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Research Article

Note on Isomorphism Theorems of Hyperrings

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There are different notions of hyperrings $(R,+,\cdot)$. In this paper, we extend the isomorphism theorems to hyperrings, where the additions and the multiplications are hyperoperations.

1. Introduction

The theory of hyperstructures was introduced in 1934 by Marty [1] at the 8th Congress of Scandinavian Mathematicians. This theory has been subsequently developed by Corsini [2–4], Mittas [5, 6], Stratigopoulos [7], and by various authors. Basic definitions and propositions about the hyperstructures are found in [3, 4, 8]. Krasner [9] has studied the notion of hyperfields, hyperrings, and then some researchers, namely, Ameri [10], Dašić [11], Davvaz [12], Gontineac [13], Massouros [14], Pianskool et al. [15], Sen and Dasgupta [16], Vougiouklis [8, 17], and others followed him.

Hyperrings are essentially rings with approximately modified axioms. There are different notions of hyperrings $(R,+,\cdot)$. If the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a binary operation, then the hyperring is called Krasner (additive) hyperring [9]. Rota [18] introduced a multiplicative hyperring, where $+$ is a binary operation and the multiplication $\cdot$ is a hyperoperation. De Salvo [19] studied hyperrings in which the additions and the multiplications were hyperoperations. These hyperrings were also studied by Barghi [20] and by Asokkumar and Velrajan [21–23]. In 2007, Davvaz and Leoreanu-Fotea [24] published a book titled Hyperring Theory and Applications. Davvaz [12] extended that the isomorphism theorems to Krasner hyperrings, provided the hyperideals considered in the isomorphism theorems are normal.

In this paper, we extend the isomorphism theorems to hyperrings, in which both the additions and the multiplications are hyperoperations. Also, it is observed that if I is an hyperideal of a hyperring R and $(I,+)$ is a normal subcanonical hypergroup of $(R,+)$, then

R/I is a ring, and hence the quotient hyperrings considered in the isomorphism theorems by Davvaz in [12] are rings.

2. Basic Definitions and Notations

This section provides some basic definitions that have been used in the sequel. A *hyperoperation* $\circ$ on a nonempty set H is a mapping of $H \times H$ into the family of nonempty subsets of H (i.e., $x \circ y \subseteq H$, for every $x, y \in H$). The definitions are found in references [3, 4, 8, 24]. A *hypergroupoid* is a nonempty set H equipped with a hyperoperation $\circ$. For any two nonempty subsets A and B of a hypergroupoid H and for $x \in H$, $A \circ B$, we mean the set $\bigcup_{a\in A, b\in B}(a \circ b)$, $A \circ x = A \circ \{x\}$, and $x \circ B = \{x\} \circ B$.

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if $x \circ (y \circ z) = (x \circ y) \circ z$ for every $x, y, z \in H$ (the associative axiom). A hypergroupoid $(H, \circ)$ is called a *quasihypergroup* if $x \circ H = H \circ x = H$ for every $x \in H$ (the reproductive axiom). A reproductive semihypergroup is called a *hypergroup* (in the sense of Marty). A comprehensive review of the theory of hypergroups appears in [3].

Definition 2.1. A nonempty set H with a hyperoperation $+$ is said to be a *canonical hypergroup* if the following conditions hold:

(i) for every $x, y \in H$, $x + y = y + x$,

(ii) for every $x, y, z \in H$, $x + (y + z) = (x + y) + z$,

(iii) there exists $0 \in H$ (called neutral element of H) such that $0 + x = \{x\} = x + 0$ for all $x \in H$,

(iv) for every $x \in H$, there exists a unique element denoted by $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,

(v) for every $x, y, z \in H$, $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$.

Example 2.2. Consider the set $H = \{0, x, y\}$. Define a hyperaddition $+$ on H as in the following table. Then, $(H, +)$ is a canonical hypergroup.

$+$	0	x	y
0	0	x	y
x	x	$\{0, x\}$	y
y	y	y	$\{0, x, y\}$

(2.1)

The following elementary facts in a canonical hypergroup H easily follow from the axioms.

(i) $-(-a) = a$ for every $a \in H$,

(ii) 0 is the unique element such that for every $a \in H$, there is an element $-a \in H$ with the property $0 \in a + (-a)$,

(iii) $-0 = 0$,

(iv) $-(a + b) = -a - b$ for all $a, b \in H$.

For any subset A of a canonical hypergroup H, $-A$ denotes the set $\{-a : a \in A\}$. A nonempty subset N of a canonical hypergroup of H is called a *subcanonical hypergroup* of H if N is a canonical hypergroup under the same hyperoperation as that of H. Equivalently, for every $x, y \in N$, $x - y \subseteq N$. In particular, for any $x \in N$, $x - x \subseteq N$. Since $0 \in x - x$, it follows that $0 \in N$.

Definition 2.3. An equivalence relation ρ defined on a canonical hypergroup $(H, +)$ is called *strongly regular* if for all $x, y \in H$ and $x\rho y$ implies that for every $p \in H$, for every $a \in x + p$ and for every $b \in y + p$ one has $a\rho b$.

