

Research Article

The Diagrammatic Soergel Category and $sl(2)$ and $sl(3)$ Foams

Pedro Vaz^{1,2}

¹ Institut de Mathématiques de Jussieu, Université Paris 7, 175 Rue du Chevaleret, 75013 Paris, France

² CAMGSD, Instituto Superior Técnico, Avenida Rovisco Pais, 1049-001 Lisboa, Portugal

Correspondence should be addressed to Pedro Vaz, pfortevaz@ualg.pt

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We define two functors from Elias and Khovanov's diagrammatic Soergel category, one targeting Clark-Morrison-Walker's category of disoriented $sl(2)$ cobordisms and the other targeting the category of (universal) $sl(3)$ foams.

1. Introduction

In this paper we define functors between the Elias-Khovanov diagrammatic version of the Soergel category \mathcal{SC} defined in [1] and the categories of universal $sl(2)$ and $sl(3)$ foams defined in [2, 3].

The Soergel category provides a categorification of the Hecke algebra and was used by Khovanov in [4] to construct a triply-graded link homology categorifying the HOMFLYPT polynomial. Elias and Khovanov constructed in [1] a category defined diagrammatically by generators and relations and showed it to be equivalent to \mathcal{SC} .

The $sl(2)$ and $sl(3)$ foams were introduced in [2, 5] and in [3, 6], respectively, to give topological constructions of the $sl(2)$ and $sl(3)$ link homologies.

This paper can be seen as a first step towards the construction of a family of functors between \mathcal{SC} and the categories of $sl(N)$ -foams for all $N \in \mathbb{Z}_+$, to be completed in a subsequent paper [7]. The functors $\mathcal{F}_{sl(2),n}$ and $\mathcal{F}_{sl(3),n}$ are not faithful. In [7] we will extend the construction of these functors to all N . The whole family of functors is faithful in the following sense: if for a morphism f in \mathcal{SC}_1 we have $\mathcal{F}_{sl(N),n}(f) = 0$ for all N , then $f = 0$. With these functors one can try to give a graphical interpretation of Rasmussen's [8] spectral sequences from the HOMFLYPT link homology to the $sl(N)$ -link homologies.

The plan of the paper is as follows. In Section 2 we give a brief description of Elias and Khovanov's diagrammatic Soergel category. In Section 3 we describe the category \mathbf{Foam}_2

of $sl(2)$ foams and construct a functor from \mathcal{SC} to \mathbf{Foam}_2 . Finally in Section 4 we give the analogue of these results for the case of $sl(3)$ foams.

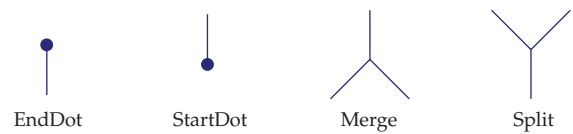
We have tried to keep this paper reasonably self-contained. Although not mandatory, some acquaintance with [1–3, 9] is desirable.

2. The Diagrammatic Soergel Category Revisited

This section is a reminder of the diagrammatics for Soergel categories introduced by Elias and Khovanov in [1]. Actually we give the version which they explained in [1, Section 4.5] and which can be found in detail in [9].

Fix a positive integer n . The category \mathcal{SC}_1 is the category whose objects are finite length sequences of points on the real line, where each point is colored by an integer between 1 and n . We read sequences of points from left to right. Two colors i and j are called adjacent if $|i - j| = 1$ and distant if $|i - j| > 1$. The morphisms of \mathcal{SC}_1 are given by generators modulo relations. A morphism of \mathcal{SC}_1 is a \mathbb{C} -linear combination of planar diagrams constructed by horizontal and vertical gluings of the following generators (by convention no label means a generic color j).

(i) Generators involving only one color are as follows:



EndDot StartDot Merge Split

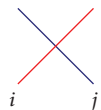
(2.1)

It is useful to define the cap and cup as


(2.2)

(ii) Generators involving two colors are as follows:

- The 4-valent vertex, with distant colors,


(2.3)

- and the 6-valent vertex, with adjacent colors i and j


(2.4)

read from bottom to top. In this setting a diagram represents a morphism from the bottom boundary to the top. We can add a new colored point to a sequence and this endows \mathcal{SC}_1 with a monoidal structure on objects, which is extended to morphisms in the obvious way. Composition of morphisms consists of stacking one diagram on top of the other.

We consider our diagrams modulo the following relations.

"Isotopy" Relations.

$$\text{cup} = | = \text{cap} \tag{2.5}$$

$$\text{cup with dot} = | = \text{cap with dot} \tag{2.6}$$

$$\text{Y-junction} = \text{X-junction} = \text{Y-junction} \tag{2.7}$$

$$\text{crossing} = \text{X} = \text{crossing} \tag{2.8}$$

$$\text{crossing with dots} = \text{X with dots} = \text{crossing with dots} \tag{2.9}$$

The relations are presented in terms of diagrams with generic colorings. Because of isotopy invariance, one may draw a diagram with a boundary on the side, and view it as a morphism in \mathcal{SC}_1 by either bending the line up or down. By the same reasoning, a horizontal line corresponds to a sequence of cups and caps.

