightarrow M^{\mathcal{A}}(\mu+\lambda) \text{ defined by } \mathrm{ev}_\lambda(a \cdot v_{\mu+}) := a \cdot v_{\mu+\lambda}, \quad \forall a \in U_q^{\mathcal{A}}(\mathfrak{gl}_N). \tag{4.8}$$

For fixed $\lambda$, the maps $\mathrm{ev}_\lambda^{\mathcal{R}}$ and $\mathrm{ev}_\lambda$ extend to a map from the subspace of $\mathbb{K}$ and ${}^{\mu}\widetilde{M} = {}^{\mu}\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{R}} \mathbb{K}$, respectively, where no denominators evaluate to 0. Where it is clear we denote both these extended maps by $\mathrm{ev}_\lambda$.

*Example 4.1.* Computing the action of $L_i$ on $v_{\mu+}$ and $v_{\mu+\lambda}$,

$$L_i \cdot v_{\mu+} = q^{(\mu,\varepsilon_i)} z_i v_{\mu+}, \qquad L_i \cdot v_{\mu+\lambda} = \mathrm{ev}_\lambda\left(q^{(\mu,\varepsilon_i)} z_i\right) v_{\mu+\lambda} = q^{(\mu,\varepsilon_i)} q^{(\lambda,\varepsilon_i)} v_{\mu+\lambda} = q^{(\mu+\lambda,\varepsilon_i)} v_{\mu+\lambda}. \tag{4.9}$$

### *4.4. Weight Decompositions*

Let $\widetilde{V}$ be a $U_q(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module. For each $\nu \in \mathfrak{h}_Z^*$, we define the *$\nu$-weight space* of $\widetilde{V}$ to be

$$\widetilde{V}_\nu := \left\{ v \in \widetilde{V} : L_i \cdot v = q^{(\nu,\varepsilon_i)} z_i v \right\}. \tag{4.10}$$

The universal Verma module ${}^{\mu}\widetilde{M}^{\mathcal{R}}$ is a $U_q(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module, where the second factor acts as multiplication. The weight space ${}^{\mu}\widetilde{M}_\eta \neq 0$ if and only if $\eta = \mu - \nu$ with $\nu$ in the positive part $Q^+$ of the root lattice. These nonzero weight spaces and the weight decomposition of ${}^{\mu}\widetilde{M}$ can be described explicitly by

$${}^{\mu}\widetilde{M}^{\mathcal{R}}_{\mu-\nu} = U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0}_{-\nu} \cdot \mathcal{R}_\mu, \qquad {}^{\mu}\widetilde{M}^{\mathcal{R}} = \bigoplus_{\nu \in Q^+} {}^{\mu}\widetilde{M}^{\mathcal{R}}_{\mu-\nu}. \tag{4.11}$$

Here, $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0}_{-\nu}$ is defined using the grading of $U_q(\mathfrak{gl}_N)^{<0}$ with $F_i \in U_q(\mathfrak{gl}_N)^{<0}_{-(\varepsilon_i-\varepsilon_{i+1})}$.

### *4.5. Tensor Products*

Let $\widetilde{V}$ be a $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module and $W$ a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-module. The tensor product $\widetilde{V} \otimes_{\mathcal{A}} W$ is a $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R}$-module, where the first factor acts via the usual coproduct and the second factor acts by multiplication on $\widetilde{V}$. In the case when $\widetilde{V}$ and $W$ both have weight space decompositions, the weight spaces of $\widetilde{V}\otimes_{\mathcal{A}}W$ are

$$\left(\widetilde{V}\otimes_{\mathcal{A}}W\right)_{\nu} = \bigoplus_{\gamma+\eta=\nu} \widetilde{V}_{\gamma}\otimes_{\mathcal{A}}W_{\eta}. \tag{4.12}$$

We also need the following.

**Proposition 4.2.** *The tensor product of a universal Verma module with a Weyl module satisfies*

$$\left({}^{\mu}\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} \Delta^{\mathcal{A}}(\nu)\right)\otimes_{\mathcal{R}}\mathbb{K} \cong \left(\bigoplus_{\gamma}\left({}^{\mu+\gamma}\widetilde{M}^{\mathcal{R}}\right)^{\oplus \dim \Delta^{\mathcal{A}}(\nu)_{\gamma}}\right)\otimes_{\mathcal{R}}\mathbb{K}. \tag{4.13}$$

*Proof.* Fix $\nu \in P^{+}$. In general, $M(\lambda+\mu)\otimes\Delta(\nu)$ has a Verma filtration (see, e.g., [28, Theorem 2.2]) and if $\lambda+\mu+\gamma$ is dominant for all $\gamma$ such that $\Delta(\nu)_{\gamma} \neq 0$ then

$$M(\lambda+\mu)\otimes\Delta(\nu) \cong \bigoplus_{\gamma} M(\lambda+\mu+\gamma)^{\oplus \dim \Delta(\nu)_{\gamma}}, \tag{4.14}$$

which can be seen by, for instance, taking central characters. The proposition follows since this is true for a Zariski dense set of weights $\lambda$. □

## 5. The Shapovalov Form and the Shapovalov Determinant

### *5.1. The Shapovalov Form*

The *Cartan involution* $\omega : U_q(\mathfrak{gl}_N) \to U_q(\mathfrak{gl}_N)$ is the $\mathbb{Q}(q)$-algebra anti-involution of $U_q(\mathfrak{gl}_N)$ defined by

$$\omega\left(L_i^{\pm 1}\right) = L_i^{\pm 1}, \qquad \omega(X_i) = Y_i L_i L_{i+1}^{-1}, \qquad \omega(Y_i) = L_i^{-1} L_{i+1} X_i. \tag{5.1}$$