Definition 2.4. A subcanonical hypergroup A of a canonical hypergroup H is said to be normal if $x + A - x \subseteq A$ for all $x \in A$.

Definition 2.5. The *heart* of a canonical hypergroup H is the union of the sums $(x_1 - x_1) + (x_2 - x_2) + (x_3 - x_3) + \cdots + (x_n - x_n)$, where $x_i \in H$ and n is a natural number and it is denoted by ω_H.

Definition 2.6. Let H_1 and H_2 be two canonical hypergroups. A mapping ϕ from H_1 into H_2 is called a *homomorphism* from H_1 into H_2 if (i) $\phi(a + b) \subseteq \phi(a) + \phi(b)$ for all $a, b \in H_1$ and (ii) $\phi(0) = 0$ hold. The mapping ϕ is called a *good or strong homomorphism* if (i) $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in H_1$ and (ii) $\phi(0) = 0$ hold.

A homomorphism (resp., strong homomorphism) ϕ from a canonical hypergroup H_1 to a canonical hypergroup H_2 is called an *isomorphism (resp., strong isomorphism)* if ϕ is one to one and onto. If H_1 is strongly isomorphic to H_2, then we denote it by $H_1 \cong H_2$.

Definition 2.7. Let ϕ be a homomorphism from canonical hypergroup H_1 into a canonical hypergroup H_2. Then, the set $\{x \in H_1 : \phi(x) = 0\}$ is called kernel of ϕ and is denoted by $\operatorname{Ker}\phi$, and the set $\{\phi(x) : x \in H_1\}$ is called Image of ϕ and is denoted by $\operatorname{Im}\phi$.

It is clear that $\operatorname{Ker}\phi$ is a subcanonical hypergroup of H_1 and $\operatorname{Im}\phi$ is a subcanonical hypergroup of H_2. The definition of a hyperring given below is equivalent to one formulated by De Salvo [19] (see Corsini [3]) and studied by Barghi [20].

Definition 2.8. A *hyperring* is a triple $(R, +, \cdot)$, where R is a nonempty set with a hyperaddition $+$ and a hypermultiplication $\cdot$ satisfying the following axioms:

(1) $(R, +)$ is a canonical hypergroup,

(2) $(R, \cdot)$ is a semihypergroup such that $x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$, (i.e, 0 is a bilaterally absorbing element),

(3) The hypermultiplication $\cdot$ is *distributive* with respect to the hyperoperation $+$. That is, for every $x, y, z \in R$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

In a hyperring if the hypermultiplication is a binary operation, then it is called as *Krasner or additive hyperring*. Also, in the Definition 2.8, if the hyperaddition is a binary operation, then it is called as*multiplicative hyperring*.

Example 2.9. Let $R = \{0, 1\}$ be a set with two hyperoperations defined as follows:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & \{0\} & \{1\} \\ 1 & \{1\} & \{0,1\} \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & \{0\} & \{0\} \\ 1 & \{0\} & \{0,1\} \end{array} \tag{2.2}$$

Then, $(R, +, \cdot)$ is a hyperring.

Definition 2.10. Let R be a hyperring, and let I be a nonempty subset of R. I is called a *left* (resp., *right*) *hyperideal* of R if $(I, +)$ is a canonical subhypergroup of R and for every $a \in I$ and $r \in R$, $ra \subseteq I$ (resp., $ar \subseteq I$). A *hyperideal* of R is one which is a left as well as a right hyperideal of R.

If I, J are left (resp., right) hyperideals of a hyperring R, then $I + J$, $I \cap J$ are left (resp., right) hyperideal of R. If I, J are hyperideals of a hyperring R, then $I + J$, $I \cap J$ are hyperideals of R.

Definition 2.11. Let R_1 and R_2 be two hyperrings. A mapping ϕ from R_1 into R_2 is called a *homomorphism* if (i) $\phi(a + b) \subseteq \phi(a) + \phi(b)$; (ii) $\phi(ab) \subseteq \phi(a)\phi(b)$ and (iii) $\phi(0) = 0$ hold for all $a, b \in R_1$. The mapping ϕ is called a *good homomorphism* or *a strong homomorphism* if (i) $\phi(a + b) = \phi(a) + \phi(b)$; (ii) $\phi(ab) = \phi(a)\phi(b)$ and (iii) $\phi(0) = 0$ hold for all $a, b \in R_1$.

Definition 2.12. A homomorphism (resp., strong homomorphism) ϕ from hyperring R_1 into a hyperring R_2 is said to be an *isomorphism (resp., strong isomorphism)* if ϕ is one to one and onto. If R_1 is strongly isomorphic to R_2, then it is denoted by $R_1 \cong R_2$.

Remark 2.13. Let ϕ be a homomorphism from a hyperring R_1 into a hyperring R_2. Then $\operatorname{Ker}\phi$ is a hyperideal of R_1 and $\operatorname{Im}\phi$ is a hyperideal of R_2.

3. Canonical Hypergroups

Let N be a subcanonical hypergroup of a canonical hypergroup H. In this section, we construct quotient canonical hypergroup H/N and prove that when N is normal, H/N is an abelian group.

