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Research Article

Generalized Order and Best Approximation of Entire Function in L^p-Norm

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The aim of this paper is the characterization of the generalized growth of entire functions of several complex variables by means of the best polynomial approximation and interpolation on a compact K with respect to the set $\Omega_r = \{z \in \mathbf{C}^n;\ \exp V_K(z) \le r\}$, where $V_K = \sup\{(1/d)\ln|P_d|,\ P_d$ polynomial of degree $\le d,\ \|P_d\|_K \le 1\}$ is the Siciak extremal function of a L-regular compact K.

1. Introduction

Let $f(z) = \sum_{k=0}^{+\infty} a_k \cdot z^{\lambda_k}$ be a no constant entire function in complex plane $\mathbb{C}$ and let

$$M(f,r) = \sup\{\left|f(z)\right|,\ |z| = r,\ r > 0\}. \tag{1.1}$$

It is well known that the function $r \to \ln(M(f,r))$ is convex and decreasing of $\ln(r)$. To estimate the growth of f, the concept of order, defined by the number ρ $(0 \le \rho \le +\infty)$, such that

$$\rho = \limsup_{r\to+\infty} \frac{\ln\ln M(f,r)}{\ln(r)} \tag{1.2}$$

has been given (see [1]).

The concept of type has been introduced to establish the relative growth of two functions having the same nonzero finite order. So an entire function, in complex plane $\mathbb{C}$, of order ρ $(0 < \rho < +\infty)$, is said to be of type σ $(0 \le \sigma \le +\infty)$ if

$$\sigma = \limsup_{r \to +\infty} \frac{\ln M(f,r)}{r^{\rho}}. \tag{1.3}$$

If f is an entire function of infinite or zero order, the definition of type is not valid and the growth of such function can not be precisely measured by the above concept. Bajpai and Juneja (see [2]) have introduced the concept of index-pair of an entire function. Thus, for $p \ge q \ge 1$, they have defined the number

$$\rho(p,q) = \limsup_{r \to +\infty} \frac{\log^{[p]}(M(f,r))}{\log^{[q]}(r)}. \tag{1.4}$$

It is easy to show that $b \le \rho(p,q) \le +\infty$ where $b = 0$ if $p > q$ and $b = 1$ if $p = q$.

The function f is said to be of index-pair (p,q) if $\rho(p-1,q-1)$ is nonzero finite number. The number $\rho(p,q)$ is called the (p,q)-order of f.

Bajpai and Juneja have also defined the concept of the (p,q)-type $\sigma(p,q)$, for $b < \rho(p,q) < +\infty$, by

$$\sigma(p,q) = \limsup_{r \to +\infty} \frac{\log^{[p-1]}(M(f,r))}{\left(\log^{[q-1]}(r)\right)^{\rho(p,q)}}. \tag{1.5}$$

In their works, the authors have established the relationship of (p,q)-growth of f with respect to the coefficients a_k in the Maclaurin series of f in complex plane $\mathbb{C}$ (for $(p,q) = (2,1)$ we obtain the classical case).

We have also many results in terms in polynomial approximation in classical case. Let K be a compact subset of the complex plane $\mathbb{C}$, of positive logarithmic capacity and f be a complex function defined and bounded on K. For $k \in \mathbb{N}$ put

$$E_k(K,f) = \|f - T_k\|_K, \tag{1.6}$$

where the norm $\|\cdot\|_K$ is the maximum on K and T_k is the kth Chebytchev polynomial of the best approximation to f on K.

It is known (see [3]) that

$$\lim_{k \to +\infty} \sqrt[k]{E_k(K,f)} = 0 \tag{1.7}$$

if and only if f is the restriction to K of an entire function g in $\mathbb{C}$.

This result has been generalized by Reddy (see [4, 5]) as follows:

$$\lim_{k \to +\infty} \sqrt[k]{E_k(K,f)} = (\rho \cdot e \cdot \sigma)2^{-\rho} \tag{1.8}$$

if and only if f is the restriction to K of an entire function g of order ρ and type σ for $K = [-1, 1]$.

In the same way Winiarski (see [6]) has generalized this result for a compact K of the complex plane $\mathbb{C}$, of positive logarithmic capacity noted $c = \operatorname{cap}(K)$ as follows.

If K be a compact subset of the complex plane $\mathbb{C}$, of positive logarithmic capacity then

$$\lim_{k \to +\infty} k^{(1/\rho)} \sqrt[k]{E_k(K, f)} = c(e\rho\sigma)^{1/\rho} \tag{1.9}$$

if and only if f is the restriction to K of an entire function of order ρ $(0 < \rho < +\infty)$ and type σ.