The map $\omega$ is also a coalgebra involution. An *ω-contravariant* form on a $U_q(\mathfrak{gl}_N)$-module $V$ is a symmetric bilinear form $(\cdot,\cdot)$ such that

$$(u, a\cdot v) = (\omega(a)\cdot u, v), \quad \text{for } u, v \in V,\ a \in U_q(\mathfrak{gl}_N). \tag{5.2}$$

It follows by the same argument used in the classical case [4] that there is an $\omega$-contravariant form (the Shapovalov form) on each Verma module $M(\lambda)$ and this is unique up to rescaling. The radical of $(\cdot,\cdot)$ is the maximal proper submodule of $M(\lambda)$, so $\Delta(\lambda) = M(\lambda)/\mathrm{Rad}(\cdot,\cdot)$ for all $\lambda \in P^{+}$. In particular, $(\cdot,\cdot)$ descends to an $\omega$-contravariant form on $\Delta(\lambda)$.

Since $\omega$ fixes $U_q^{\mathcal{A}}(\mathfrak{gl}_N) \subseteq U_q(\mathfrak{gl}_N)$, there is a well-defined notion of an $\omega$-contravariant form on a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ module. In particular, the restriction of the Shapovalov form on $\Delta(\lambda)$ to $\Delta^{\mathcal{A}}(\lambda)$ is $\omega$-contravariant.

### *5.2. Universal Shapovalov Forms*

There are surjective maps of $\mathcal{A}$-algebras $p_- : U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \to \mathbb{Q}(q)$ and $p_+ : U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0} \to \mathbb{Q}(q)$ defined by $p_-(F_i) = 0$ and $p_+(E_i) = 0$, for $1 \le i \le N$. Using the triangular decomposition (2.7), there is an $\mathcal{A}$-linear surjection

$$\pi_0 := p_- \otimes \mathrm{Id} \otimes p_+ : U_q^{\mathcal{A}}(\mathfrak{gl}_N) \cong U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{0} \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{>0} \longrightarrow U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{0}. \quad (5.3)$$

The *standard universal Shapovalov form* is the $\mathcal{R}$-bilinear form $(\cdot,\cdot)_{{}^{\mu}\widetilde{M}^{\mathcal{R}}} : {}^{\mu}\widetilde{M}^{\mathcal{R}} \otimes {}^{\mu}\widetilde{M}^{\mathcal{R}} \to \mathcal{R}$ defined by

$$\left(a_1 \cdot v_{\mu+}, a_2 \cdot v_{\mu+}\right)_{{}^{\mu}\widetilde{M}^{\mathcal{R}}} = \left(\sigma_\mu \circ \psi \circ \pi_0\right)\left(\omega(a_2) a_1\right) \quad (5.4)$$

for all $a_1, a_2 \in U_q^{\mathcal{R}}(\mathfrak{gl}_N)^{<0}$. Here, $\psi$ and $\sigma_\mu$ are as in (4.1) and (4.2). Since

$$\left(a_1 a_2 \cdot v_{\mu+}, a_3 \cdot v_{\mu+}\right)_{{}^{\mu}\widetilde{M}^{\mathcal{R}}} = \left(\sigma_\mu \circ \psi \circ \pi_0\right)\left(\omega(a_2)\omega(a_1) a_3\right) = \left(a_2 \cdot v_{\mu+}, \omega(a_1) a_3 \cdot v_{\mu+}\right)_{{}^{\mu}\widetilde{M}^{\mathcal{R}}} \quad (5.5)$$

for $a_1, a_2, a_3 \in U_q(\mathfrak{gl}_N)$, the form $(\cdot,\cdot)_{{}^{\mu}\widetilde{M}^{\mathcal{R}}}$ is $\omega$-contravariant. As with the usual Shapovalov form, distinct weight spaces are orthogonal, where weight spaces are defined as in Section 4.4.

Evaluation at $\lambda$ gives an $\mathcal{A}$-valued $\omega$-contravariant form $(\cdot,\cdot)_{M^{\mathcal{A}}(\mu+\lambda)}$ on $M^{\mathcal{A}}(\mu+\lambda)$ by

$$\left(\mathrm{ev}_\lambda(u_1), \mathrm{ev}_\lambda(u_2)\right)_{M^{\mathcal{A}}(\mu+\lambda)} = \mathrm{ev}_\lambda\left((u_1, u_2)_{{}^{\mu}\widetilde{M}^{\mathcal{R}}}\right) \quad \text{for } u_1, u_2 \in {}^{\mu}\widetilde{M}^{\mathcal{R}}. \quad (5.6)$$

The form $(\cdot,\cdot)_{{}^{\mu}\widetilde{M}^{\mathcal{R}}}$ can be extended by linearity to an $\omega$-contravariant form $(\cdot,\cdot)_{{}^{\mu}\widetilde{M}}$ on ${}^{\mu}\widetilde{M}$.

### *5.3. The Shapovalov Determinant*

Let $\widetilde{V}$ be a $(U_q^{\mathcal{A}}(\mathfrak{gl}_N) \otimes_{\mathcal{A}} \mathcal{R})$-module with a chosen $\omega$-contravariant form. Let $B_\eta$ be an $\mathcal{R}$ basis for the $\eta$-weight space $\widetilde{V}_\eta$ of $\widetilde{V}$. Let $\det \widetilde{V}_{B_\eta}$ be the determinant of the form evaluated on the basis $B_\eta$. Changing the basis $B_\eta$ changes the determinant by a unit in $\mathcal{R}$, and we sometimes write $\det \widetilde{V}_\eta$ to mean the determinant calculated on an unspecified basis ($\det \widetilde{V}_\eta$ which is only defined up to multiplication by unit in $\mathcal{R}$). The *Shapovalov determinant* is

$$\det \widetilde{M}_\eta^{\mathcal{R}} := \det\left(\left(b_i, b_j\right)_{\widetilde{M}^{\mathcal{R}}}\right)_{b_i, b_j \in B_\eta}. \quad (5.7)$$

Define the *partition function* $p : \mathfrak{h}^* \to \mathbb{Z}_{\ge 0}$ by

$$p(\gamma) := \dim M(0)_\gamma. \quad (5.8)$$

Then, $p(\gamma) = \dim M(\lambda)_{\gamma+\lambda}$ for any $\lambda$, and $\eta \notin Q^-$ implies that $p(\eta) = 0$ and $\det \widetilde{M}_\eta^{\mathcal{R}} = 1$.