Proposition 3.1. *Let H be a canonical hypergroup, and let N be a subcanonical hypergroup of H. For any two elements $a, b \in H$, if we define a relation $a \sim b$ if $a \in b + N$, then $\sim$ is an equivalence relation on H.*

Proof. Let $a \in H$. Since $a = a + 0 \in a + N$, the relation $\sim$ is reflexive. Let $a, b \in H$. If $a \in b + N$, then $a \in b + n$ for some $n \in N$. That is, $b \in a - n \subseteq a + N$. So, $\sim$ is a symmetric relation. Suppose that $a, b, c \in H$ such that $a \sim b$ and $b \sim c$, then $a \in b + N$ and $b \in c + N$. Therefore, $a \in b + n$, and $b \in c + m$, for some $n, m \in N$. So, $a \in c + m + n \subseteq c + N$. Hence $a \sim c$. Therefore, the relation $\sim$ is transitive. □

Remark 3.2. Let N be a subcanonical hypergroup of a canonical hypergroup H. We denote the equivalence class determined by the element $x \in H$ by the equivalence relation $\sim$ by $\overline{x}$. It is clear that $\overline{x} = x + N$.

Proposition 3.3. *Let H be a canonical hypergroup, and let N be a normal subcanonical hypergroup of H. Then, for $x, y \in N$, the following are equivalent:*

(1) $y \in x + N$,

(2) $x - y \subseteq N$,

(3) $(x - y) \cap N \neq \emptyset$.

Proof. (1) implies (2).

Since $y \in x+N$, we have $y-x \subseteq x+N-x$. Since N is normal subcanonical hypergroup of H, we get $x + N - x \subseteq N$. Thus, $y - x \subseteq N$. That is, $-(y - x) \subseteq N$, and hence $x - y \subseteq N$.

(2) implies (3) is obvious.

(3) implies (1).

Since $(x - y) \cap N \neq \emptyset$, there exists $a \in x - y$ and $a \in N$. Therefore, $-y + x \subseteq -y + a + y \subseteq N$. If $z \in -y + x$, then $z \in N$. Therefore, $-y \in z - x$. That is, $y \in x - z \subseteq x + N$. □

Remark 3.4. Let H be a canonical hypergroup, and let N be a subcanonical hypergroup of H. When N is normal, the equivalence relation defined in the Proposition 3.1 coincides with the the equivalence relation defined by Davvaz [12]. Further, the Propositions 3.1 and 3.3 are true when the hyperaddition on the canonical hypergroup H is not commutative. Also, for any $x \in H$, we have $(\overline{-x}) = -(\overline{x})$.

Theorem 3.5. *Let H be a canonical hypergroup, N be a subcanonical hypergroup of H. Then for $x, y \in H$, the sets $A = \{\overline{z} : z \in x + y\}$, $B = \{\overline{z} : z \in \overline{x} + \overline{y}\}$ and $C = \{\overline{z} : \overline{z} \subseteq \overline{x} + \overline{y}\}$ are equal.*

Proof. Let $\overline{z} \in A$. Then $z \in x + y$. Since $x \in \overline{x}$ and $y \in \overline{y}$ we have $z \in \overline{x} + \overline{y}$. Thus $A \subseteq B$. Suppose $\overline{z} \in B$, then $z \in \overline{x} + \overline{y}$. That is, $z \in t + n$ for some $t \in x + y$ and $n \in N$. Therefore $\overline{z} = \overline{t}$, where $t \in x + y$. Since $\overline{t} \in A$, we get $\overline{z} \in A$. Thus $B \subseteq A$. Hence $A = B$.

If $\overline{z} \in A$, then $z \in x+y$. Therefore, $\overline{z} \subseteq x+y+N = x+N+y+N = \overline{x}+\overline{y}$. Hence $A \subseteq C$. On the other hand if $\overline{z} \in C$, then $\overline{z} \subseteq \overline{x} + \overline{y}$. Since $z \in \overline{z} \subseteq \overline{x} + \overline{y}$, we get $z \in s + n$ for some $s \in x + y$ and $n \in N$. Thus $\overline{z} = \overline{s}$. Since $\overline{s} \in A$, we get $C \subseteq A$. Hence $A = C$. □

Remark 3.6. Let H be a canonical hypergroup, and let N be a subcanonical hypergroup of H. Then, we denote the collection of all equivalence classes $\{\overline{x} : x \in H\}$ induced by the equivalence relation $\sim$ by H/N.