One Color Relations.

$$\text{Y-junction} = \text{X-junction} \tag{2.10}$$

$$\text{circle} = 0 \tag{2.11}$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} | \\ \bullet \\ | \\ \bullet \end{array} = 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \quad (2.12)$$

Two Distant Colors.

$$\begin{array}{c} \text{blue} \\ \diagdown \\ \text{red} \\ \diagup \\ \text{blue} \end{array} = \begin{array}{c} | \\ | \\ | \end{array} \quad (2.13)$$

$$\begin{array}{c} \text{blue} \\ \diagdown \\ \text{red} \\ \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \\ / \\ \text{red} \end{array} \quad (2.14)$$

$$\begin{array}{c} \text{blue} \\ \diagdown \\ \text{red} \\ \diagup \\ \text{blue} \end{array} = \begin{array}{c} \text{blue} \\ \diagdown \\ \text{red} \\ \diagup \\ \text{blue} \end{array} \quad (2.15)$$

Two Adjacent Colors.

$$\begin{array}{c} \text{red} \\ \diagdown \\ \text{blue} \\ \diagup \\ \text{red} \end{array} = \begin{array}{c} \bullet \\ | \\ \text{red} \end{array} + \begin{array}{c} \text{red} \\ | \\ \text{blue} \end{array} + \begin{array}{c} \bullet \\ \diagdown \\ \text{red} \\ \diagup \\ \text{blue} \end{array} \quad (2.16)$$

$$\begin{array}{c} | \\ | \\ | \end{array} = \begin{array}{c} \text{red} \\ \diagdown \\ \text{blue} \\ \diagup \\ \text{red} \end{array} - \begin{array}{c} \bullet \\ | \\ \text{red} \end{array} - \begin{array}{c} \text{red} \\ | \\ \text{blue} \end{array} \quad (2.17)$$

$$\begin{array}{c} \text{red} \\ \diagdown \\ \text{blue} \\ \diagup \\ \text{red} \end{array} = \begin{array}{c} \text{red} \\ \diagdown \\ \text{blue} \\ \diagup \\ \text{red} \end{array} \quad (2.18)$$

$$\begin{array}{c} \bullet \\ | \\ \text{red} \end{array} - \begin{array}{c} \text{red} \\ | \\ \bullet \end{array} = \frac{1}{2} \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right) \quad (2.19)$$

Relations Involving Three Colors: (Adjacency is determined by the vertices which appear)


(2.20)


(2.21)


(2.22)

Introduce a q -grading on \mathcal{SC}_1 declaring that dots have degree 1, trivalent vertices have degree -1 and $4-$, and 6-valent vertices have degree 0.

Definition 2.1. The category \mathcal{SC}_2 is the category containing all direct sums and grading shifts of objects in \mathcal{SC}_1 and whose morphisms are the grading-preserving morphisms from \mathcal{SC}_1 .

Definition 2.2. The category \mathcal{SC} is the Karoubi envelope of the category \mathcal{SC}_2 .

Elias and Khovanov’s main result in [1] is the following theorem.

Theorem 2.3 (Elias-Khovanov). *The category \mathcal{SC} is equivalent to the Soergel category in [10].*

From Soergel’s results from [10] we have the following corollary.

Corollary 2.4. *The Grothendieck algebra of \mathcal{SC} is isomorphic to the Hecke algebra.*

Notice that \mathcal{SC} is an additive category but not abelian and we are using the (additive) split Grothendieck algebra.

In Sections 3 and 4 we will define functors from \mathcal{SC}_1 to the categories of $sl(2)$ and $sl(3)$ foams. These functors are grading preserving, so they obviously extend uniquely to \mathcal{SC}_2 . By the universality of the Karoubi envelope, they also extend uniquely to functors between the respective Karoubi envelopes.

3. The $sl(2)$ Case

3.1. Clark-Morrison-Walker’s Category of Disoriented $sl(2)$ Foams

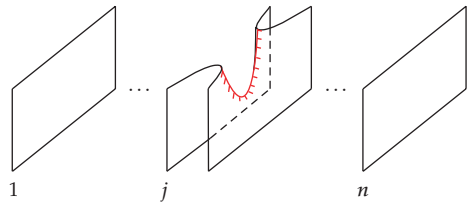
In this subsection we review the category \mathbf{Foam}_2 of $sl(2)$ foams following Clark et al. construction in [2]. This category was introduced in [2] to modify Khovanov’s link homology theory making it properly functorial with respect to link cobordisms. Actually we will use

the version with dots of Clark-Morrison-Walker’s original construction in [2]. Recall that we obtain one from the other by replacing each dot by $1/2$ times a handle.