Recall that $\operatorname{cap}([-1, 1]) = 1/2$ and capacity of a disk unit is $\operatorname{cap}(D(O, 1)) = 1$.

The authors considered the Taylor development of f with respect to the sequence $(z_n)_n$ and the development of f with respect to the sequence $(W_n)_n$ defined by

$$W_n(z) = \prod_{j=1}^{j=n} (z - \eta_{nj}), \tag{1.10}$$

where $(a_{nj})_n$ is the th$n$th extremal points system of K.

The aim of this paper is to establish relationship between the rate at which $(\pi_k^p(K, f))^{1/k}$ tends to zero in terms of best approximation in L^p-norm, and the generalized growth of entire functions of several complex variables for a compact subset K of $\mathbb{C}^n$, where K is a compact well-selected. In this work we give the generalization of these results in $\mathbb{C}^n$, replacing the circle $\{z \in \mathbb{C}; |z| = r\}$ by the set $\{z \in \mathbb{C}^n; \exp(V_E(z)) < r\}$, where V_E is the Siciak's extremal function of E a compact of $\mathbb{C}^n$ which will be defined later satisfying some properties.

2. Definitions and Notations

Before we give some definitions and results which will be frequently used in this paper.

Definition 2.1 (see Siciak [7]). Let K be a compact set in $\mathbb{C}^n$ and let $\|\cdot\|_K$ denote the maximum norm on K. The function

$$V_K = \sup\left\{ \frac{1}{d} \log|P_d|,\ P_d \text{ polynomial of degree } \leq d,\ \|P_d\|_K \leq 1,\ d \in \mathbb{N} \right\} \tag{2.1}$$

is called the Siciak's extremal function of the compact K.

Definition 2.2. A compact K in $\mathbb{C}^n$ is said to be L-regular if the extremal function, V_K, associated to K is continuous on $\mathbb{C}^n$.

Regularity is equivalent to the following Bernstein-Markov inequality (see [8]).

For any $\epsilon > 0$, there exists an open $U \supset K$ such that for any polynomial P

$$\|P\|_U \leq e^{\epsilon \cdot \deg(P)} \|P\|_K. \tag{2.2}$$

In this case we take $U = \{z \in \mathbb{C}^n;\ V_K(z) < \epsilon\}$.

Regularity also arises in polynomial approximation. For $f \in C(K)$, we let

$$\epsilon_d(K, f) = \inf\{\|f - P\|_K,\ P \in \mathcal{P}_k(\mathbb{C}^n)\}, \tag{2.3}$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the set of polynomials of degree at most d. Siciak (see [7]) showed the following.

If K is L-regular, then

$$\limsup_{d \to +\infty} (\varepsilon_d(K, f))^{1/d} = \frac{1}{r} < 1 \tag{2.4}$$

if and only if f has an analytic continuation to $\{z \in \mathbf{C}^n;\ V_K(z) < \log(1/r)\}$.

Let f be a function defined and bounded on K. For $k \in \mathbb{N}$ put

$$\pi_k^p(K, f) = \inf\left\{\|f - P\|_{L^p(K,\mu)},\ P \in \mathcal{P}_k(\mathbb{C}^n)\right\}, \tag{2.5}$$

where $\mathcal{P}_k(\mathbb{C}^n)$ is the family of all polynomial of degree $\leq k$ and μ the well-selected measure (The equilibrium measure $\mu = (dd^c V_K)^n$ associated to a L-regular compact E) (see [9]) and $L^p(K, \mu)$, $p \geq 1$, is the class of all function such that:

$$\|f\|_{L^p(K,\mu)} = \left(\int_K |f|^p d\mu\right)^{1/p} < \infty. \tag{2.6}$$

For an entire function $f \in \mathbb{C}^n$ we establish a precise relationship between the general growth with respect to the set:

$$\Omega_r = \{\exp(V_K) < r\} \tag{2.7}$$

and the coefficients of the development of f with respect to the sequence $(A_k)_k$, called extremal polynomial (see [10]). Therefore, we use these results to give the relationship between the generalized growth of f and the sequence $(\pi_k^p(K, f))_k$.

Recall that the subset of $\mathbb{C}^n$, $\Omega_r = \{\exp(V_K) < r\}$, replaces the unit disc $D(O, r) = \{|z| < r\}$ in the classical case.