Theorem 3.7. *Let H be a canonical hypergroup, and let N be a subcanonical hypergroup of H. If we define $\overline{x} \oplus \overline{y} = \{\overline{z} : z \in x + y\}$ for all $\overline{x}, \overline{y} \in H/N$, then H/N is a canonical hypergroup.*

Proof. If $x_1, y_1, x_2, y_2 \in H$ such that $\overline{x_1} = \overline{x_2}$ and $\overline{y_1} = \overline{y_2}$, then $x_2 \in x_1 + N$ and $y_2 \in y_1 + N$. Let $z_2 \in x_2 + y_2 \subseteq (x_1 + N) + (y_1 + N)$. Since H is commutative, $z_2 \in z_1 + i$ for some $z_1 \in x_1 + y_1$ and for some $i \in N$. That is, $z_2 + N = z_1 + N$. Hence, $\overline{x_2} \oplus \overline{y_2} \subseteq \overline{x_1} \oplus \overline{y_1}$. Also, since $x_1 \in x_2 + N$ and $y_1 \in y_2 + N$, by a similar argument, we get, $\overline{x_1} \oplus \overline{y_1} \subseteq \overline{x_2} \oplus \overline{y_2}$. Hence, $\overline{x_1} \oplus \overline{y_1} = \overline{x_2} \oplus \overline{y_2}$. Thus, hyperaddition $\oplus$ is well defined.

Let $\overline{x}, \overline{y}, \overline{z} \in H/N$. If $\overline{u} \in (\overline{x} \oplus \overline{y}) \oplus \overline{z}$, then $\overline{u} \in \overline{p} \oplus \overline{z}$ for some $\overline{p} \in \overline{x} \oplus \overline{y}$. That is, $\overline{u} = \overline{a}$ for some $a \in p + z$. Also, $\overline{p} = \overline{b}$ for some $b \in x + y$. Now, $a \in p + z \subseteq b + N + z = b + z + N$. That is, $a \in v+N$ for some $v \in b+z \subseteq (x+y)+z = x+(y+z)$. So, $v \in x+t$ for some $t \in y+z$. This means that $\overline{a} = \overline{v}$ and $\overline{v} \in \overline{x} \oplus \overline{t}$. Since $\overline{t} \in \overline{y} \oplus \overline{z}$, we have $\overline{u} = \overline{a} = \overline{v} \in \overline{x} \oplus \overline{t} \subseteq \overline{x} \oplus (\overline{y} \oplus \overline{z})$. This means that $\overline{u} \in \overline{x} \oplus (\overline{y} \oplus \overline{z})$. Hence $(\overline{x} \oplus \overline{y}) \oplus \overline{z} \subseteq \overline{x} \oplus (\overline{y} \oplus \overline{z})$. Similarly, we get $\overline{x} \oplus (\overline{y} \oplus \overline{z}) \subseteq (\overline{x} \oplus \overline{y}) \oplus \overline{z}$. Hence, $\overline{x} \oplus (\overline{y} \oplus \overline{z}) = (\overline{x} \oplus \overline{y}) \oplus \overline{z}$. Thus, the hyperaddition is associative.

Consider the element $\overline{0} = 0 + N \in H/N$. Now, for any $x \in H$, we have $\overline{x} \oplus \overline{0} = \{\overline{z} : z \in x + 0\} = \overline{x}$. Similarly, $\overline{0} \oplus \overline{x} = \overline{x}$. Thus, $\overline{0}$ is the zero element of H/N.

Let $x \in H$, then $\overline{x} \oplus (-\overline{x}) = \{\overline{z} : z \in x + (-x) = x - x\}$. Since $0 \in x - x$, we get $\overline{0} \in \overline{x} \oplus (-\overline{x})$. Similarly, we can show that $\overline{0} \in (-\overline{x}) \oplus \overline{x}$. Let $\overline{x} \in H/N$, and suppose that $\overline{y} \in H/N$ is such that $\overline{0} \in \overline{y} \oplus \overline{x}$, then $\overline{0} = \overline{a}$, where $a \in y + x$. That is, $y \in a - x \subseteq N - x$, and hence $\overline{y} = -\overline{x}$. Thus, the element $\overline{x} \in H/N$ has a unique inverse $-\overline{x} \in H/N$.

Suppose that $\overline{z} \in \overline{x} \oplus \overline{y}$, then $\overline{z} = \overline{a}$, where $a \in x + y$. This implies $x \in a - y \subseteq z + N - y$. That is, $x \in r + N$, where $r \in z - y$. Thus, $\overline{x} = \overline{r} \in \overline{z} \oplus (-\overline{y})$. Similarly, we can show $\overline{y} \in (-\overline{x}) \oplus \overline{z}$. Since H is commutative, it is obvious that H/N is also commutative. Thus, H/N is a canonical hypergroup. □

Corollary 3.8. *Let ϕ be a strong homomorphism from canonical hypergroup H_1 into a canonical hypergroup H_2, then $H_1/\operatorname{Ker}\phi$ is a canonical hypergroup.*

Remark 3.9. Let H be a canonical hypergroup, and let A be a subcanonical hypergroup of H. We denote the subset $\{x \in H : x - x \subseteq A\}$ of H by S_A.

Proposition 3.10. *Let H be a canonical hypergroup, and let A be a subcanonical hypergroup of H. Then, S_A is a subcanonical hypergroup of H containing A.*

Proof. Let $x \in A$. Since A is a subcanonical hypergroup of H, $x - x \subseteq A$. This implies $x \in S_A$. Therefore, $A \subseteq S_A$. Since $A \neq \emptyset$, the set S_A is nonempty.