**Theorem 5.1** (see [5, Proposition 1.9A], [6, Theorem 3.4], [4]). *For any weight* $\eta$,

$$\det \widetilde{M}_\eta^{\mathcal{R}} = c_\eta \prod_{\substack{1\le i<j\le N \\ m>0}} \left( z_i z_j^{-1} - q^{2m+2i-2j} z_i^{-1} z_j \right)^{p(\eta+m\varepsilon_i-m\varepsilon_j)}, \tag{5.9}$$

*where* $c_\eta$ *is a unit in* $\mathcal{R}\otimes_{\mathcal{A}} \mathbb{Q}(q) = Q(q)[z_1^{\pm1},\dots,z_N^{\pm1}]$.

**Proposition 5.2.** *Fix* $\mu,\eta \in \mathfrak{h}_{\mathbb{Z}}^*$ *with* $\eta-\mu \in Q^-$. *Choose an* $\mathcal{A}$*-basis* $B_{\eta-\mu}$ *for* $U_q^{\mathcal{A}}(\mathfrak{gl}_N)_{\eta-\mu}$. *Consider the* $\mathcal{R}$*-bases* $\widetilde{B}_{\eta-\mu} := \{b\cdot v_+ \mid b \in B_{\eta-\mu}\}$ *for* $\widetilde{M}_{\eta-\mu}^{\mathcal{R}}$ *and* ${}^\mu\widetilde{B}_\eta := \{b \cdot v_{\mu+} \mid b\in B_{\eta-\mu}\}$ *for* ${}^\mu\widetilde{M}_\eta^{\mathcal{R}}$. *Then* $\det {}^\mu\widetilde{M}^{\mathcal{R}}_{({}^\mu\widetilde{B}_\eta)} = \sigma_\mu(\det \widetilde{M}^{\mathcal{R}}_{\widetilde{B}_{\eta-\mu}})$.

*Proof.* For $b,b' \in B_{\eta-\mu}$,

$$(b\cdot v_{\mu+}, b'\cdot v_{\mu+})_{{}^\mu\widetilde{M}^{\mathcal{R}}} = \sigma_\mu \circ \psi \circ \pi_0(\omega(b')b) = \sigma_\mu\left((b\cdot v_{0+}, b'\cdot v_{0+})_{\widetilde{M}^{\mathcal{R}}}\right). \tag{5.10}$$

The result follows by taking determinants. □

### *5.4. Contravariant Forms on Tensor Products*

If $V$ and $W$ are $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$-modules with $\omega$-contravariant forms $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$, define an $\mathcal{A}$-bilinear form $(\cdot,\cdot)_{W\otimes V}$ by $(w_1\otimes v_1, w_2\otimes v_2)_{W\otimes V} = (w_1,w_2)_W(v_1,v_2)_V$. Similarly, for a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)\otimes_{\mathcal{A}}\mathcal{R}$ module $\widetilde{W}$ with $\mathcal{R}$-bilinear $\omega$-contravariant form $(\cdot,\cdot)_{\widetilde{W}}$, define a $\mathcal{R}$-bilinear form $(\cdot,\cdot)_{\widetilde{W}\otimes_{\mathbb{Q}(q)}V}$ on $\widetilde{W}\otimes_{\mathbb{Q}(q)}V$ by

$$(u_1\otimes v_1, u_2\otimes v_2)_{\widetilde{W}\otimes_{\mathbb{Q}(q)}V} = (u_1,u_2)_{\widetilde{W}}(v_1,v_2)_V. \tag{5.11}$$

Since $\omega$ is a coalgebra involution (i.e., $\Delta(\omega(a)) = (\omega\otimes\omega)\Delta(a)$, for $a\in U_q(\mathfrak{gl}_N)$), the forms $(\cdot,\cdot)_{V\otimes W}$ and $(\cdot,\cdot)_{{}^\mu\widetilde{M}\otimes_{Q(q)}V}$ are $\omega$-contravariant.

In the case when $\widetilde{W} = {}^\mu\widetilde{M}^{\mathcal{R}}$, evaluation of the $\omega$-contravariant form $(\cdot,\cdot)_{{}^\mu\widetilde{M}^{\mathcal{R}}\otimes_{\mathcal{A}}V}$ at $\lambda$ gives an $\omega$-contravariant form $(\cdot,\cdot)_{M^{\mathcal{A}}(\mu+\lambda)\otimes_{\mathcal{A}}V}$:

$$\begin{aligned}(u_1\otimes v_1, u_2\otimes v_2)_{M^{\mathcal{A}}(\mu+\lambda)\otimes_{\mathcal{A}}V} &= \mathrm{ev}_\lambda\left((u_1\otimes v_1, u_2\otimes v_2)_{{}^\mu\widetilde{M}\otimes_{\mathcal{A}}V}\right) \\ &= (\mathrm{ev}_\lambda(u_1)\otimes v_1, \mathrm{ev}_\lambda(u_2)\otimes v_2)_{M(\mu+\lambda)\otimes_{\mathcal{A}}V},\end{aligned} \tag{5.12}$$

for $u_1,u_2 \in {}^\mu\widetilde{M}$ and $v_1,v_2\in V$. As in Section 4.3, this form can be extended to the $\mathcal{A}$-submodule of the rational module where no denominators evaluate to zero.

