Research Article
Normed Domains of Holomorphy

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We treat the classical concept of domain of holomorphy in $\mathbb{C}^n$ when the holomorphic functions considered are restricted to lie in some Banach space. Positive and negative results are presented. A new view of the case $n = 1$ is considered.

1. Introduction

In this paper a *domain* $\Omega \subseteq \mathbb{C}^n$ is a connected open set. We let $\mathcal{O}(\Omega)$ denote the algebra of holomorphic functions on Ω.

We will use the following notation: D denotes the unit disc in the complex plane. We let $D^2 = D \times D$ denote the bidisc, and $D^n = D \times D \times \cdots \times D$ the polydisc in $\mathbb{C}^n$. The symbol $B = B_n$ is the unit ball in $\mathbb{C}^n$.

A domain $\Omega \subseteq \mathbb{C}^n$ is said to be *Runge* if any holomorphic f on Ω is the limit, uniformly on compact subsets of Ω, of polynomials.

In the classical function theory of several complex variables there are two fundamental concepts: domain of holomorphy and pseudoconvex domain. The Levi problem, which was solved comprehensively in the 1940s and 1950s, asserts that these two concepts are equivalent: a domain $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy if and only if it is pseudoconvex. These matters are discussed in some detail in [1].

Roughly speaking, if Ω is a domain of holomorphy, then there is a holomorphic function f on Ω such that f cannot be analytically continued to any larger domain. Generally speaking one cannot say much about the nature of this f—whether it is bounded, or satisfies some other growth condition.

In the paper [2], Sibony presents the following remarkable example.

Example 1.1. There is a bounded pseudoconvex Runge domain $\Omega_S \subseteq \mathbb{C}^2$, with Ω_S being a proper subset of the bidisc $D^2 = D \times D$, such that any *bounded* holomorphic function φ on Ω analytically continues to all of D^2.

Of course this result of Sibony can be extended to $\mathbb{C}^n$ in a variety of ways. For one thing, one may take the product of the Sibony domain with the polydisc in $\mathbb{C}^{n-2}$ to obtain an example in $\mathbb{C}^n$. Alternatively, one may replace the first (or z) variable in the Sibony construction with a tuple in $\mathbb{C}^{n-1}$ to obtain a counterexample in $\mathbb{C}^n$. The Sibony example and its implications are studied extensively in [3]. See also [4].

It is interesting to note that in some sense, the Sibony example is generic. In fact we have the following proposition.

Proposition 1.2. *The collection of domains* $\Omega \subseteq D^2$, *with* $\Omega \neq D^2$, *such that any bounded holomorphic function on* Ω *analytically continues to all of* D^2 *(as in the Sibony example above) is uncountable.*

Proof. We very quickly review the key steps of the Sibony construction.

Let p_j be a countable collection of points in the unit disc D, with no interior accumulation point, so that every boundary point of D is an accumulation point of the set $\{p_j\}$. Now define

$$\varphi(\zeta) = \sum_j \lambda_j \log\left|\frac{\zeta - p_j}{2}\right|. \tag{1.1}$$

Here $\{\lambda_j\}$ is a summable sequence of positive, real numbers. Notice that the function φ—being the sum of subharmonic functions—is subharmonic. Define

$$V_0(\zeta) = \exp(\varphi(\zeta)). \tag{1.2}$$

Then V_0 has the properties:

(i) V_0 is subharmonic;

(ii) $0 < V_0(\zeta) \leq 1$ for all $\zeta \in D$;

(iii) the function V_0 is continuous.

The last property holds because the sequence $\{p_j\}$ is discrete and V_0 takes the value 0 only at the p_j.

Now define the domain

$$M(D, V_0) = \left\{(z, w) \in \mathbb{C}^2 : z \in D, \ w \in \mathbb{C}, \ |w| < \exp(-V_0(z))\right\}. \tag{1.3}$$

Since V_0 is positive, we see that this definition makes sense and that $M(D, V_0)$ is a proper subset of D^2.

The remainder of Sibony's argument shows that any bounded, holomorphic function on $M(D, V_0)$ analytically continues to a bounded, holomorphic function on D^2. We will not repeat it but refer the reader to [2, 5].

The key fact in the Sibony construction is that the points $\{p_j\}$ form a discrete set that accumulates at every boundary point of D. Apart from this property, there is complete freedom in choosing the p_j. We begin by showing how to construct two biholomorphically distinct instances of Sibony domains and then consider at the end how to produce uncountably many biholomorphically distinct domains.