A *disoriented arc* is an arc composed by oriented segments with oppositely oriented segments separated by a mark pointing to one of these segments. A *disoriented diagram* consists of a collection D of disoriented arcs in the strip in \mathbb{R}^2 bounded by the lines $y = 0, 1$ containing the boundary points of D . We allow diagrams containing oriented and disoriented circles. Disoriented diagrams can be composed vertically, which endows \mathbf{Foam}_2 with a monoidal structure on objects. For example, the diagrams 1_n and u_j for $(1 < j < n)$ are disoriented diagrams:

$$1_n = \begin{array}{c} \uparrow \\ | \\ \uparrow \\ | \\ \dots \\ \uparrow \\ | \\ n \end{array} \quad u_j = \begin{array}{c} \uparrow \\ | \\ \dots \\ \uparrow \\ | \\ \text{↖ ↗} \\ | \\ j \quad j+1 \\ \text{↙ ↘} \\ \uparrow \\ | \\ \dots \\ \uparrow \\ | \\ n \end{array} \tag{3.1}$$

A *disoriented cobordism* between disoriented diagrams is a $2D$ cobordism which can be decorated with dots and with seams separating differently oriented regions and such that the vertical boundary of each cobordism is a set (possibly empty) of vertical lines. Disorientation seams can have one out of two possible orientations which we identify with a fringe. We read cobordisms from bottom to top. For example,



$$\tag{3.2}$$

is a disoriented cobordism from 1_n to u_j .

Cobordism composition consists of placing one cobordism on top of the other and the monoidal structure is given by vertical composition which corresponds to placing one cobordism behind the other in our pictures. Let $\mathbb{C}[t]$ be the ring of polynomials in t with coefficients in \mathbb{C} .

Definition 3.1. The category \mathbf{Foam}_2 is the category whose objects are disoriented diagrams, and whose morphisms are $\mathbb{C}[t]$ -linear combinations of isotopy classes of disoriented cobordisms, modulo some relations:

- (i) the disorientation relations

$$\begin{array}{c} \text{⊙} \\ \text{⊙} \end{array} = i \quad \begin{array}{c} \text{⊙} \\ \text{⊙} \end{array} = -i \tag{3.3}$$

$$\begin{array}{c} \text{⌒} \\ \text{⌒} \end{array} = -i \quad \begin{array}{c} \text{⌒} \\ \text{⌒} \end{array} \tag{3.4}$$

$$\begin{array}{|} \hline \bullet \\ \hline \end{array} = - \begin{array}{|} \hline \bullet \\ \hline \end{array} \tag{3.5}$$

where i is the imaginary unit,

(ii) and the Bar-Natan (BN) relations

$$\begin{array}{|} \hline \bullet \bullet \\ \hline \end{array} = t \begin{array}{|} \hline \\ \hline \end{array} \tag{3.6}$$

$$\begin{array}{|} \hline \\ \hline \end{array} = 0 \quad \begin{array}{|} \hline \bullet \\ \hline \end{array} = 1 \tag{3.7}$$

$$\begin{array}{|} \hline \\ \hline \end{array} = \begin{array}{|} \hline \bullet \\ \hline \end{array} + \begin{array}{|} \hline \\ \hline \end{array} + \begin{array}{|} \hline \\ \hline \end{array} + \begin{array}{|} \hline \bullet \\ \hline \end{array} \tag{3.8}$$

which are only valid away from the disorientations.

The universal theory for the original Khovanov homology contains another parameter h , but we have to put $h = 0$ in the Clark-Morrison-Walker’s cobordism theory over a field of characteristic zero. Suppose that we have a cylinder with a transversal disoriented circle. Applying (3.8) on one side of the disorientation circle followed by the disoriented relation (3.3) gives a cobordism that is independent of the side chosen to apply (3.8) only if $h = 0$ over a field of characteristic zero.

Define a q -grading on $\mathbb{C}[t]$ by $q(1) = 0$ and $q(t) = 4$. We introduce a q -grading on \mathbf{Foam}_2 as follows. Let f be a cobordism with $|\bullet|$ dots and $|b|$ vertical boundary components. The q -grading of f is given by

$$q(f) = -\chi(f) + 2|\bullet| + \frac{1}{2}|b|, \tag{3.9}$$

where χ is the Euler characteristic. For example, the degree of a saddle is 1 while the degree of a cap or a cup is -1 . The category \mathbf{Foam}_2 is additive and monoidal. More details about \mathbf{Foam}_2 can be found in [2].

3.2. The Functor $\mathcal{F}_{sl(2),n}$

In this subsection we define a monoidal functor $\mathcal{F}_{sl(2),n}$ between the categories \mathcal{SC} and \mathbf{Foam}_2 . *On Objects.* $\mathcal{F}_{sl(2),n}$ sends the empty sequence to 1_n and the one-term sequence (j) to u_j with $\mathcal{F}_{sl(2),n}(jk)$ given by the vertical composite $u_j u_k$.