## 6. The Misra-Miwa Formula for $F_{\bar{i}}$ from $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$ Representation Theory

Let us prepare the setting for our main result (Theorem 6.1). Fix $\ell \geq 2$ and a partition $\lambda$. Let $N$ be a positive integer greater than the number of parts of $\lambda$. All calculations below are in terms of representations of $U_q^{\mathcal{A}}(\mathfrak{gl}_N)$.

(1) Let $V = \Delta^{\mathcal{A}}(\varepsilon_1)$ be the standard $N$-dimensional module. Since $\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}\mathbb{Q}(q) = \Delta(\lambda)$, (3.1) implies

$$\left(\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}V\right)\otimes_{\mathcal{A}}\mathbb{Q}(q) \simeq \bigoplus \Delta^{\mathcal{A}}\left(\lambda+\varepsilon_{k_j}\right)\otimes_{\mathcal{A}}\mathbb{Q}(q), \tag{6.1}$$

where the sum is over those indices $1 = k_1 < k_2 < \cdots < k_{m_\lambda} \leq N$ for which $\lambda + \varepsilon_{k_j}$ is a partition. For ease of notation, let $\mu^{(j)} = \lambda + \varepsilon_{k_j}$.

(2) Fix an $\mathcal{A}$-basis $\{v_1, \ldots, v_N\}$ of $V$ where $v_k$ has weight $\varepsilon_k$ and $Y_i(v_k) = \delta_{i,k}v_{k+1}$. Recursively, define singular weight vectors $v_{\mu^{(j)}}$ in $(\Delta^{\mathcal{A}}(\lambda)\otimes V)\otimes_{\mathcal{A}}\mathbb{Q}(q)$ by

  (i) $v_{\mu^{(1)}} = v_\lambda \otimes v_1$

  (ii) for each $k$, the submodule $W_k$ of $(\Delta(\lambda)\otimes_{\mathcal{A}}V)\otimes_{\mathcal{A}}\mathbb{Q}(q)$ generated by $\{v_\lambda\otimes v_i \mid 1\leq i\leq k\}$ contains all weight vectors of $(\Delta(\lambda)\otimes_{\mathcal{A}}V)\otimes_{\mathcal{A}}\mathbb{Q}(q)$ of weight greater than or equal to $\lambda+\varepsilon_k$. Thus, using (6.1), for each $1\leq j\leq m_\lambda$ there is a one-dimensional space of singular vectors of weight $\mu^{(j)}$ in $W_{k_j}$, and this is not contained in $W_{k_{j-1}}$ (since $k_j > k_{j-1}$). This implies that there unique singular vector $v_{\mu^{(j)}}$ of weight $\mu^{(j)}$ in

$$v_\lambda\otimes v_{k_j} + \bigoplus_{1\leq i<j} U_q(\mathfrak{gl}_N)v_{\mu^{(i)}} \subseteq \left(\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}V\right)\otimes_{\mathcal{A}}\mathbb{Q}(q), \tag{6.2}$$

  where we recall that $U_q(\mathfrak{gl}_N) = U_q^{\mathcal{A}}(\mathfrak{gl}_N)\otimes_{\mathcal{A}}\mathbb{Q}(q)$.

(3) There is a unique $\omega$-contravariant form on $\Delta^{\mathcal{A}}(\lambda)$ normalized so that $(v_\lambda, v_\lambda) = 1$ and a unique $\omega$-contravariant form on $V$ normalized so that $(v_1, v_1) = 1$. As in Section 5.4, define a $\omega$-contravariant form on $(\Delta^{\mathcal{A}}(\lambda)\otimes_{\mathcal{A}}V)\otimes_{\mathcal{A}}\mathbb{Q}(q)$ by $(u_1\otimes w_1, u_2\otimes w_2) = (u_1,u_2)(w_1,w_2)$. For each $1\leq j\leq m_\lambda$, define an element $r_j(\lambda)\in\mathbb{Q}(q)$ by

$$r_j(\lambda) := \left(v_{\mu^{(j)}}, v_{\mu^{(j)}}\right). \tag{6.3}$$

**Theorem 6.1.** *The Misra-Miwa operators $F_{\bar{i}}$ from Section 3.3 satisfy*

$$F_{\bar{i}}|\lambda\rangle = \sum_{\bar{c}\left(b^{(j)}\right)=\bar{i}} v^{val_{\phi_{2e}} r_j(\lambda)} \left|\mu^{(j)}\right\rangle, \tag{6.4}$$

*where $b^{(j)}$ is the box $\mu^{(j)}/\lambda$, $\overline{c}(b^{(j)})$ is the color of box $b^{(j)}$ as in Figure 1, $\phi_{2\ell}$ is the $2\ell$th cyclotomic polynomial in $q$, and $\mathrm{val}_{\phi_{2\ell}} r$ is the number of factors of $\phi_{2\ell}$ in the numerator of $r$ minus the number of factors of $\phi_{2\ell}$ in the denominator of $r$.*

The proof of Theorem 6.1 will occupy the rest of this section. We will first prove a similar statement, Proposition 6.6, where the role of the Weyl modules is played by the universal Verma modules from Section 4. For ease of notation, let $\widetilde{M}^{\mathcal{R}}$ denote the module ${}^{0}\widetilde{M}^{\mathcal{R}}$ from Section 4.2.