It is known that if K is an compact L-regular of $\mathbb{C}^n$, there exists a measure μ, called extremal measure, having interesting properties (see [7, 8]), in particular, we have:

(P_1) Bernstein-Markov Inequality.

For all $\epsilon > 0$, there exists $C = C_\varepsilon$ is a constant such that

$$(\mathrm{BM}) : \|P_d\|_K = C(1 + \varepsilon)^{s_k} \|P_d\|_{L^2(K,\mu)}, \tag{2.8}$$

for every polynomial of n complex variables of degree at most d.

(P_2) Bernstein-Waish (B.W) Inequality.

For every set L-regular K and every real $r > 1$, we have:

$$\|f\|_K \leq M \cdot r^{\deg(f)} \left(\int_K |f|^p \cdot d\mu \right)^{1/p}. \tag{2.9}$$

Note that the regularity is equivalent to the Bernstein-Markov inequality.

Let $\alpha : \mathbb{N} \to \mathbb{N}^n, k \to \alpha(k) = (\alpha_1(k), \ldots, \alpha_n(k))$ be a bijection such that

$$|\alpha(k+1)| \geq |\alpha(k)| \quad \text{where } |\alpha(k)| = \alpha_1(k) + \cdots + \alpha_n(k). \tag{2.10}$$

Zériahi (see [10]) has constructed according to the Hilbert-Shmidt method a sequence of monic orthogonal polynomial according to a extremal measure (see [8]), $(A_k)_k$, called extremal polynomial, defined by

$$A_k(z) = z^{\alpha(k)} + \sum_{j=1}^{k-1} a_j z^{\alpha(j)} \tag{2.11}$$

such that

$$\|A_k\|_{L^p(K,\mu)} = \left[\inf\left\{ \left\| z^{\alpha(k)} + \sum_{j=1}^{k-1} a_j z^{\alpha(j)} \right\|_{L^2_{(K,\mu)}}, \ (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n \right\} \right]^{1/\alpha_k}. \tag{2.12}$$

We need the following notations which will be used in the sequel:

(N_1) $\nu_k = \nu_k(K) = \|A_k\|_{L^2(K,\mu)}$.

(N_2) $a_k = a_k(K) = \|A_k\|_K = \max_{z \in K} |A_k(z)|$ and $\tau_k = (a_k)^{1/s_k}$, where $s_k = \deg(A_k)$.

With that notations and (B.W) inequality, we have

$$\|A_k\|_{\overline{\Omega}_r} \leq a_k \cdot r^{s_k}, \tag{2.13}$$

where $s_k = \deg(A_k)$. For more details (see [9]).

Let α and β be two positives, strictly increasing to infinity differentiable functions $]0, +\infty[$ to $]0, +\infty[$ such that for every $c > 0$:

$$\begin{gathered} \lim_{x \to +\infty} \frac{\alpha(cx)}{\alpha(x)} = 1, \\ \lim_{x \to +\infty} \frac{\beta(1 + x\omega(x))}{\beta(x)} = 1, \end{gathered} \tag{2.14}$$

where ω a function such that $\lim_{x \to +\infty} \omega(x) = 0$.

Assume that, for every $c > 0$, there exists two constants a and b such that for every $x \geq a$:

$$\left| \frac{d(\beta^{-1}(c\alpha(x)))}{\alpha(\log(x))} \right| \leq b, \tag{2.15}$$

where $d(u)$ means the differential of u.

Definition 2.3. Let K be a compact L-regular, we put

$$\Omega_r = \left\{ z \in \mathbb{C}^n;\ \exp V_K(z) \leq r \right\}. \tag{2.16}$$

If f is an entire function we define the (α, β)-order and the (α, β)-type of f (or generalized order and generalized type), respectively, by

$$\begin{aligned} \rho(\alpha, \beta) &= \limsup_{r \to +\infty} \frac{\alpha\left(\log\left(\|f\|_{\overline{\Omega}_r}\right)\right)}{\beta(\log(r))}, \\ \sigma(\alpha, \beta) &= \limsup_{r \to +\infty} \frac{\alpha\left(\|f\|_{\overline{\Omega}_r}\right)}{[\beta(r)]^{\rho(\alpha,\beta)}}, \end{aligned} \tag{2.17}$$

where $\|f\|_{\overline{\Omega}_r} = \sup_{\overline{\Omega}_r} |f(z)|$.

Note that in the classical case $\alpha(x) = \log(x)$ and $\beta(x) = x$.

In this paper we will consider a more generalized growth to extend the classical results to a large class of entire functions of several variables.

We need the following lemma, see [10].