On Morphisms

(i) The empty diagram is sent to n parallel vertical sheets:

(3.10)

(ii) The vertical line colored j is sent to the identity cobordism of u_j :

(3.11)

The remaining $n-2$ vertical parallel sheets on the r.h.s. are not shown for simplicity, a convention that we will follow from now on.

(iii) The *StartDot* and *EndDot* morphisms are sent to saddle cobordisms:

(3.12)

(iv) *Merge* and *Split* are sent to cup and cap cobordisms:

(3.13)

(v) The 4-valent vertex with distant colors is given as follows. For $j + 1 < k$ we have

$$\begin{array}{ccc}
 \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ k \quad j \end{array} & \mapsto & \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ j \quad j+1 \quad \dots \quad k \quad k+1 \end{array}
 \end{array}
 \tag{3.14}$$

The case $j > k + 1$ is given by reflection in a horizontal plane.

(vi) The 6-valent vertices are sent to zero:

$$\begin{array}{ccc}
 \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} & \mapsto & 0
 \end{array}
 \tag{3.15}$$

Notice that $\mathcal{F}_{sl(2),n}$ respects the gradings of the morphisms. Taking the quotient of \mathcal{SC} by the 6-valent vertex gives a diagrammatic category \mathcal{TL} categorifying the Temperley-Lieb algebra. According to [11] relations (2.16) and (2.17) can be replaced by a single relation in \mathcal{TL} . The functor $\mathcal{F}_{sl(2),n}$ descends to a functor between \mathcal{TL} and \mathbf{Foam}_2 .

Proposition 3.2. $\mathcal{F}_{sl(2),n}$ is a monoidal functor.

Proof. The assignment given by $\mathcal{F}_{sl(2),n}$ clearly respects the monoidal structures of \mathcal{SC}_1 and \mathbf{Foam}_2 . So we only need to show that $\mathcal{F}_{sl(2),n}$ is a functor, that is, it respects the relations (2.5) to (2.22) of Section 2.

“Isotopy Relations”

Relations (2.5) to (2.8) are straightforward to check and correspond to isotopies of their images under $\mathcal{F}_{sl(2),n}$ which respect the disorientations. Relation (2.9) is automatic since $\mathcal{F}_{sl(2),n}$ sends all terms to zero. For the sake of completeness we show the first equality in (2.5). We have

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ j \quad j+1 \end{array} \cong \begin{array}{c} \text{---} \\ \text{---} \\ j \quad j+1 \end{array} = \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right)
 \tag{3.16}$$

One Color Relations

For relation (2.10) we have

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \cong \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right) \cong \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right),
 \tag{3.17}$$

where the first equivalence follows from relations (2.5) and (2.7) and the second from isotopy of the cobordisms involved.

For relation (2.11) we have

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} \bigcirc \\ | \\ j \end{array} \right) = \begin{array}{c} \text{[Diagram: A cobordism with two sheets labeled } j \text{ and } j+1. \text{ A red circle with a dot is on sheet } j+1. \text{ A dashed line indicates a cut.}] \\ j \quad j+1 \end{array} = 0 \quad \text{by relations (3.3) and (3.7).} \quad (3.18)$$

Relation (2.12) requires some more work. We have

$$\begin{aligned} \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \bullet \\ | \\ j \\ | \\ \bullet \\ | \\ j \end{array} \right) &= \begin{array}{c} \text{[Diagram: A cobordism with two sheets labeled } j \text{ and } j+1. \text{ A red curve connects the sheets.}] \\ j \quad j+1 \end{array} = -i \begin{array}{c} \text{[Diagram: A cobordism with two sheets labeled } j \text{ and } j+1. \text{ A red curve connects the sheets, different orientation.}] \\ j \quad j+1 \end{array} \\ &= -i \left(\begin{array}{c} \text{[Diagram: Sheet } j \text{ with a red vertical line and a dot.}] \\ j \quad j+1 \end{array} + \begin{array}{c} \text{[Diagram: Sheet } j+1 \text{ with a red vertical line and a dot.}] \\ j \quad j+1 \end{array} \right), \end{aligned} \quad (3.19)$$

where the second equality follows from the disoriented relation (3.4) and the third follows from the BN relation (3.8). We also have

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} \bullet \\ | \\ j \\ | \\ \bullet \\ | \\ j \end{array} \right) = \begin{array}{c} \text{[Diagram: A cobordism with two sheets labeled } j \text{ and } j+1. \text{ A red tube connects them.}] \\ j \quad j+1 \end{array} = -i \begin{array}{c} \bullet \\ | \\ j \end{array} \begin{array}{c} | \\ j+1 \end{array} + i \begin{array}{c} | \\ j \end{array} \begin{array}{c} \bullet \\ | \\ j+1 \end{array} \quad (3.20)$$

and therefore

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} | \\ | \\ \bullet \\ | \\ \bullet \\ | \\ | \end{array} \right) = -2i \begin{array}{c} \text{[Diagram: A cobordism with two sheets labeled } j \text{ and } j+1. \text{ A red vertical line with a dot on sheet } j+1. \text{ A dashed line indicates a cut.}] \\ j \quad j+1 \end{array} \quad (3.21)$$