Define, for $\ell = 1, 2, \ldots,$

$$S_\ell = \left\{ z \in D : |z| = 1 - 2^{-\ell-1} \right\}. \tag{1.4}$$

Set

$\{p_j^1\}$ = (the sequence consisting of 4 equally spaced points on S_1, 8 equally spaced points on S_2, 16 equally spaced points on S_3, etc.),

$\{p_j^2\}$ = (the sequence consisting of 8 equally spaced points on S_1, 16 equally spaced points on S_2, 32 equally spaced points on S_3, etc.).

Define a domain Ω_1 using the Sibony construction, as above, with the sequence $\{p_j^1\}$ and define a domain Ω_2 using the Sibony construction with the sequence $\{p_j^2\}$. We claim that Ω_1 and Ω_2 are biholomorphically inequivalent.

To see this, suppose the contrary. So there is a biholomorphic mapping $\Phi : \Omega_1 \rightarrow \Omega_2$. By the usual classical arguments (see the proof of Proposition 11.1.2 in [1]), we see that Φ must commute with rotations in the w variable. It follows that any disc in Ω_1 of the form

$$\mathbf{d}_z = \left\{ (z, w) : |w| < \exp(-V_0(z)) \right\}, \tag{1.5}$$

for $z \in D$ fixed, must be mapped to a similar disc in Ω_2.

Further observe that each of the discs $\mathbf{d}_{p_j} \subseteq \Omega_1$ is a totally geodesic submanifold in the Kobayashi metric. This assertion follows immediately from the existence of the maps

$$D \xrightarrow{i} \Omega_1 \xrightarrow{\pi} D, \tag{1.6}$$

where i is the injection

$$i(w) = (p_j, w), \tag{1.7}$$

and π is the projection

$$\pi(p_j, w) = w. \tag{1.8}$$

Observe that $\pi \circ i = \text{id}$. Similar reasoning shows that the disc $\mathbf{d}^* = \{(z, 0)\} \subseteq \Omega_1$ is totally geodisc. Of course similar remarks apply to the corresponding discs in Ω_2.

Now it is essential to notice that, in Ω_1, the vertical discs of the form $\mathbf{d}_z$ for z not one of the p_j are *not* totally geodesic. This follows because, at the point $(z, 0)$, the Kobayashi extremal disc in the vertical direction $\langle 0, 1 \rangle$ will not be the rigid disc $\zeta \mapsto (0, \zeta)$ but rather a disc that curves into one of the spikes above a nearby p_j.

A final observation that we need to make is this. The vertical totally geodesic discs will be mapped to each other by the biholomorphic mapping Φ. But more is true. Because $\mathbf{d}^*$ is totally geodesic, in fact the totally geodesic discs located at the points p_j that lie on S_1 in Ω_1 must be mapped to the totally geodesic discs located at the points p_j that lie on S_1 in Ω_2 (because both circles consist of points that have the same Kobayashi distance from the

origin). This is impossible because S_1 in Ω_1 has four such points while S_1 in Ω_2 has eight such points. That is the required contradiction.

Now it is clear how to construct uncountably many inequivalent domains of Sibony type. For if $\mathcal{A} = \{\alpha_j\}$ is any exponentially increasing sequence of positive integers, then, we may associate to it a domain $\tilde{\Omega}$ of Sibony-type with α_1 points on S_1, α_2 points on S_2, and so forth. The preceding argument shows that different choices of $\mathcal{A}$ result in biholomorphically inequivalent domains. And there are clearly uncountably many such sequences. That completes the argument.   □

Recall that if $\Omega \subseteq \mathbb{C}^n$ is a domain and $\mathcal{F}$ is a family of functions on Ω, then we say that Ω is convex with respect to $\mathcal{F}$ if, whenever $K \subseteq \Omega$ is relatively compact, then

$$\hat{K} \equiv \left\{ z \in \Omega : |f(z)| \leq \sup_{w \in K}|f(w)| \right\} \tag{1.9}$$

is also relatively compact in Ω. It is well known that any domain of holomorphy (i.e., any pseudoconvex domain) is convex with respect to the family of holomorphic functions on the domain.

It is natural in our discussion to consider a domain Ω and the family $\mathcal{G}$ of all *bounded holomorphic functions* on Ω.