Let $x, y \in S_A$. For $r \in x - y$, we get $r - r \subseteq (x - y) - (x - y) = (x - x) + (y - y) \subseteq A + A = A$. Hence, $r \in S_A$. That is, $x - y \subseteq S_A$. Therefore, S_A is a subcanonical hypergroup of H containing A. □

Definition 3.11. Let $(H, +)$ be a canonical hypergroup, and let A be a subcanonical hypergroup of H. A is called a subgroup of H if $(A, +)$ is a group. That is, $x + y$ is a singleton set for all $x, y \in A$.

Example 3.12. The set $H = \{0, a, b, c\}$ with the following hyperoperation $+$ is a canonical hypergroup

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0,b\}$	$\{a,c\}$	$\{b\}$
b	$\{b\}$	$\{a,c\}$	$\{0,b\}$	$\{a\}$
c	$\{c\}$	$\{b\}$	$\{a\}$	$\{0\}$

(3.1)

In this example $\{0, c\}, \{0\}$ are subgroups of H and $\omega_H = \{0, b\}$ whereas in the Example 2.2, $\{0\}$ is the subgroup of H and $\omega_H = H$.

Proposition 3.13. *Let H be a canonical hypergroup. Then, $S_{\{0\}}$ is the subgroup of H containing all subgroups of H.*

Proof. By the Proposition 3.10, $S_{\{0\}}$ is the subcanonical hypergroup of H. Let $x, y \in S_{\{0\}}$. Consider the set $x + y$. If $u, v \in x + y$, then $u - v \subseteq (x + y) - (x + y) = (x - x) + (y - y) = 0 + 0 = 0$.

Hence, $u = v$. This means that the set $x + y$ has only one element. Thus, $S_{\{0\}}$ is a subgroup of H. Suppose, A is any subgroup of H, then for any $x \in A$ that we have $x - x = 0$. That is, $x \in S_{\{0\}}$. Hence, $A \subseteq S_{\{0\}}$. Thus, $S_{\{0\}}$ contains all subgroups of H. □

Corollary 3.14. *Let H be a canonical hypergroup. Then, H is an abelian group if and only if $S_{\{0\}} = H$.*

Proposition 3.15. *Let H be a canonical hypergroup, and let A be a subcanonical hypergroup of H. Then, A is normal if and only if $S_A = H$.*

Proof. Let A be normal. Then, for $x \in H$, $x + 0 - x \subseteq A$. That is, $x \in S_A$. Hence, $S_A = H$. Conversely, if $S_A = H$, then for $x \in H$, we get $x + A - x = x - x + A \subseteq A + A = A$. Thus, A is normal. □

Proposition 3.16. *The heart ω_H of a canonical hypergroup H is a normal subcanonical hypergroup of H.*

Proof. If $x, y \in \omega_H$, then $x \in (x_1 - x_1) + (x_2 - x_2) + (x_3 - x_3) + \cdots + (x_n - x_n)$ and $y \in (y_1 - y_1) + (y_2 - y_2) + (y_3 - y_3) + \cdots + (y_m - y_m)$, where $x_i, y_j \in H$ and m, n are natural numbers. Thus $x - y \in (x_1 - x_1) + (x_2 - x_2) + (x_3 - x_3) + \cdots + (x_n - x_n) + (y_1 - y_1) + (y_2 - y_2) + (y_3 - y_3) + \cdots + (y_m - y_m) \subseteq \omega_H$. Now, for any element $h \in \omega_H$, there exists natural number n and elements $x_i \in H$ such that $h \in (x_1 - x_1) + (x_2 - x_2) + (x_3 - x_3) + \cdots + (x_n - x_n)$. Then, for any $x \in H$, $x + h - x = x - x + h \subseteq x - x + (x_1 - x_1) + (x_2 - x_2) + (x_3 - x_3) + \cdots + (x_n - x_n) \subseteq \omega_H$. Hence, heart ω_H is a normal subcanonical hypergroup of H. □

Proposition 3.17. *A subcanonical hypergroup A of a canonical hypergroup H is normal if and only if A contains the heart ω_H of the canonical hypergroup H.*

Proof. Let A be a normal subcanonical hypergroup of the canonical hypergroup H. Then $x + i - x \subseteq A$ for every $x \in H$, and $i \in A$. In particular, when $i = 0 \in A$, we get $x - x \subseteq A$ for every $x \in H$. Since A is a subcanonical hypergroup of H, the union of the sums $(x_1 - x_1) + (x_2 - x_2) + (x_3 - x_3) + \cdots + (x_n - x_n) \subseteq A$ for $x_i \in H$ and n is a natural number. That is, $\omega_H \subseteq A$. Conversely, assume that subcanonical hypergroup A contains the heart ω_H of the canonical hypergroup H. For $x \in H$ and $i \in A$, $x + i - x = x - x + i \subseteq \omega_H + A \subseteq A + A = A$. Hence, A is a normal subcanonical hypergroup. □

From Propositions 3.16 and 3.17, we have the following proposition.