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} | \\ \bullet \\ | \\ \bullet \\ | \\ j \end{array} \right) = -2i \begin{array}{c} \text{[Diagram of a cobordism with two vertical planes and a red curve]} \\ j \quad j+1 \end{array} . \quad (3.22)$$

We thus have that

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} | \\ \bullet \\ | \\ \bullet \\ | \end{array} \right) + \mathcal{F}_{sl(2),n} \left(\begin{array}{c} | \\ \bullet \\ | \\ \bullet \\ | \end{array} \right) = 2\mathcal{F}_{sl(2),n} \left(\begin{array}{c} | \\ \bullet \\ | \\ \bullet \\ | \end{array} \right) . \quad (3.23)$$

Two Distant Colors

Relations (2.13) to (2.15) correspond to isotopies of the cobordisms involved and are straightforward to check.

Adjacent Colors

We prove the case where "blue" corresponds to j and "red" corresponds to $j+1$. The relations with colors reversed are proved the same way. To prove relation (2.16) we first notice that

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} \text{[Diagram of a blue curve with a dot]} \\ \bullet \\ \text{[Diagram of a red curve]} \end{array} \right) \cong \begin{array}{c} \text{[3D diagram of a cobordism with three vertical planes and red curves]} \\ j \quad j+1 \quad j+2 \end{array} \quad (3.24)$$

which means that

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} \text{[Diagram of a red curve with a dot]} \\ \bullet \\ \text{[Diagram of a blue curve]} \end{array} \right) \cong \begin{array}{c} \text{[3D diagram of a cobordism with three vertical planes and blue curves]} \\ j \quad j+1 \quad j+2 \end{array} \quad (3.25)$$

On the other side we have

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} \downarrow \\ \bullet \\ | \\ \downarrow \\ \bullet \end{array} \right) \cong \text{Diagram} \tag{3.26}$$

$j \quad j+1 \quad j+2$

which, using isotopies and the disorientation relation (3.4) twice, can be seen to be equivalent to

$$- \text{Diagram} \tag{3.27}$$

$j \quad j+1 \quad j+2$

which equals

$$-\mathcal{F}_{sl(2),n} \left(\begin{array}{c} \downarrow \\ \bullet \\ \cup \\ \downarrow \\ \bullet \end{array} \right). \tag{3.28}$$

This implies that

$$0 = \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \diagup \quad \diagdown \\ \bullet \\ \diagdown \quad \diagup \end{array} \right) = \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \downarrow \\ \bullet \\ | \\ \downarrow \\ \bullet \end{array} \right) + \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \downarrow \\ \bullet \\ \cup \\ \downarrow \\ \bullet \end{array} \right). \tag{3.29}$$

We now prove relation (2.17). We have isotopy equivalences

$$\mathcal{F}_{sl(2),n} \left(\begin{array}{c} | \\ | \\ | \end{array} \right) \cong \text{Diagram} \tag{3.30}$$

$j \quad j+1 \quad j+2$

$$\begin{aligned}
 \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \downarrow \\ \text{Y} \\ \uparrow \end{array} \right) &\cong \text{Diagram 1} \\
 &= - \text{Diagram 2}
 \end{aligned}
 \tag{3.31}$$

Therefore we see that

$$0 = \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \text{X} \\ \text{Y} \\ \text{X} \end{array} \right) = \mathcal{F}_{sl(2),n} \left(\begin{array}{c} | \\ | \\ | \end{array} \right) + \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \downarrow \\ \text{Y} \\ \uparrow \end{array} \right).
 \tag{3.32}$$

The functor $\mathcal{F}_{sl(2),n}$ sends both sides of relation (2.18) to zero and so there is nothing to prove here. To prove relation (2.19) we start with the equivalence

$$\begin{aligned}
 \mathcal{F}_{sl(2),n} \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right) &= \text{Diagram 1} \\
 &\cong -i \text{Diagram 2} + i \text{Diagram 3}
 \end{aligned}
 \tag{3.33}$$

which is a consequence of the neck-cutting relation (3.8) and the disorientation relations (3.3) and (3.5). We also have

$$\mathcal{F}_{sl(2),n} \left(\left| \begin{array}{c} | \\ \bullet \\ | \\ \bullet \\ | \end{array} \right. \right) \cong -i \left(\text{diagram} \right) + i \left(\text{diagram} \right) \tag{3.34}$$