*Definition 6.2.* Recursively define singular weight vectors $v_{\varepsilon_k+} \in (\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V) \otimes_{\mathcal{R}} \mathbb{K}$ and elements $s_k \in \mathbb{K}$ for $1 \leq k \leq N$ by

(i) $v_{\varepsilon_1+} = v_+ \otimes v_1$,

(ii) since $\{v_+ \otimes v_j \mid 1 \leq j \leq N\}$ generates $\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V$ as a $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{\leq 0}$ module, Proposition 4.2 implies that, for each $1 \leq k \leq N$, there is a unique singular vector $v_{\varepsilon_k+}$ in $v_+ \otimes v_k + \oplus_{1 \leq j < k} U_q^{\mathbb{K}}(\mathfrak{gl}_N) v_{\varepsilon_j+} \subseteq (\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V) \otimes_{\mathcal{R}} \mathbb{K}$, where $U_q^{\mathbb{K}}(\mathfrak{gl}_N) := U_q(\mathfrak{gl}_N) \otimes_{(q)} \mathbb{K}$ and the factor of $\mathbb{K}$ acts by multiplication on $\widetilde{M}^{\mathcal{R}}$.

Let $s_k = (v_{\varepsilon_k+}, v_{\varepsilon_k+})$.

The $s_k$ are quantized versions of the Jantzen numbers first calculated in [11, Section 5] and quantized in [12]. It follows immediately from the definition that $s_1 = 1$.

**Lemma 6.3.** *For any weight $\eta$, up to multiplication by a power of $q$,*

$$\prod_{1 \leq k \leq N} s_k^{p(\eta - \varepsilon_k)} = \prod_{1 \leq k \leq N} \frac{\det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}}{\sigma_{\varepsilon_k} \det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}}, \tag{6.5}$$

*where, as in Section 5.3, $\det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}$ is the determinant of the Shapovalov form evaluated on an $\mathcal{R}$-basis for the $\eta - \varepsilon_k$ weight space of $\widetilde{M}^{\mathcal{R}}$.*

*Comment 2.* In order for Lemma 6.3 to hold as stated, for each $1 \leq k \leq N$, one must calculate the $\det \widetilde{M}^{\mathcal{R}}_{\eta - \varepsilon_k}$ in the numerator and denominator with respect to the same $\mathcal{R}$-basis. The power of $q$ which appears depends on this choice of $\mathcal{R}$-bases.

*Proof of Lemma 6.3.* For each $\gamma \in \operatorname{span}_{\mathbb{Z}_{\leq 0}}(R^+)$ fix an $\mathcal{R}$-basis $B_\gamma$ for $U_q^{\mathcal{R}}(\mathfrak{gl}_N)^{<0}_\gamma$. Consider the following three $\mathbb{K}$-bases for $((\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_\eta) \otimes_{\mathcal{R}} \mathbb{K}$:

$$\begin{aligned} A_\eta &:= \left\{(b \cdot v_+) \otimes v_k \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\right\}, \\ C_\eta &:= \left\{b \cdot (v_+ \otimes v_k) \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\right\}, \\ D_\eta &:= \left\{b \cdot v_{\varepsilon_k+} \mid b \in B_{\eta - \varepsilon_k}, 1 \leq k \leq N\right\}. \end{aligned} \tag{6.6}$$

Let $\det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_B$ denote the determinant of $(\cdot,\cdot)_{(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_\eta}$ calculated on $B$, where $B$ is one of $A_\eta, C_\eta$, or $D_\eta$. Let $\det {}^{\nu}\widetilde{M}^{\mathcal{R}}_{B_{\eta-\nu}}$ denote $\det {}^{\nu}\widetilde{M}^{\mathcal{R}}_{\eta}$ calculated with respect to the basis $B_{\eta-\nu} \cdot v_{\nu+}$.

By the definition of the $\omega$-contravariant form on $\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V$ (see Section 4.5),

$$\det\left(\widetilde{M}^{\mathcal{R}} \otimes V\right)_{A_\eta} = \prod_{k=1}^{N} \left(\det \widetilde{M}^{\mathcal{R}}_{B_{\eta-\varepsilon_k}}\right)^{\dim V_{\varepsilon_k}} \left(\det V_{\varepsilon_k}\right)^{\dim \widetilde{M}^{\mathcal{R}}_{\eta-\varepsilon_k}}. \tag{6.7}$$

For $1 \leq k \leq N$, $V_{\varepsilon_k}$ is one dimensional and $\det V_{\varepsilon_k}$ is a power of $q$. Hence, up to multiplication by a power of $q$, (6.7) simplifies to

$$\det\left(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V\right)_{A_\eta} = \prod_{k=1}^{N} \det \widetilde{M}^{\mathcal{R}}_{B_{\eta-\varepsilon_k}}. \tag{6.8}$$

Notice that $U_q^{\mathcal{A}}(\mathfrak{gl}_N)^{<0} \cdot v_{\varepsilon_k +}$ is isomorphic to ${}^{\varepsilon_k}\widetilde{M}$, and $D_\eta$ is the union of $\mathcal{R}$-bases for each of these submodules. For each $1 \leq k \leq N$, and each $\eta \in \mathfrak{h}_Z^*$ define an $\mathcal{R}$ basis of ${}^{\varepsilon_k}\widetilde{M}_\eta$ by

$${}^{\varepsilon_k}\widetilde{B}_\eta := \left\{ b \cdot v_{\varepsilon_k +} \mid b \in B_{\eta-\varepsilon_k} \right\}. \tag{6.9}$$

Using $(v_{\varepsilon_k +}, v_{\varepsilon_k +}) = s_k$,

$$\det\left(\widetilde{M}^{\mathcal{R}} \otimes V\right)_{D_\eta} = \prod_{k=1}^{N} s_k^{\dim ({}^{\varepsilon_k}\widetilde{M}^{\mathcal{R}}_\eta)} \det {}^{\varepsilon_k}\widetilde{M}^{\mathcal{R}}_{({}^{\varepsilon_k}\widetilde{B}_\eta)} = \prod_{k=1}^{N} s_k^{p(\eta-\varepsilon_k)} \sigma_{\varepsilon_k}\left(\det \widetilde{M}^{\mathcal{R}}_{\widetilde{B}_{\eta-\varepsilon_k}}\right), \tag{6.10}$$

where the last equality uses Proposition 5.2. Here, as in Section 5.3, $\det {}^{\varepsilon_k}\widetilde{M}^{\mathcal{R}}_{({}^{\varepsilon_k}\widetilde{B}_\eta)}$ is the Shapovalov determinant calculated with respect to the basis ${}^{\varepsilon_k}\widetilde{B}_\eta$.