Proposition 1.3. *The Sibony domain Ω_S is convex with respect to the family $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be as above for the domain Ω_S and let $\mathcal{F}$ be the usual family of *all* holomorphic functions on Ω_S. Let K be a compact subset of Ω_S and $\hat{K}_{\mathcal{F}}$ the hull with respect to $\mathcal{F}$. Then, since Ω_S is a domain of holomorphy, $\hat{K}_{\mathcal{F}}$ is still compact in Ω_S. Let $z \in \Omega_S$ be a point that does not lie in $\hat{K}_{\mathcal{F}}$. Then there is a holomorphic function f on Ω_S so that $|f(z)| > \sup_{w \in \hat{K}_{\mathcal{F}}}|f(w)|$. Let $\epsilon = |f(z)| - \sup_{w \in \hat{K}_{\mathcal{F}}}|f(w)| > 0$. Since Ω_S is Runge, there is a polynomial p so that $|p(m) - f(m)| < \epsilon/3$ on $K \cup \{z\}$. But then p is bounded on Ω_S and, by the triangle inequality, $|p(z)| > \sup_{w \in \hat{K}_{\mathcal{F}}}|f(w)|$. Thus $\hat{K}_{\mathcal{F}} = \hat{K}_{\mathcal{G}}$. This shows that Ω_S is convex with respect to the family of bounded holomorphic functions on Ω_S.   □

In fact the argument just presented (for which I thank Erik Løw) shows that any bounded, pseudoconvex, Runge domain is convex with respect to the family of bounded, holomorphic functions.

The Sibony result has an interesting and important interpretation in terms of the corona problem. We have the following proposition.

Proposition 1.4. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Suppose that X is a Banach space of holomorphic functions on Ω that contains $H^\infty(\Omega)$. Let Ω' be a strictly larger domain that contains Ω. Assume that any element of X analytically continues to a holomorphic function on Ω' (one often assumes that the extended function satisfies a similar norm estimate to that specified by the norm on X, but that is not necessary and one does not impose that condition at this time). Then the corona problem cannot be solved in the space X. That is to say, if $f_1, f_2, \ldots, f_k$ are holomorphic functions in X with no common zero, then there do not exists elements $g_1, g_2, \ldots, g_k \in X$ such that*

$$f_1 g_1 + f_2 g_2 + \cdots f_k g_k \equiv 1 \tag{1.10}$$

on Ω.

Proof. Assume to the contrary that such $g_1, g_2, \ldots, g_k$ exist. Of course each g_j analytically continues to Ω'.

Let $P = (p_1, p_2, \ldots, p_n)$ be a point of $\Omega' \setminus \Omega$. Set $f_j(z) = z_j - p_j$. Then the f_j have no common zero in Ω. So, by hypothesis, the g_j exist. And these functions extend analytically to Ω'. But then

$$f_1 g_1 + f_1 g_2 + \cdots f_n g_n \equiv 1 \tag{1.11}$$

on Ω'. Since the f_j all vanish at P, we see that, at P, the left-hand side of this last equation vanishes. That is clearly a contradiction. Hence the g_j do not exist. $\qquad \square$

Of course this last proposition means in particular that the point evaluations on Ω are not weak-$*$ dense in the maximal ideal space of $H^\infty(\Omega)$; see [6] for more on these matters.

By contrast to Sibony's result, Catlin [7] has shown that any smoothly bounded, pseudoconvex domain in $\mathbb{C}^n$ supports a bounded holomorphic function that cannot be analytically continued to any larger domain. In fact he has proved something sharper.

Theorem 1.5. *Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Then there is a function in $C^\infty(\overline{\Omega})$, holomorphic on the interior, which cannot be analytically continued to any larger domain.*

Hakim and Sibony [8] have proved something even more decisive.

Theorem 1.6. *Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Then the maximal ideal space (or spectrum) of the algebra $C^\infty(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ is in fact $\overline{\Omega}$.*

It should be stressed that the proofs of the last two results use an algebraic formalism of Hörmander [9] which entails the loss of some derivatives; so it is essential to be working with functions that are C^∞ on $\overline{\Omega}$. Attempts to adapt the arguments to other function spaces are doomed to failure.

Pflug and Zwonek [10] have shown that the situation for L^2 holomorphic functions is very neat and elegant.

Theorem 1.7. *Let $\Omega \subseteq \mathbb{C}^n$ be any pseudoconvex domain. There is an L^2 holomorphic function on Ω that cannot be continued to any larger domain if and only if, for all $w \in \partial U$ and all neighborhoods U of w, $U \setminus \Omega$ is not pluripolar.*

There is also a characterization in terms of geometric regularity of the boundary, expressed terms of external balls (see [11]).