Proposition 3.18. *In a canonical hypergroup H, ω_H is the smallest normal subcanonical hypergroup.*

Proposition 3.19. *Let A, B be subcanonical hypergroups of a canonical hypergroup H such that $A \subseteq B$, then $S_A \subseteq S_B$.*

Proof. Let $x \in S_A$. Then, $x - x \subseteq A$. That is, $x \in S_B$. Hence, $S_A \subseteq S_B$. □

Proposition 3.20. *Let A, B be subcanonical hypergroups of a canonical hypergroup H such that $A \subseteq B$. If A is normal, then B is also normal.*

Proof. If A is normal, then by Proposition 3.15, $S_A = H$. Since $A \subseteq B$, by Proposition 3.19, $S_A \subseteq S_B$. Hence, $H = S_B$. By Proposition 3.15, B is normal. □

Corollary 3.21. *Let A, B be subcanonical hypergroups of a canonical hypergroup H such that A is normal, then the subcanonical hypergroup $A + B$ is also normal.*

Corollary 3.22. *Let H be a canonical hypergroup such that* (0) *is normal, then all the subcanonical hypergroups are normal.*

Theorem 3.23. *Let H be a canonical hypergroup. Then, the following are equivalent:*

(i) *H is an abelian group,*

(ii) (0) *is a normal subcanonical hypergroup of H,*

(iii) $\omega_H = (0)$.

Proof. By Corollary 3.14, a canonical hypergroup H is an abelian group if and only if $S_{\{0\}} = H$. By Proposition 3.15, $S_{\{0\}} = H$ if and only if (0) is a normal subcanonical hypergroup of H. Hence, a canonical hypergroup H is an abelian group if and only if (0) is a normal subcanonical hypergroup of H.

By Proposition 3.18, ω_H is the smallest normal subcanonical hypergroup of H. Therefore, (0) is normal if and only if $\omega_H = (0)$. □

Corollary 3.24. *H is an abelian group if and only if all subcanonical hypergroups of H are normal.*

Theorem 3.25. *Let H be a canonical hypergroup, and let N be a normal subcanonical hypergroup of H. Then, H/N is an abelian group.*

Proof. For the quotient canonical hypergroup H/N, the zero element is N. Since $(x + N) + N + (-x + N) = (x + N - x) + N \subseteq N + N = N$ for all $x \in H$, we have $\{N\}$ is a normal subcanonical hypergroup in H/N. By Theorem 3.23, H/N is an abelian group. □

Remark 3.26. If N is a normal subcanonical hypergroup of a canonical hypergroup H, then the relation $\sim$ defined in Proposition 3.1, is a strongly regular equivalence relation. Hence, by Theorem 31 in [3], H/N is an abelian group. However, we have proved Theorem 3.25 in a different way.

4. Isomorphism Theorems of Canonical Hypergroups

In this section, we prove the isomorphism theorems of canonical hypergroups.

Theorem 4.1 (First Isomorphism Theorem). *Let ϕ be a strong homomorphism from a canonical hypergroup H_1 into a canonical hypergroup H_2 with kernel K. Then, H_1/K is strongly isomorphic to* $\operatorname{Im}\phi$.

Proof. Define a map $f : H_1/K \to \operatorname{Im}\phi$ by $f(\overline{x}) = \phi(x)$ for all $x \in H_1$. Suppose that $\overline{x} = \overline{y}$, where $x, y \in H$, then $x \in \overline{y}$. That is, $x \in y + k$ for some $k \in K$. Hence, $\phi(x) \in \phi(y + k) = \phi(y) + \phi(k) = \phi(y) + 0 = \phi(y)$. So $\phi(x) = \phi(y)$. Hence, $f(\overline{x}) = f(\overline{y})$. Thus, the map f is well defined.

If $x, y \in H_1$, then

$$\begin{aligned} f(\overline{x} \oplus \overline{y}) &= f(\{\overline{z} : z \in x + y\}) \\ &= \{f(\overline{z}) : z \in x + y\} \\ &= \{\phi(z) : z \in x + y\}. \end{aligned} \tag{4.1}$$

Also,

$$\begin{aligned} f(\overline{x}) + f(\overline{y}) &= \phi(x) + \phi(y) \\ &= \phi(x + y) \\ &= \{\phi(z) : z \in x + y\}. \end{aligned} \tag{4.2}$$

Thus, $f(\overline{x} \oplus \overline{y}) = f(\overline{x}) + f(\overline{y})$. Moreover, $f(\overline{0}) = \phi(0) = 0$. Hence, f is a strong homomorphism.