The change of basis from $A_\eta$ to $C_\eta$ is unitriangular and the change of basis from $C_\eta$ to $D_\eta$ is unitriangular. Thus, $\det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_{A_\eta} = \det(\widetilde{M}^{\mathcal{R}} \otimes_{\mathcal{A}} V)_{D_\eta}$, and so the right sides of (6.8) and (6.10) are equal. The lemma follows from this equality by rearranging. □

**Lemma 6.4.** *Up to multiplication by a power of $q$,*

$$s_k = \prod_{1 \leq j < k} \left( \frac{z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k}{\sigma_{\varepsilon_j}\left(z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k\right)} \right). \tag{6.11}$$

*Proof.* Fix $1 \leq k \leq N$. Setting $\eta = \varepsilon_k$ in Lemma 6.3 and applying Theorem 5.1 we see that, up to multiplication by a power of $q$,

$$\prod_{1\leq x\leq N} s_x^{p(\varepsilon_k-\varepsilon_x)} = \prod_{1\leq x\leq N} \frac{\det \widetilde{M}^{\mathcal{R}}_{\varepsilon_k-\varepsilon_x}}{\sigma_{\varepsilon_x} \det \widetilde{M}^{\mathcal{R}}_{\varepsilon_k-\varepsilon_x}}$$
$$= \prod_{1\leq x\leq N} \prod_{\substack{1\leq i<j\leq N\\ m>0}} \left( \frac{c_{\varepsilon_k-\varepsilon_x}\left(z_iz_j^{-1} - q^{2m+2i-2j}z_i^{-1}z_j\right)}{\sigma_{\varepsilon_x}(c_{\varepsilon_k-\varepsilon_x})\sigma_{\varepsilon_x}\left(z_iz_j^{-1} - q^{2m+2i-2j}z_i^{-1}z_j\right)} \right)^{p(\varepsilon_k-\varepsilon_x+m\varepsilon_i-m\varepsilon_j)}, \tag{6.12}$$

where, for each $1 \leq x \leq N$, $c_{\varepsilon_k-\varepsilon_x}$ is a unit in $\mathbb{Q}(q)[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$. The value $p(\varepsilon_k - \varepsilon_x + m\varepsilon_i - m\varepsilon_j)$ is 0 unless $m = 1$ and $x \leq i < j \leq k$. If $i > x$, then $\sigma_{\varepsilon_x}$ acts as the identity on $z_iz_j^{-1} - q^{2+2i-2j}z_i^{-1}z_j$, so the corresponding factors in the numerator and denominator cancel. Hence, we need only consider factors on the right hand side where $m = 1$, $i = x$, and $x < j \leq k$. If $x > k$, then $\varepsilon_k - \varepsilon_x \notin Q^-$, and hence $p(\varepsilon_k - \varepsilon_x) = 0$, so on the left hand since we only need to consider those factors where $1 \leq x \leq k$. Up to multiplication by a power of $q$, the expression reduces to

$$\prod_{1\leq x\leq k} s_x^{p(\varepsilon_k-\varepsilon_x)} = \prod_{1\leq x<k} \left( \frac{c_{\varepsilon_k-\varepsilon_x}}{\sigma_{\varepsilon_x}(c_{\varepsilon_k-\varepsilon_x})} \right)^{p(\varepsilon_k-\varepsilon_j)} \prod_{x<j\leq k} \left( \frac{z_xz_j^{-1} - q^{2+2x-2j}z_x^{-1}z_j}{\sigma_{\varepsilon_x}\left(z_xz_j^{-1} - q^{2+2x-2j}z_x^{-1}z_j\right)} \right)^{p(\varepsilon_k-\varepsilon_j)}$$
$$= \prod_{1<j\leq k} \left( \prod_{1\leq x<j} \frac{z_xz_j^{-1} - q^{2+2x-2j}z_x^{-1}z_j}{\sigma_{\varepsilon_x}\left(z_xz_j^{-1} - q^{2+2x-2j}z_x^{-1}z_j\right)} \right)^{p(\varepsilon_k-\varepsilon_j)}. \tag{6.13}$$

The last two expressions are equal because they are each a product over pairs $(x, j)$ with $1 \leq x < j \leq k$, and the factors of $c_{\varepsilon_k-\varepsilon_x}/(\sigma_{\varepsilon_x}(c_{\varepsilon_k-\varepsilon_x}))$ have been dropped because they are powers of $q$. Using the fact that $s_1 = 1$ and making the change of variables $j \to x$ and $x \to j$ on the right side, (6.13) becomes

$$\prod_{1<x\leq k} s_x^{p(\varepsilon_k-\varepsilon_x)} = \prod_{1<x\leq k} \left( \prod_{1\leq j<x} \frac{z_jz_x^{-1} - q^{2+2j-2x}z_j^{-1}z_x}{\sigma_{\varepsilon_j}\left(z_jz_x^{-1} - q^{2+2j-2x}z_j^{-1}z_x\right)} \right)^{p(\varepsilon_k-\varepsilon_x)}. \tag{6.14}$$

For $k \geq 2$, the lemma now follows by induction. For $k = 1$, the result simply says that $s_1 = 1$, which we already know. □