Lemma 2.4. *Let K be a compact L-regular subset of $\mathbb{C}^n$. Then for every $\theta > 1$, there exists an integer $N_\theta \geq 1$ and a constant C_θ such that:*

$$\pi_k^p(K, f) \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^k}. \tag{2.18}$$

Let $f = \sum_{k=0}^{+\infty} f_k \cdot A_k$ be an entire function. Then for every $\theta > 1$, there exists $N_\theta \in \mathbb{N}$ and $C_\theta > 0$ such that

$$|f_k| \nu_k \leq C_\theta \frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}} \frac{\|f\|_{\overline{\Omega}_{r\theta}}}{r^{s_k}}, \tag{2.19}$$

for every $k \geq 1$ and $r > 1$.

Note that the second assertion of the lemma is a consequence of the first assertion and it replaces Cauchy inequality for complex function defined on the complex plane $\mathbb{C}$.

3. Generalized Growth and Coefficients of the Development with Respect to Extremal Polynomial

The purpose of this section is to establish this relationship of the generalized growth of an entire function with respect to the set

$$\Omega_r = \{\exp(V_K) < r\} \tag{3.1}$$

and the coefficient of entire function $f \in \mathbb{C}^n$ of the development with respect to the sequence of extremal polynomials.

Let $(A_k)_k$ be a basis of extremal polynomial associated to the set K defined the relation (2.11). We recall that $(A_k)_k$ is a basis of $\mathcal{O}(\mathbb{C}^n)$ (the set of entire functions on $\mathbb{C}^n$). So if f is an entire function then

$$f = \sum_{k\geq 1} f_k \cdot A_k \tag{3.2}$$

and we have the following results.

Theorem 3.1. *If $f = \sum_{k\geq 1} f_k \cdot A_k$ then the (α,β)-order $\rho(\alpha,\beta)$ of f is given by formula*

$$\rho(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left(-(1/s_k)\ln\left|f_k\right|\cdot\tau_k^{s_k}\right)} < +\infty. \tag{3.3}$$

To prove theorem we need the following lemmas.

Lemma 3.2. *Let K be a compact L-regular subset of $\mathbb{C}^n$. Then*

$$\lim_{k\to+\infty}\left[\frac{|A_k(z)|}{\nu_k}\right]^{1/s_k} = \exp(V_K(z)), \tag{3.4}$$

for every $z \in \mathbb{C}^n \setminus \widehat{K}$ the connected component of $\mathbb{C}^n \setminus K$,

$$\lim_{k\to+\infty}\left[\frac{\|A_k\|_K}{\nu_k}\right]^{1/s_k} = 1. \tag{3.5}$$

Lemma 3.3. *For every $r > 1$ the maximum of the function*

$$x \longrightarrow \omega(x,r) = r^x \exp\left[-x\beta^{-1}\left(\frac{1}{\mu}\alpha(x)\right)\right] \tag{3.6}$$

is reached for $x = x_r$ solution of the equation

$$x = \alpha^{-1}\left\{\mu\beta\left[\log(r) - \frac{d\left(\beta^{-1}((1/\mu)\alpha(x))\right)}{d(\log(x))}\right]\right\}. \tag{3.7}$$

Consequence. This relation is equivalent to

$$x \leq \alpha^{-1}\left(\beta\left(\log(r) + b\right)\right). \tag{3.8}$$

Indeed, the relations

$$\begin{gathered}\left|\frac{d\left(\beta^{-1}(c\alpha(x))\right)}{d\left(\log(x)\right)}\right| \leq b, \\ \beta^{-1}\left(\frac{\alpha(x)}{\mu}\right) = \log(r) - \frac{d\left(\beta^{-1}(c\alpha(x))\right)}{d\left(\log(x)\right)}\end{gathered} \tag{3.9}$$

give

$$\log(r) - b \leq \log(r) - \frac{d\left(\beta^{-1}(c\alpha(x))\right)}{d\left(\log(x)\right)} \leq \log(r) + b. \tag{3.10}$$

Then

$$\begin{gathered}\beta^{-1}\left(\frac{\alpha(x)}{\mu}\right) \leq \log(r) + b, \\ x \leq \alpha^{-1}\left(\beta\left(\log(r) + b\right)\right).\end{gathered} \tag{3.11}$$

We verify easily with the relation

$$\left|\frac{d\left(\beta^{-1}(c\alpha(x))\right)}{\alpha\left(\log(x)\right)}\right| \leq b, \tag{3.12}$$

that

$$r^n \exp\left[-n\beta^{-1}\left(\frac{1}{\mu}\alpha(n)\right)\right] \leq \exp\left\{b\alpha^{-1}\left(\mu \cdot \beta\left(\log(r) + b\right)\right)\right\} \tag{3.13}$$

for every $n > n_0$ and $r > 0$.