Suppose that $\overline{x}, \overline{y} \in H_1/K$ such that $f(\overline{x}) = f(\overline{y})$, then $\phi(x) = \phi(y)$. This means that $0 \in \phi(x) - \phi(y) = \phi(x - y)$. That is, $\phi(z) = 0$ for some $z \in x - y$. Since $\phi(z) = 0$, we get $z \in K$. Now, $z \in x - y \Rightarrow x \in z + y \Rightarrow x \in y + K$. Then, by Proposition 3.3 $\overline{x} = \overline{y}$ and hence f is one to one. Clearly, f is onto. Thus, f is a strong isomorphism. That is, H_1/K is strongly isomorphic to $\operatorname{Im} \phi$. □

Corollary 4.2. *Let ϕ be a strong homomorphism from a canonical hypergroup H_1 onto a canonical hypergroup H_2 with kernel K. Then, H_1/K is isomorphic to H_2.*

Theorem 4.3 (Second Isomorphism Theorem). *If M and N are subcanonical hypergroups of a canonical hypergroup H, then $N/(M \cap N) \cong (M + N)/M$.*

Proof. It is clear that we can consider the subcanonical hypergroup $M + N$ of the canonical hypergroup H as a canonical hypergroup $M + N$ for which M is a subcanonical hypergroup. Similarly, the subcanonical hypergroup N of the canonical hypergroup H as a canonical hypergroup N for which $(M \cap N)$ is a subcanonical hypergroup.

Define $g : N \to (M + N)/M$ by $g(b) = b + M$ for every $b \in N$. For all $a, b \in N$, $g(a + b) = g(\{x : x \in a + b\}) = \{g(x) : x \in a + b\} = \{x + M : x \in a + b\} = (a + M) \oplus (b + M) = g(a) \oplus g(b)$. Moreover, $g(0) = 0$. Thus, g is a strong homomorphism.

Now, $x + M \in (M + N)/M$ implies that $x \in y + M$ for some $y \in M + N$. That is, $y \in a + b$ for some $a \in M$, $b \in N$. Since $y \in b + M$, we get $y + M = b + M$. Thus, $g(b) = b + M = y + M = x + M$. Thus, g is onto. Let $b \in N$. Then, $b \in \operatorname{Ker} g \Leftrightarrow g(b) = 0 \Leftrightarrow b + M = 0 + M \Leftrightarrow b \in M$. Thus, $b \in \operatorname{Ker} g$ if and only if $b \in M \cap N$. Hence, by the First Isomorphism Theorem, $N/(M \cap N) \cong (M + N)/M$. □

Theorem 4.4 (Third Isomorphism Theorem). *If M and N are subcanonical hypergroup of a canonical hypergroup H such that $M \subseteq N$, then $H/N \cong (H/M)/(N/M)$.*

Proof. Define a map $h : H/N \to H/M$ by $h(x + N) = x + M$. Then, h is a strong onto homomorphism of canonical hypergroup with kernel N/M. Therefore, by the First Isomorphism Theorem of canonical hypergroups, $H/N \cong (H/M)/(N/M)$. □

5. Isomorphism Theorems of Hyperrings

Let R be a hyperring, and let I be a hyperideal of R. Since I is a subcanonical hypergroup of $R, R/I = \{\overline{x} : x \in R\}$ is a canonical hypergroup under the hyperaddition defined in the Theorem 3.7. In this section, we define a hypermultiplication on R/I and prove that R/I is a hyperring.

Theorem 5.1. *If we define $\overline{x} \otimes \overline{y} = \{\overline{z} : z \in xy\}$ for all $\overline{x}, \overline{y} \in R/I$, then R/I is a hyperring.*

Proof. If $x_1, y_1, x_2, y_2 \in R$ such that $\overline{x_1} = \overline{x_2}$ and $\overline{y_1} = \overline{y_2}$, then $x_2 \in x_1 + I$ and $y_2 \in y_1 + I$. Let $z_2 \in x_2 y_2 \subseteq (x_1 + I)(y_1 + I) \subseteq x_1 y_1 + I$. Then, $z_2 \in z_1 + i$ for some $z_1 \in x_1 y_1$ and for some $i \in I$. That is, $z_2 + I = z_1 + I$ and so $\overline{x_2} \otimes \overline{y_2} \subseteq \overline{x_1} \otimes \overline{y_1}$. Similarly, we get, $\overline{x_1} \otimes \overline{y_1} \subseteq \overline{x_2} \oplus \overline{y_2}$. Hence, $\overline{x_1} \oplus \overline{y_1} = \overline{x_2} \otimes \overline{y_2}$. Thus, hypermultiplication $\otimes$ is well defined.

Suppose, $\overline{x}, \overline{y}, \overline{z} \in R/I$. Then,

$$\begin{aligned}
\overline{x} \otimes (\overline{y} \otimes \overline{z}) &= \overline{x} \otimes \{\overline{a} : a \in yz\} \\
&= \{\overline{s} : s \in xa, a \in yz\} \\
&= \{\overline{s} : s \in x(yz)\} \\
&= \{\overline{s} : s \in (xy)z\} \\
&= \{\overline{s} : s \in bz, b \in xy\} \\
&= \{\overline{s} : s \in xy\} \otimes \overline{z} \\
&= (\overline{x} \otimes \overline{y}) \otimes \overline{z}
\end{aligned} \tag{5.1}$$