Theorem 1.8. *Suppose that Ω in $\mathbb{C}^n$ is a domain of holomorphy and that, for each $z^0 \in \partial\Omega$, there is a sequence $w^\nu \in {}^c\overline{\Omega}$ such that $w^\nu \to z^0$ and there are $0 < r \leq 1$ and $\alpha > 0$ such that $B(w^\nu, r|z^0 - w^\nu|^\alpha) \cap \overline{\Omega} = \emptyset$. Then there is an L^2 holomorphic function on Ω that cannot be analytically continued to any larger domain.*

It is natural to ask for a characterization of those domains Ω which are domains of holomorphy in the traditional sense but *not* domains of holomorphy for bounded holomorphic functions. One would also like to know whether there are analogous results for L^p holomorphic functions, $1 \leq p < \infty$.

The purpose of the present paper is to consider these matters. While we cannot provide a full answer to the questions just posed, we can certainly give some useful partial results, and point in some new directions. The work in [3] contains a detailed consideration of questions of this kind for the case of H^∞.

We mention in passing that the paper [12] contains some results that bear on the questions posed here. The arguments presented in [12] appear to be incomplete.

2. Some Notation

Let us say that a domain $\Omega \subseteq \mathbb{C}^n$ is of type HL^p, $1 \leq p \leq \infty$, if there is a holomorphic function f on Ω, $f \in L^p(\Omega)$, which cannot be analytically continued to any larger domain. We instead say that Ω is of type EL^p if there is a strictly larger domain $\widehat{\Omega}$ so that every holomorphic L^p function on Ω analytically continues to $\widehat{\Omega}$. Obviously HL^p and EL^p are disjoint.

We are interested in giving an extrinsic description of those domains which are of type HL^p and those which are of type EL^p. It is not the case in higher dimensions, for instance, that HL^∞ domains are the same as domains that are convex with respect to $\mathcal{G} = H^\infty$ (see [3]). So we seek other characterizations.

3. The Situation in the Complex Plane

Matters in one complex variable are fairly well understood.

First of all, we should note the example of

$$\Omega = D \setminus \{0\}. \tag{3.1}$$

Of course, by the Riemann removable singularities theorem, any bounded holomorphic function on Ω analytically continues to all of D. So Ω is *not* a domain of type HL^∞. It *is* a domain of type EL^∞.

In fact it may be noted (for the domain Ω in the last paragraph) that, if $p \geq 2$, then any holomorphic function that is $L^p(\Omega)$ will analytically continue to all of D (see [13]). So this Ω is a domain of type EL^p. By contrast, if $p < 2$, then the function $f(\zeta) = 1/\zeta$ is holomorphic on Ω and in $L^p(\Omega)$. But of course this f *does not* analytically continue to the full disc D. So, for $p < 2$, the domain is of type HL^p.

The treatment in [13] of the matter just discussed is rather abstract, and it is worthwhile to have a traditional function-theoretic treatment of these matters. We provide one now. We thank Richard Rochberg for a helpful conversation about this topic. So let f be holomorphic on $D \setminus \{0\}$ and assume that $f \in L^2(D)$ (the case $p > 2$ follows immediately from this one). We write $f(\zeta) = \sum_{j=-\infty}^{\infty} a_j \zeta^j$. For $0 < a < b < 1$ and k a negative integer, consider the expression

$$A = \int_a^b r \int_0^{2\pi} f(\zeta) e^{-ik\theta} d\theta \, dr, \tag{3.2}$$

where it is understood that $\zeta = re^{i\theta}$.

On the one hand,

$$|A| \leq \int_{a \leq |\zeta| \leq b} |f(\zeta)| \, dA(\zeta)$$
$$\leq \|f\|_{L^2} \cdot |\{\zeta : a \leq |\zeta| \leq b\}|^{1/2} \tag{3.3}$$
$$= \|f\|_{L^2} \cdot \left[\pi\left(b^2 - a^2\right)\right]^{1/2}.$$

On the other hand,

$$|A| = \left| \int_a^b r a_k r^k \, dr \right|$$
$$= \left| a_k \cdot \left[\frac{r^{k+2}}{k+2} \right]_a^b \right| \tag{3.4}$$
$$= \left| a_k \left(\frac{b^{k+2}}{k+2} - \frac{a^{k+2}}{k+2} \right) \right|.$$

If $k < -2$ and $a = b/2$, this gives a contradiction as $b \to 0^+$. Of course the cases $k = -2$ and $k = -1$ can be handled separately because ζ^{-2} and ζ^{-1} are certainly *not* in L^2.