Proof of Theorem 3.1. Put $\gamma = \limsup_{k\to+\infty}(\alpha(s_k)/\beta(-(1/s_k)\ln|f_k|\cdot\tau_k^{s_k}))$ and show that $\gamma = \rho(\alpha,\beta)$.

(1) Show that $\gamma \leq \rho(\alpha,\beta)$.

By definition of γ, we have, for all $\epsilon > 0$, $\exists k_\epsilon$ such that for all $k > k_\epsilon$

$$\frac{\alpha(s_k)}{\beta(-(1/s_k)\ \ln\left|f_k\right| \cdot \nu_k^{s_k})} < \gamma + \epsilon. \tag{3.14}$$

From the second assertion of the Lemma 3.2, we have for every $\theta > 1$, there exists $N_\theta \in \mathbb{N}$ and $C_\theta > 0$ such that

$$\begin{aligned} \log\left(\left|f_k\right|\tau_k^{s_k}\right) \leq \log(C_\theta) + \log\left(\frac{(r+1)^{N_\theta}}{(r-1)^{2N-1}}\right) \\ - s_k \log(r) + \alpha^{-1}\left[(\rho+\epsilon)\cdot\beta\left(\log(r\theta)\right)\right] = \varphi(r) \end{aligned} \tag{3.15}$$

or

$$\begin{aligned} -\frac{1}{s_k}\log\left(\left|f_k\right|\tau_k^{s_k}\right) \geq -\frac{1}{s_k}\log(C_\theta) - \frac{N_\theta}{s_k}\log(r+1) - \frac{2N-1}{s_k}\log(r-1) + \log(r) \\ - \frac{1}{s_k}\alpha^{-1}\left((\rho+\epsilon)\beta\left(\log(r\theta)\right)\right) = \varphi(r,k). \end{aligned} \tag{3.16}$$

But $\lim_{k\to+\infty}\varphi(r,k) = \log(r)$ so $\varphi(r,k) \sim \log(r)$ for r sufficiently large.
Then

$$-\frac{1}{s_k}\log\left(\left|f_k\right|\tau_k^{s_k}\right) \geq (1+o(1))\log(r). \tag{3.17}$$

Let r_k be a real satisfying

$$r_k = \exp\left[\beta^{-1}\left(\frac{1}{\rho+\epsilon}\alpha(s_k)\right)\right]. \tag{3.18}$$

Then

$$-\frac{1}{s_k}\log\left(\left|f_k\right|\tau_k^{s_k}\right) \geq \beta^{-1}\left(\frac{1}{\rho+\epsilon}\cdot\alpha(s_k)\right)(1+o(1)). \tag{3.19}$$

This is equivalent to

$$\frac{\beta\left[-(1/s_k)\log\left(\left|f_k\right|\tau_k^{s_k}\right)\right]}{\alpha(s_k)} \geq \frac{1}{\rho+\epsilon}, \tag{3.20}$$

or

$$\frac{\alpha(s_k)}{\beta\left[-(1/s_k)\log\left(\left|f_k\right|\tau_k^{s_k}\right)\right]} \leq \rho+\epsilon, \tag{3.21}$$

hence

$$\gamma \leq \rho(\alpha,\beta). \tag{3.22}$$

(2) Show that $\gamma \geq \rho(\alpha,\beta)$. According to the definition of γ, we have for every $\epsilon > 0$ there exists k_ϵ such that for all $k > k_\epsilon$

$$|f_k|\tau_k^{s_k} \leq \exp\left[-s_k\beta^{-1}\left(\frac{\alpha(s_k)}{\gamma+\epsilon}\right)\right] \tag{3.23}$$

for $k \geq k_\epsilon$ sufficiently large.