Thus, we get $\overline{x} \otimes (\overline{y} \otimes \overline{z}) = (\overline{x} \otimes \overline{y}) \otimes \overline{z}$. Hence, hypermultiplication is associative. Further,

$$\begin{aligned}
\overline{x} \otimes (\overline{y} \oplus \overline{z}) &= \overline{x} \otimes \{p + I : p \in y + z\} \\
&= \{q + I : q \in xp, p \in y + z\} \\
&= \{q + I : q \in x(y + z)\} \\
&= \{q + I : q \in xy + xz\}.
\end{aligned} \tag{5.2}$$

Also,

$$\begin{aligned}
(\overline{x} \otimes \overline{y}) \oplus (\overline{x} \otimes \overline{z}) &= \{a + I : a \in xy\} \oplus \{b + I : b \in xz\} \\
&= \{c + I : c \in \mathrm{a} + b, a \in xy, b \in xz\} \\
&= \{c + I : c \in xy + xz\}.
\end{aligned} \tag{5.3}$$

Hence, $\overline{x} \otimes (\overline{y} \oplus \overline{z}) = (\overline{x} \otimes \overline{y}) \oplus (\overline{x} \otimes \overline{z})$. Similarly, we can show that $(\overline{x} \oplus \overline{y}) \otimes \overline{z} = (\overline{x} \otimes \overline{z}) \oplus (\overline{x} \otimes \overline{z})$. Therefore, hypermultiplication is distributive with respect to the hyperaddition. Thus, R/I is a hyperring. □

Corollary 5.2. *Let ϕ be a strong homomorphism from hyperring R_1 into a hyperring R_2, then $R_1 / \operatorname{Ker} \phi$ is a hyperring.*

Remark 5.3. If R is a Krasner hyperring and I is a hyperideal of R, then R/I is also a Krasner hyperring. Further if $(I, +)$ is a normal subcanonical hypergroup of R, then by the Theorems 3.23 and 5.1, R/I is a ring. Hence, the quotient hyperrings considered in [12] are just rings. So, in the isomorphism theorems proved in [12], all the quotient hyperrings considered are rings. However, we prove the isomorphism theorems of hyperrings in which the additions and the multiplications are hyperoperations.

If R is a hyperring, and I is a hyperideal of R, and $(I,+)$ is a normal subcanonical hypergroup of R, then R/I is a multiplicative hyperring.

Theorem 5.4 (First Isomorphism Theorem). *Let ϕ be a strong homomorphism from a hyperring R_1 into a hyperring R_2 with kernel K. Then, R_1/K is strongly isomorphic to* $\operatorname{Im}\phi$.

Proof. Define a map $f : R_1/K \to \operatorname{Im}\phi$ by $f(\overline{x}) = \phi(x)$ for all $x \in R_1$.

By Theorem 4.1, this map f is a strong isomorphism from canonical hypergroup R_1/K onto $\operatorname{Im}\phi$. Now,

$$\begin{aligned} f(\overline{x}\otimes\overline{y}) &= f(\{\overline{z} : z \in xy\}) \\ &= \{f(\overline{z}) : z \in xy\} \\ &= \{\phi(z) : z \in xy\}, \\ f(\overline{x})f(\overline{y}) &= \phi(x)\phi(y) \\ &= \phi(xy) \\ &= \{\phi(z) : z \in xy\}. \end{aligned} \tag{5.4}$$

Thus, $f(\overline{x}\otimes\overline{y}) = f(\overline{x})f(\overline{y})$. Hence, f is a strong hyperring isomorphism. □

Corollary 5.5. *Let ϕ be a strong homomorphism from a hyperring R_1 onto a hyperring R_2 with kernel K. Then, R_1/K is strongly isomorphic to R_2.*

Theorem 5.6 (Second Isomorphism Theorem). *If I and J are hyperideals of a hyperring R, then $J/(I \cap J) \cong (I + J)/I$.*

Proof. We can consider the hyperideal $I + J$ of the hyperring R as a hyperring $I + J$ for which I is a hyperideal. Similarly, hyperideal J of the hyperring R as a hyperring J for which $(I \cap J)$ is a hyperideal.

Define $g : J \to (I + J)/I$ by $g(b) = b + I$ for every $b \in J$. By Theorem 4.3, g is strong isomorphism from canonical hypergroup J onto the canonical hypergroup $(I + J)/I$. Now, $g(ab) = g(\{x : x \in ab\}) = \{g(x) : x \in ab\} = \{x+I : x \in ab\} = (a+I)(b+I) = g(a)g(b)$. Thus, g is strong isomorphism from hyperring J onto the hyperring $(I+J)/I$. Also, from Theorem 4.3, $\operatorname{Ker} g = I \cap J$. Hence, by First Isomorphism Theorem of hyperrings, $J/(I \cap J) \cong (I + J)/I$. □

Theorem 5.7 (Third Isomorphism Theorem). *If I and J are hyperideals of a hyperring R such that $I \subseteq J$, then $R/J \cong (R/I)/(J/I)$.*

Proof. Define a map $h : R/J \to R/M$ by $h(x + I) = x + J$. Then, h is a strong onto homomorphism of hyperring with kernel J/I. Therefore, by the First Isomorphism Theorem of hyperrings, $R/J \cong (R/I)/(J/I)$. □

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