Comparing with (3.21) and (3.22) and using the disoriented relation (3.5), we get

$$\mathcal{F}_{sl(2),n} \left(\left| \begin{array}{c} | \\ \bullet \\ | \\ \bullet \\ | \end{array} \right. \right) - \mathcal{F}_{sl(2),n} \left(\left| \begin{array}{c} | \\ \bullet \\ | \\ \bullet \\ | \end{array} \right. \right) = \frac{1}{2} \mathcal{F}_{sl(2),n} \left(\left| \begin{array}{c} | \\ \bullet \\ | \\ \bullet \\ | \end{array} \right. \right) - \frac{1}{2} \mathcal{F}_{sl(2),n} \left(\left| \begin{array}{c} | \\ \bullet \\ | \\ \bullet \\ | \end{array} \right. \right). \tag{3.35}$$

Relations Involving Three Colors

Functor $\mathcal{F}_{sl(2),n}$ sends to zero both sides of relations (2.20) and (2.22). Relation (2.21) follows from isotopies of the cobordisms involved. □

4. The $sl(3)$ Case

4.1. The Category \mathbf{Foam}_3 of $sl(3)$ Foams

In this subsection we review the category \mathbf{Foam}_3 of $sl(3)$ foams introduced by the author and Mackaay in [3]. This category was introduced to universally deform Khovanov’s construction in [6] leading to the $sl(3)$ -link homology theory.

We follow the conventions and notation of [3]. Recall that a *web* is a trivalent planar graph, where near each vertex all edges are oriented away from it or all edges are oriented towards it. We also allow webs without vertices, which are oriented loops. A *pre-foam* is a cobordism with singular arcs between two webs. A singular arc in a prefoam f is the set of points of f which has a neighborhood homeomorphic to the letter Y times an interval. Singular arcs are disjoint. Interpreted as morphisms, we read prefoams from bottom to top by convention; foam composition consists of placing one prefoam on top of the other. The orientation of the singular arcs is by convention as in the *zip* and the *unzip*:

$$\left(\text{zip diagram} \right) \text{ and } \left(\text{unzip diagram} \right) \tag{4.1}$$

respectively. Pre-foams can have dots which can move freely on the facet to which they belong but are not allowed to cross singular arcs. A *foam* is an isotopy class of pre-foams. Let $\mathbb{C}[a, b, c]$ be the ring of polynomials in a, b, c with coefficients in \mathbb{C} .

We impose the set of relations $\ell = (3D, CN, S, \Theta)$ on foams, as well as the *closure relation*, which are explained below.

$$\begin{array}{|c|} \hline \bullet \bullet \bullet \\ \hline \end{array} = a \begin{array}{|c|} \hline \bullet \bullet \\ \hline \end{array} + b \begin{array}{|c|} \hline \bullet \\ \hline \end{array} + c \begin{array}{|c|} \hline \\ \hline \end{array} \tag{3D}$$

$$\text{Cylinder} = - \begin{array}{c} \text{cup} \\ \text{cup} \\ \text{cup} \end{array} + a \left(\begin{array}{c} \text{cup} \\ \text{cup} \end{array} \right) + b \begin{array}{c} \text{cup} \\ \text{cup} \end{array} \tag{CN}$$

$$\text{Sphere} = \text{Sphere} = 0, \quad \text{Sphere} = -1 \tag{S}$$

$$\begin{array}{c} \alpha \\ \beta \\ \delta \end{array} \text{Sphere} \begin{cases} 1 & (\alpha, \beta, \delta) = (1, 2, 0) \text{ or a cyclic permutation} \\ -1 & (\alpha, \beta, \delta) = (2, 1, 0) \text{ or a cyclic permutation} \\ 0 & \text{else} \end{cases} \tag{\Theta}$$

The *closure relation* says that any $\mathbb{C}[a, b, c]$ -linear combination of foams, all of which having the same boundary, is equal to zero if and only if any common way of closing these foams yields a $\mathbb{C}[a, b, c]$ -linear combination of closed foams whose evaluation is zero.

Using the relations ℓ , one can prove the identities below (for detailed proofs see [3]).

$$\text{Bamboo} = - \begin{array}{c} \text{cup} \\ \text{cup} \end{array} \tag{Bamboo}$$

$$\text{RD} = \begin{array}{c} \text{cup} \\ \text{cup} \end{array} - \begin{array}{c} \text{cup} \\ \text{cup} \end{array} \tag{RD}$$

$$\text{Bubble} = 0 \tag{Bubble}$$

$$\text{DR} = \begin{array}{c} \text{cup} \\ \text{cup} \end{array} - \begin{array}{c} \text{cup} \\ \text{cup} \end{array} \tag{DR}$$

(SqR)

(Dot Migration)

In this paper we will work with open webs and open foams.

Definition 4.1. **Foam₃** is the category whose objects are webs Γ inside a horizontal strip in \mathbb{R}^2 bounded by the lines $y = 0, 1$ containing the boundary points of Γ and whose morphisms are $\mathbb{C}[a, b, c]$ -linear combinations of foams inside that strip times the unit interval such that the vertical boundary of each foam is a set (possibly empty) of vertical lines.