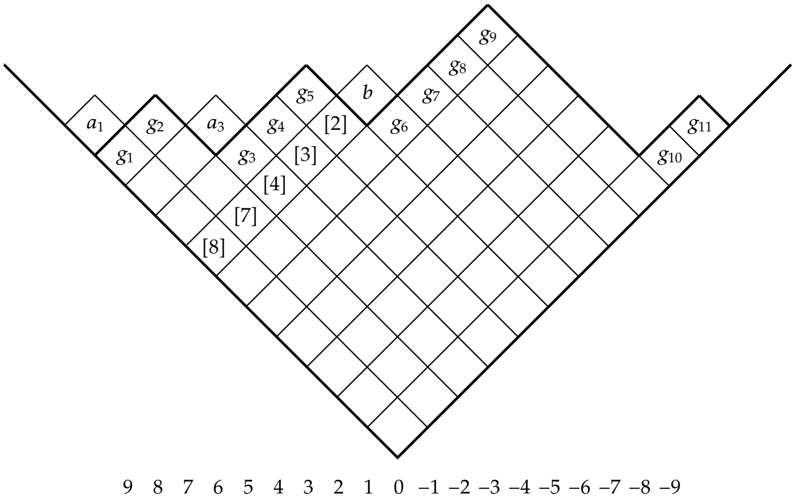



**Figure 2:** The partition enclosed by the thick lines is $\lambda = (10, 10, 8, 8, 8, 6, 6, 6, 6, 1, 1)$. If $k = 6$ then $A(\lambda, < 6) = \{a_1, a_3\}$, $R(\lambda, < 6) = \{g_2, g_5\}$, and $\mathrm{ev}_\lambda(s_6) = ([2]/[3])([3]/[4])([4]/[5])([7]/[8])([8]/[9]) = ([2]/[5])([7]/[9]) = ([c(g_5)-c(b)][c(g_2)-c(b)])/([c(a_3)-c(b)][c(a_1)-c(b)])$. The factors in the numerator of the first expression are displayed. These are the $q$-integers corresponding to the hook lengths of the boxes in the same column as the addable box $b$ in row 6.

**Proposition 6.5.** *Let $\lambda$ be a partition. Let $A(\lambda, < k)$ (resp. $R(\lambda, < k)$) be the set of boxes which can be added to (resp. removed from) $\lambda$ on rows $\lambda_j$ with $j < k$ such that the result is still a partition. Let $b = (\lambda + \varepsilon_k)/\lambda$ and let $c(\cdot)$ be as in Figure 1. Then, up to multiplication by a power of $q$,*

$$ev_\lambda(s_k) = \begin{cases} \dfrac{\prod_{r\in R(\lambda,<k)}[c(r) - c(b)]}{\prod_{a\in A(\lambda,<k)}[c(a) - c(b)]}, & \text{if } \lambda + \varepsilon_k \text{ is a partition,} \\ 0, & \text{if } \lambda + \varepsilon_k \text{ is not a partition.} \end{cases} \tag{6.15}$$

*Proof.* For $1 \le j \le N$, let $g_j$ be the last box in row $j$ of $\lambda$. By Lemma 6.4, up to multiplication by a power of $q$,

$$\mathrm{ev}_\lambda(s_k) = \mathrm{ev}_\lambda\left(\prod_{1\le j<k} \frac{z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k}{\sigma_{\varepsilon_j}\left(z_j z_k^{-1} - q^{2+2j-2k} z_j^{-1} z_k\right)}\right) = \prod_{1\le j<k} \frac{[c(g_j) - c(b)]}{[c(g_j) - c(b) + 1]}, \tag{6.16}$$

where the last equality is a simple calculation from definitions. The denominator on the right side is never zero, and the numerator is zero exactly when $\lambda_k = \lambda_{k-1}$, so that $\lambda + \varepsilon_k$ is no longer a partition. If $\lambda_j = \lambda_{j+1}$ for any $j < k$, then there is cancellation, giving (6.15). See Figure 2. □

**Proposition 6.6.** *Let $N^l_{\bar{j}}(\mu/\lambda)$ be as in Section 3.3. For any partition $\lambda$,*

$$\begin{aligned} &val_{\phi_{2e}} ev_\lambda(s_k) = N^l_{\bar{i}}(\mu/\lambda), \quad \text{if } \mu = \lambda + \varepsilon_k \text{ is a partition, and } \mu/\lambda \text{ is an } \bar{i} \text{ colored box,} \\ &ev_\lambda(s_k) = 0, \quad \text{otherwise.} \end{aligned} \tag{6.17}$$

*Proof.* By Proposition 6.5, $\mathrm{ev}_\lambda(s_k) = 0$ if $\lambda + \varepsilon_k$ is not a partition. If $\lambda + \varepsilon_k$ is a partition, then

$$\begin{aligned}\{b \in A(\lambda, < k) : \overline{c}(b) = \overline{c}(\mu/\lambda)\} &= \{b \in A_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\},\\ \{b \in R(\lambda, < k) : \overline{c}(b) = \overline{c}(\mu/\lambda)\} &= \{b \in R_{\bar{i}}(\lambda) \mid b \text{ is to the left of } \mu/\lambda\},\end{aligned} \tag{6.18}$$

where the notation is as in Section 3.3. Since

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = q^{-x}\left(q - q^{-1}\right)^{-1} \prod_{d|2x} \phi_d, \tag{6.19}$$

$[x]$ is divisible by $\phi_{2\ell}$ if and only if $x$ is divisible by $\ell$, and $[x]$ is never divisible by $\phi_{2\ell}^2$. The result now follows from Proposition 6.5. □

*Proof of Theorem 6.1.* Fix $\lambda$ and $1 \leq k \leq m_\lambda$. From definitions, $(\mathrm{ev}_\lambda \otimes 1)v_{\varepsilon_{k_j}+} = v_{\mu^{(j)}}$. Thus, using (5.12),

$$r_j(\lambda) = \left(v_{\mu^{(j)}}, v_{\mu^{(j)}}\right) = \left((\mathrm{ev}_\lambda \otimes 1)v_{\varepsilon_{k_j}+}, (\mathrm{ev}_\lambda \otimes 1)v_{\varepsilon_{k_j}+}\right) = \mathrm{ev}_\lambda\left(v_{\varepsilon_{k_j}+}, v_{\varepsilon_{k_j}+}\right) = \mathrm{ev}_\lambda\left(s_{k_j}\right). \tag{6.20}$$

The result now follows from Proposition 6.6. □

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## References

[1] V. G. Kac, *Infinite-Dimensional Lie Algebras*, Cambridge University Press, Cambridge, UK, 3rd edition, 1990.