According to the first assertion of Lemma 3.2 and (BM) and (BW) inequalities, we have

$$\|A_k\|_{\overline{\Omega}_r} \leq a_k \cdot r^{s_k} \tag{3.24}$$

and for every entire function

$$f(z) = \sum_{k\geq 0} f_k \cdot A_k(z),$$
$$\|f\|_{\overline{\Omega}_r} \leq \underbrace{\sum_{k=0}^{k_\epsilon} a_k \cdot r^{s_k}}_{(1)} + \underbrace{C_\epsilon \sum_{k=k_\epsilon+1}^{k_r} |f_k| \cdot \nu^{s_k} \cdot (1+\epsilon)r^{s_k}}_{(2)} + \underbrace{C_\epsilon \sum_{k=k_r+1}^{+\infty} |f_k| \cdot \tau_k^{s_k} \cdot r^{s_k}}_{(3)}. \tag{3.25}$$

The term (1) is a constant denoted C_0, and

$$\sum_{k=k_\epsilon+1}^{k_r} |f_k| \cdot \tau_k^{s_k} \cdot (1+\epsilon)r^{s_k} \leq ((1+\epsilon)r)^{s_{k(r)}} \underbrace{\sum_{k=0}^{+\infty} |f_k| \cdot \tau_k^{s_k}}_{(4)}. \tag{3.26}$$

The series (4) is convergent.

Let, for r sufficiently,

$$N(r) = E\left[\alpha^{-1}\left((\gamma+\epsilon)\beta\left(\log 2(1+\epsilon)r\right)\right)\right], \tag{3.27}$$

where $E(x)$ means the integer part of x.

Then

$$|f_k| \cdot \nu_k \cdot ((1+\epsilon)r)^{s_k} \leq \exp\left[-s_k \cdot \beta^{-1}\left(\frac{\alpha(s_k)}{\gamma+\epsilon}\right)\right]((1+\varepsilon)r)^{s_k}. \tag{3.28}$$

Applying the the relation (3.13) with $\mu = \gamma+\epsilon$ and if we replace r by $(1+\varepsilon)r$, we obtain:

$$\|f\|_{\overline{\Omega}_r} \leq \exp\left[b \cdot \alpha^{-1}\left((\gamma+\epsilon) \cdot \beta\left(\log((1+\varepsilon)r) + b\right)\right)\right]. \tag{3.29}$$

And so

$$\|f\|_{\overline{\Omega}_r} \leq C \cdot ((1+\varepsilon)r)^{s_k} \cdot \exp\left[-s_k \cdot \beta^{-1}\left(\frac{\alpha(s_k)}{\gamma+\epsilon}\right)\right], \tag{3.30}$$

or

$$\|f\|_{\overline{\Omega}_r} \leq C \cdot \exp b \cdot \alpha^{-1}\left((\gamma+\epsilon) \cdot \beta\left(\log((1+\varepsilon)r) + b\right)\right). \tag{3.31}$$

Thus

$$\log\left(\|f\|_{\overline{\Omega}_r}\right) \leq \log(C) + b \cdot \alpha^{-1}\left((\gamma+\epsilon) \cdot \beta\left(\log((1+\varepsilon)r) + b\right)\right) \tag{3.32}$$

and for r sufficiently large, we have

$$\frac{\alpha\left(b^{-1}\log\left(\|f\|_{\overline{\Omega}_r}\right)\right)}{\beta(\log((1+\varepsilon)r) + b)} \leq \gamma + \epsilon. \tag{3.33}$$

But $\alpha(c \cdot x) \sim \alpha(x)$ near to the infinity, thus

$$\alpha\left(b^{-1}\log\left(|f|_{\overline{\Omega}_r}\right)\right) \sim \alpha\left(\log\left(\|f\|_{\overline{\Omega}_r}\right)\right). \tag{3.34}$$

If we put $x = \log((1+\varepsilon)r)$, then $\beta(\log((1+\epsilon)r) + b) \sim \beta(\log((1+\epsilon)r))$ and,

$$\frac{\alpha\left(\log\left(\|f\|_{\overline{\Omega}_r}\right)\right)}{\beta(\log(r))} \leq \gamma + \epsilon. \tag{3.35}$$

This is true for every $\epsilon > 0$ hence $\rho(\alpha, \beta) \leq \gamma$. Thus the assertion is proved. $\square$

4. Best Approximation Polynomial in L^p-Norm

To our knowledge, no similar result is known according to polynomial approximation in L^p-norm ($1 \leq p \leq \infty$) with respect to a measure μ on K in $\mathbb{C}^n$.

The purpose of this paragraph is to give the relationship between the generalized order and speed of convergence to 0 in the best polynomial. We need the following lemma.

Lemma 4.1. *Let $f = \sum_{k\geq 0} f_k \cdot A_k$ an element of $L^p(K,\mu)$, for $p \geq 1$, then*

$$\limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[-(1/k)\log\left(\pi_k^p(K,f)\right)\right]} = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta\left[-(1/s_k)\log\left(|f_k| \cdot \tau_k^{s_k}\right)\right]}. \tag{4.1}$$

Proof of Lemma 4.1. The proof is done in two steps $p \geq 2$ and $1 < p < 2$.