For example, the diagrams 1_n and v_j are objects of **Foam₃**:

$$1_n = \begin{array}{c} \uparrow \quad \uparrow \quad \dots \quad \uparrow \\ 1 \quad 2 \quad \dots \quad n \end{array} \quad v_j = \begin{array}{c} \uparrow \quad \dots \quad \uparrow \quad \uparrow \quad \uparrow \quad \dots \quad \uparrow \\ 1 \quad \dots \quad j \quad j+1 \quad \dots \quad n \end{array} \tag{4.2}$$

The category **Foam₃** is additive and monoidal, with the monoidal structure given as in **Foam₂**. The category **Foam₃** is also additive and graded. The q -grading in $\mathbb{C}[a, b, c]$ is defined as

$$q(1) = 0, \quad q(a) = 2, \quad q(b) = 4, \quad q(c) = 6 \tag{4.3}$$

and the degree of a foam f with $|\bullet|$ dots and $|b|$ vertical boundary components is given by

$$q(f) = -2\chi(f) + \chi(\partial f) + 2|\bullet| + |b|, \tag{4.4}$$

where χ denotes the Euler characteristic and ∂f is the boundary of f .

4.2. The Functor $\mathcal{F}_{sl(3),n}$

In this subsection we define a monoidal functor $\mathcal{F}_{sl(3),n}$ between the categories \mathcal{SC} and \mathbf{Foam}_3 .

On Objects

$\mathcal{F}_{sl(3),n}$ sends the empty sequence to 1_n and the one-term sequence (j) to v_j with $\mathcal{F}_{sl(3),n}(jk)$ given by the vertical composite $v_j v_k$.

On Morphisms

(i) As before the empty diagram is sent to n parallel vertical sheets:

(ii) The vertical line colored j is sent to the identity foam of v_j :

(iii) The *StartDot* and *EndDot* morphisms are sent to the zip and the unzip, respectively:

(iv) *Merge* and *Split* are sent to the digon annihilation and creation, respectively:

(v) The 4-valent vertex with distant colors is shown as follows. For $j + 1 < k$ we have.

(4.9)

The case $j > k + 1$ is given by reflection around a horizontal plane.

(vi) For the 6-valent vertex we have

(4.10)

The case with the colors switched is given by reflection in a vertical plane.

Notice that $\mathcal{F}_{sl(3),n}$ respects the gradings of the morphisms.

Proposition 4.2. $\mathcal{F}_{sl(3),n}$ is a monoidal functor.

Proof. The assignment given by $\mathcal{F}_{sl(3),n}$ clearly respects the monoidal structures of \mathcal{SC}_1 and \mathbf{Foam}_3 . To prove that it is a monoidal functor we need only to show that it is actually a functor, that is, it respects relations (2.5) to (2.22) of Section 2.

Isotopy Relations

Relations (2.5) to (2.9) correspond to isotopies of their images under $\mathcal{F}_{sl(3),n}$, and we leave its check to the reader.

One-Color Relations

Relation (2.10) is straightforward and left to the reader. For relation (2.11) we have

(4.11)

the last equality following from the (Bubble) relation.

For relation (2.12) we have

$$\mathcal{F}_{sl(3),n} \left(\begin{array}{c} | \\ j \\ \bullet \\ | \\ j \\ \bullet \\ | \end{array} \right) = \begin{array}{c} \text{Diagram 1} \\ j \quad j+1 \end{array} = \begin{array}{c} \text{Diagram 2} \\ j \quad j+1 \end{array} - \begin{array}{c} \text{Diagram 3} \\ j \quad j+1 \end{array} \tag{4.12}$$

where the second equality follows from the (DR) relation. We also have

$$\mathcal{F}_{sl(3),n} \left(\begin{array}{c} | \\ j \\ \bullet \\ | \\ j \\ \bullet \\ | \end{array} \right) = \begin{array}{c} \text{Diagram 4} \\ j \quad j+1 \end{array} = \begin{array}{c} \text{Diagram 5} \\ j \quad j+1 \end{array} - \begin{array}{c} \text{Diagram 6} \\ j \quad j+1 \end{array}, \tag{4.13}$$

which is given by (RD). Using (Dot Migration) one obtains

$$\mathcal{F}_{sl(3),n} \left(\begin{array}{c} | \\ j \\ \bullet \\ | \\ j \\ \bullet \\ | \end{array} \right) = 2 \begin{array}{c} \text{Diagram 7} \\ j \quad j+1 \end{array} + \begin{array}{c} \text{Diagram 8} \\ j \quad j+1 \end{array} + a \begin{array}{c} \text{Diagram 9} \\ j \quad j+1 \end{array} \tag{4.14}$$

$$\mathcal{F}_{sl(3),n} \left(\begin{array}{c} \bullet \\ | \\ j \\ \bullet \\ | \\ j \\ \bullet \\ | \end{array} \right) = -2 \begin{array}{c} \text{Diagram 10} \\ j \quad j+1 \end{array} - \begin{array}{c} \text{Diagram 11} \\ j \quad j+1 \end{array} - a \begin{array}{c} \text{Diagram 12} \\ j \quad j+1 \end{array} \tag{4.15}$$