[2] T. Hayashi, "$q$-analogues of Clifford and Weyl algebras—spinor and oscillator representations of quantum enveloping algebras," *Communications in Mathematical Physics*, vol. 127, no. 1, pp. 129–144, 1990.

[3] K. Misra and T. Miwa, "Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}_n)$," *Communications in Mathematical Physics*, vol. 134, no. 1, pp. 79–88, 1990.

[4] N. N. Šapovalov, "A certain bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra," *Funkcional'nyi Analiz i ego Priloženija*, vol. 6, no. 4, pp. 65–70, 1972.

[5] C. de Concini and V. G. Kac, "Representations of quantum groups at roots of 1," in *Modern Quantum Field Theory (Bombay, 1990)*, pp. 333–335, World Scientific Publishing, River Edge, NJ, USA, 1991.

[6] S. Kumar and G. Letzter, "Shapovalov determinant for restricted and quantized restricted enveloping algebras," *Pacific Journal of Mathematics*, vol. 179, no. 1, pp. 123–161, 1997.

[7] M. Kashiwara, "The universal Verma module and the $b$-function," in *Algebraic Groups and Related Topics (Kyoto/Nagoya, 1983)*, vol. 6 of *Advanced Studies in Pure Mathematics*, pp. 67–81, North-Holland, Amsterdam, The Netherlands, 1985.

[8] A. Kamita, "The $b$-functions for prehomogeneous spaces of commutative parabolic type and universal Verma modules II—quantum cases," preprint.

[9] A. S. Kleshchev, “Branching rules for modular representations of symmetric groups. II,” *Journal für die Reine und Angewandte Mathematik*, vol. 459, pp. 163–212, 1995.
[10] J. Brundan and A. Kleshchev, “Some remarks on branching rules and tensor products for algebraic groups,” *Journal of Algebra*, vol. 217, no. 1, pp. 335–351, 1999.
[11] J. C. Jantzen, “Zur charakterformel gewisser darstellungen halbeinfacher gruppen und lie-algebren,” *Mathematische Zeitschrift*, vol. 140, pp. 127–149, 1974.
[12] E. Wiesner, *Translation functors and the Shapovalov determinant*, thesis, University of Wisconsin-Madison, Madison, Wis, USA, 2005.
[13] J. Brundan, “Modular branching rules and the Mullineux map for Hecke algebras of type $A$,” *Proceedings of the London Mathematical Society*, vol. 77, no. 3, pp. 551–581, 1998.
[14] S. Ariki, “On the decomposition numbers of the Hecke algebra of $G(m,1,n)$,” *Journal of Mathematics of Kyoto University*, vol. 36, no. 4, pp. 789–808, 1996.
[15] A. Lascoux, B. Leclerc, and J.-Y. Thibon, “Hecke algebras at roots of unity and crystal bases of quantum affine algebras,” *Communications in Mathematical Physics*, vol. 181, no. 1, pp. 205–263, 1996.
[16] S. Ryom-Hansen, “Grading the translation functors in type A,” *Journal of Algebra*, vol. 274, no. 1, pp. 138–163, 2004.
[17] J. Brundan and A. Kleshchev, “Graded decomposition numbers for cyclotomic Hecke algebras,” *Advances in Mathematics*, vol. 222, no. 6, pp. 1883–1942, 2009.
[18] S. Ariki, “Graded $q$-Schur algebras,” http://arxiv.org/abs/0903.3453.
[19] M. Khovanov and A. D. Lauda, “A diagrammatic approach to categorification of quantum groups. I,” *Representation Theory*, vol. 13, pp. 309–347, 2009.
[20] R. Rouquier, “2-Kac-Moody algebras,” http://arxiv.org/abs/0812.5023.
[21] P. Shan, “Graded decomposition matrices of $v$-Schur algebras via Jantzen filtration,” http://arxiv.org/abs/1006.1545.
[22] A. Beĭlinson and J. Bernstein, “A proof of Jantzen conjectures,” in *I. M. Gel′fand Seminar*, vol. 16 of *Advances in Soviet Mathematics*, pp. 1–50, American Mathematical Society, Providence, RI, USA, 1993.
[23] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, UK, 1995.
[24] G. Lusztig, “Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra,” *Journal of the American Mathematical Society*, vol. 3, no. 1, pp. 257–296, 1990.
[25] S. Ariki, *Representations of Quantum Algebras and Combinatorics of Young Tableaux*, vol. 26 of *University Lecture Series*, American Mathematical Society, Providence, RI, USA, 2002.
[26] W. Fulton and J. Harris, *Representation Theory. A First Course, Readings in Mathematic*, vol. 129 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1991.
[27] M. Kashiwara, T. Miwa, and E. Stern, “Decomposition of $q$-deformed Fock spaces,” *Selecta Mathematica. New Series*, vol. 1, no. 4, pp. 787–805, 1995.
[28] J. C. Jantzen, *Moduln mit Einem Höchsten Gewicht*, vol. 750 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1979.