Step 1. If $f \in L^p(K,\mu)$ where $p \geq 2$, then $f = \sum_{k=0}^{+\infty} f_k \cdot A_k$ with convergence in $L^2(K,\mu)$, hence for $k \geq 0$

$$f_k = \frac{1}{\nu_k^2} \int_K f \cdot \overline{A}_k d\mu \tag{4.2}$$

and therefore

$$f_k = \frac{1}{\nu_k^2} \int_K \left(f - P_{k-1}\right) \cdot \overline{A}_k d\mu \tag{4.3}$$

(because $\deg(A_k) = s_k$).

Since the relation,

$$\left|f_k\right| \leq \frac{1}{\nu_k^2} \int_K \left|f - P_{k-1}\right| \cdot \left|\overline{A}_k\right| \mu \tag{4.4}$$

satisfied, is easily verified by using inequalities Bernstein-walsh and Holder that, we have for all $\varepsilon > 0$

$$\left|f_k\right| \cdot \nu_k \leq C_\varepsilon \cdot (1+\varepsilon)^{s_k} \cdot \pi^p_{s_k-1}(K,f). \tag{4.5}$$

for all $k \geq 0$.

Step 2. If $1 \leq p < 2$, let p' such that $1/p + 1/p' = 1$, we have $p' \geq 2$. According to the inequality of Hölder, we have:

$$\left|f_k\right| \cdot \nu_k^2 \leq \left\|f - P_{k-1}\right\|_{L^p(K,\mu)} \cdot \left\|A_k\right\|_{L^{p'}(K,\mu)}. \tag{4.6}$$

But,

$$\left\|A_k\right\|_{L^{p'}(K,\mu)} \leq C \cdot \left\|A_k\right\|_K = C \cdot a_k(K). \tag{4.7}$$

This shows, according to inequality (BM), that:

$$\left|f_k\right| \cdot \nu_k^2 \leq C \cdot C_\varepsilon \cdot (1+\varepsilon)^{s_k} \cdot \left\|f - P_{s_k-1}\right\|_{L^p(K,\mu)}. \tag{4.8}$$

Hence the result

$$\left|f_k\right| \cdot \nu_k^2 \leq C'_\varepsilon \cdot (1+\varepsilon)^{s_k} \cdot \pi^p_{s_k}(K,f). \tag{4.9}$$

In both cases, we have therefore

$$\left|f_k\right| \cdot v_k^2 \leq A_\varepsilon \cdot (1+\varepsilon)^{s_k} \cdot \pi_{s_k}^p(K, f), \tag{4.10}$$

where A_ε is a constant which depends only on ε.

After passing to the upper limit in the relation (4.10) and Applying the relation (3.5) of the Lemma 3.2 we get

$$\limsup_{k \to +\infty} \frac{\alpha(k)}{\beta\left[-(1/k) \log\left(\pi_k^p(K, f)\right)\right]} \geq \limsup_{k \to +\infty} \frac{\alpha(s_k)}{\beta\left[-(1/s_k) \log\left(\left|f_k\right| \cdot \tau_k^{s_k}\right)\right]}. \tag{4.11}$$

To prove the other inequality we consider the polynomial of degree s_k,

$$P_k(z) = \sum_{s_j=0}^{k} f_j \cdot A_j \tag{4.12}$$

then

$$\pi_{s_k-1}^p(K, f) \leq \sum_{s_j=s_k}^{+\infty} \left|f_j\right| \cdot \left\|A_j\right\|_{L^p(K,\mu)} \leq C_0 \sum_{s_j=s_k}^{+\infty} \left|f_j\right| \cdot \left\|A_j\right\|_K. \tag{4.13}$$

By Bernstein-Walsh inequality, we have

$$\pi_k^p(K, f) \leq C_\epsilon \sum_{s_j=s_k}^{+\infty} (1+\epsilon)^{s_j} \left|f_j\right| \cdot v_j \tag{4.14}$$

for $k \geq 0$ and $p \geq 1$. If we take as a common factor $(1+\varepsilon)^{s_k} \cdot |f_k| \cdot v_k$ the other factor is convergent thus, we have

$$\pi_k^p(K, f) \leq C(1+\epsilon)^{s_k} \cdot \left|f_k\right| \cdot v_k \tag{4.15}$$

and by (3.5) of Lemma 3.2, we have, then

$$\pi_k^p(K, f) \leq C(1+\epsilon)^{2s_k} \cdot \left|f_k\right| \cdot \tau_k^{s_k}. \tag{4.16}$$