and therefore, we have that

$$\mathcal{F}_{sl(3),n} \left(\begin{array}{c} \bullet \\ | \\ j \\ \bullet \\ | \\ j \\ \bullet \\ | \end{array} \right) + \mathcal{F}_{sl(3),n} \left(\begin{array}{c} | \\ j \\ \bullet \\ | \\ j \\ \bullet \\ | \end{array} \right) = 2\mathcal{F}_{sl(3),n} \left(\begin{array}{c} \bullet \\ | \\ j \\ \bullet \\ | \\ j \\ \bullet \\ | \end{array} \right) \tag{4.16}$$

Two Distant Colors

Relations (2.13) to (2.15) correspond to isotopies of the foams involved and are straightforward to check.

Adjacent Colors

We prove the case where "blue" corresponds to j and "red" corresponds to $j+1$. The relations with colors reversed are proved the same way. To prove relation (2.16) we first notice that

$$\mathcal{F}_{sl(3),n} \left(\begin{array}{c} \text{blue} \\ \text{red} \end{array} \right) = \text{foam diagram} \tag{4.17}$$

$j \quad j+1 \quad j+2$

$$\mathcal{F}_{sl(3),n} \left(\begin{array}{c} \text{red} \\ \text{blue} \end{array} \right) = \text{foam diagram} \tag{4.18}$$

$j \quad j+1 \quad j+2$

We also have an isotopy equivalence

$$\mathcal{F}_{sl(3),n} \left(\begin{array}{c} \text{red} \\ \text{blue} \end{array} \right) \cong - \text{foam diagram} \tag{4.19}$$

$j \quad j+1 \quad j+2$

which in turn is isotopy equivalent to the foam obtained by putting

$$\text{foam diagram} \text{ on top of } - \text{foam diagram} \tag{4.20}$$

The common boundary of these two foams contains two squares. Putting (SqR) on the square on the right glued with the identity foam everywhere else gives two terms, one isotopic to $\mathcal{F}_{sl(3),n} \left(\begin{array}{c} \text{red} \\ \text{blue} \end{array} \right)$ and the other isotopic to $\mathcal{F}_{sl(3),n} \left(\begin{array}{c} \text{blue} \\ \text{red} \end{array} \right)$.

We now prove relation (2.17). We have

$$\mathcal{F}_{sl(3),n} \left(\begin{array}{c} | \\ | \\ | \\ | \end{array} \right) \cong \text{foam diagram} \tag{4.21}$$

Applying (SqR) to the middle square we obtain two terms. One is isotopic to $-\mathcal{F}_{sl(3),n} (\otimes)$ and the other gives $\mathcal{F}_{sl(3),n} (\otimes)$ after using the (Bamboo) relation.

We now prove relation (2.18) in the form

$$\text{braid diagram} = \text{braid diagram} \tag{4.22}$$

The images of the l.h.s. and r.h.s. under $\mathcal{F}_{sl(3),n}$ are isotopic to

$$\text{foam diagram} \text{ and } \text{foam diagram} \tag{4.23}$$

respectively, and both give the same foam after applying the (Bamboo) relation.

Relation (2.19) follows from a straightforward computation and is left to the reader.

Relations Involving Three Colors

Relations (2.20) and (2.21) follow from isotopies of the cobordisms involved.

We prove relation (2.22) in the form

$$\text{braid diagram} = \text{braid diagram} \tag{4.24}$$

We claim that $\mathcal{F}_{sl(3),n}$ sends both sides to zero. Since the images of both sides of (4.24) can be obtained from each other using a symmetry relative to a vertical plane placed between the sheets labelled $j+1$ and $j+2$, it suffices to show that one side of (4.24) is sent to zero. The foams involved are rather complicated and hard to visualize. To make the computations easier we use movies (two dimensional diagrams) for the whole foam and implicitly translate some bits to three-dimensional foams to apply isotopy equivalences or relations from Section 4.1. The r.h.s. corresponds to

$$f_1 = \text{[Diagram of foam } f_1 \text{ with 12 cells]} \tag{4.25}$$

followed by

$$f_2 = \text{[Diagram of foam } f_2 \text{ with 10 cells]} \tag{4.26}$$

The foam f_2 is isotopic to

$$\text{[Diagram of foam } f_2 \text{ with 7 cells]} \tag{4.27}$$

Using this, we can also see that the foams corresponding with

$$\text{[Diagram of foam with 7 cells]} \tag{4.28}$$

$$\text{[Diagram of foam with 6 cells]} \tag{4.29}$$

are isotopic. We see that the foam we have contains

$$\text{[Diagram of foam with 5 cells]} \tag{4.30}$$

which corresponds to a foam containing [Bubble symbol] , which is zero by the (Bubble) relation. \square

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