We deduce

$$\limsup_{k \to +\infty} \frac{\alpha(k)}{\beta\left[-(1/k) \log\left(\pi_k^p(K, f)\right)\right]} = \limsup_{k \to +\infty} \frac{\alpha(s_k)}{\beta\left[-(1/s_k) \log\left(\left|f_k \cdot \tau_k^{s_k}\right|\right)\right]}. \tag{4.17}$$

This inequality is a direct consequence of the relation (4.10) and the inequality on coefficients $|f_k|$ given by

$$\begin{gathered}|f_k| \cdot \tau_k^{s_k} \leq \exp\left[-s_k\beta^{-1}\left(\alpha(s_k)^{1/(\rho+\epsilon)}\right)\right],\\ |f_k| \cdot \nu^2 \leq C_\varepsilon \cdot (1+\varepsilon)^{s_k} \cdot \pi_{s_k}^p(K,f).\end{gathered} \tag{4.18}$$

□

Applying this lemma we get the following main result:

Theorem 4.2. *Let $f \in L^p(K,\mu)$, then f is μ-almost-surely the restriction to K of an entire function in $\mathbb{C}^n$ of finite generalized order $\rho(\alpha,\beta)$ finite if and only if*

$$\rho(\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[-(1/k)\log\left(\pi_k^p(K,f)\right)\right]} < +\infty. \tag{4.19}$$

Proof. Suppose that f is μ-almost-surely the restriction to K of an entire function g of general order ρ $(0<\rho<+\infty)$ and show that $\rho=\rho(\alpha,\beta)$.

We have $g \in L^p(K,\mu)$, $p \geq 2$ and $g = \sum_{k\geq 0} g_k \cdot A_k$ in $L^2(K,\mu)$ Since g is an element of $L^2(K,\mu)$ then $g = \sum_{k=0}^{+\infty} g_k \cdot A_k$ and according to the Theorem 3.1.

$$\rho(g,\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta[-(1/s_k)\log(|g_k| \cdot \tau_k^{s_k})]} \tag{4.20}$$

and with the Lemma 4.1, we have

$$\limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left(-(1/k)\log\left(\pi_k^p(K,g)\right)\right)} = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta[-(1/s_k)\log(|g_k| \cdot \tau_k^{s_k})]}. \tag{4.21}$$

But $g=f$ on K hence

$$\rho = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta(-(1/k)\log(\pi_k^t(K,f)))} < +\infty. \tag{4.22}$$

Suppose now that f is a function of $L^p(K,\mu)$ such that the relation (4.19) is verified.

(1) Let $p \geq 2$, then $f = \sum_{k=0}^{+\infty} f_k \cdot A_k$, because f is an element of $L^2(K,\mu)$ $((L^p(K,\mu))_{p\geq 1}$ is decreasing sequence).

Consider in $\mathbb{C}^n$ the series $\sum f_k \cdot A_k$, $k \geq 0$, we verify easily that this series converges normally on all compact of $\mathbb{C}^n$ to an entire function denoted f_1. We have $f_1 = f$, obviously, μ-almost surly on K, and by Theorem 3.1, we have

$$\rho(f_1) = \limsup_{k\to+\infty} \frac{\alpha(s_k)}{\beta[-(1/s_k)\log(|f_k| \cdot \tau_k^{s_k})]} = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta[-(1/k)\log(\pi_k^t(K,f))]} < +\infty. \tag{4.23}$$

Applying the Lemma 4.1, we find

$$\rho(f_1) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[-(1/k)\log\left(\pi_k^t(K,f)\right)\right]} < +\infty. \tag{4.24}$$

Consider the function $f_1 = \sum_{k\geq 0} f_k \cdot A_k$, we have $f_1(z) = f(z)\mu$-almost surely for every z in K. Therefore the (α,β)-order of f_1 is:

$$\rho(f_1,\alpha,\beta) = \limsup_{k\to+\infty} \frac{\alpha(k)}{\beta\left[-(1/k)\log\left(\pi_k^t(K,f)\right)\right]} < +\infty \tag{4.25}$$

(see Theorem 3.1).

By Lemma 4.1, we check $\rho(f_1) = \rho$ so the proof is completed. □

(2) Now let $p \in [1,2[$ and $f \in L^p(K,\mu)$.

By (BM) inequality and Hölder inequality, we have again the inequality the relation (4.10) and by the previous arguments we obtain the result.

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