Remark 3.1. We note that the proof goes through for $p < 2$ up until the very end. One must note that ζ^{-1} in fact *does* lie in L^p for $p < 2$. So there is no removable singularities theorem for this range of p.

Remark 3.2. A standard result coming from potential theory is that, if $\Omega \subseteq \mathbb{C}$, $P \in \Omega$, and f is holomorphic on $\Omega \setminus \{P\}$, then $|f(z)| = o(\log[1/|z - P|])$ (where we are using Landau's notation) implies that f continues analytically to all of Ω. The philosophy here is that a function f satisfying this growth hypothesis has a singularity at P that is milder than the singularity of the Green's function. This point of view is particularly useful in the study of removable singularities for harmonic functions. This result is not of any particular interest for us because it is not formulated in the language of Lebesgue spaces. In any event, it is weaker than the result presented above for L^2 because the logarithm function is certainly square integrable. It is a pleasure to thank Al Baernstein and David Minda for helpful remarks about these ideas.

The enemy in the results discussed at the beginning of this section is that Ω is not equal to the interior of its closure. In fact we have the following proposition.

Proposition 3.3. *Suppose that the bounded domain $\Omega \subseteq \mathbb{C}$ is the interior of its closure. Then Ω is a domain of type HL^p for $1 \leq p \leq \infty$.*

Proof. The proof that we now present is an adaptation and simplification of an argument from [4].

Let $\{p_j\}$ be a countable, dense subset of ${}^c\overline{\Omega}$. For each j, the function $\varphi_j(\zeta) = 1/(\zeta - p_j)$ is holomorphic and bounded on Ω and does not analytically continue past p_j.

Now, for each j, let $\mathbf{d}_j$ be an open disc centered at p_j which has nontrivial intersection with Ω. Consider the linear mapping

$$I_j : \mathcal{O}(\Omega \cup \mathbf{d}_j) \cap L^p(\Omega \cup \mathbf{d}_j) \longrightarrow \mathcal{O}(\Omega) \cap L^p(\Omega) \qquad (3.5)$$

given by restriction. Of course each of the indicated spaces is equipped with the L^p norm, and is therefore a Banach space. We note that the example above of φ_j shows that I_j is not surjective. As a result, the open mapping principle tells us that the image $\mathcal{M}_j$ of I_j is of first category in $\mathcal{O}(\Omega) \cap L^p(\Omega)$. Therefore, by the Baire category theorem,

$$\mathcal{M} \equiv \bigcup_j \mathcal{M}_j \qquad (3.6)$$

is of first category in $\mathcal{O}(\Omega) \cap L^p(\Omega)$. But this just says that the set of L^p holomorphic functions on Ω that can be analytically continued to some p_j is of first category. Therefore the set of L^p holomorphic functions that *cannot* be analytically continued across the boundary is dense in $\mathcal{O}(\Omega) \cap L^p(\Omega)$. That completes the proof. $\qquad\qquad\square$

The key point of the proof just presented is that, for each point not in the closure of the given domain, there is a function holomorphic on the domain (and in the given function space) that does not analytically continue past the point. Such functions are trivial to construct in one complex variable, not so in higher dimensions.

We note in passing that when $\Omega \subseteq \mathbb{C}$ is the unit disc D then it is easy to construct a bounded holomorphic function that does not analytically continue to a larger domain. Let $\{p_j\}$ be a discrete set in D that accumulates at every boundary point and so that

$$\sum_j 1 - |p_j| < \infty. \qquad (3.7)$$

For example, take

p_1, p_2, p_3, p_4 to be equally spaced points at distance $1/4$ from ∂D,

$p_5, p_6, \dots, p_{12}$ to be equally spaced points at distance $1/8$ from ∂D,

$p_{13}, p_6, \dots, p_{28}$ to be equally spaced points at distance $1/16$ from ∂D,

and so forth. Then the Blaschke product with zeros at the p_j will do the job. If Ω is a simply connected domain having a Jordan curve as its boundary, then conformal mapping together with Carathéodory's theorem about continuous boundary extension will give a bounded, holomorphic, non-continuable function on this Ω.

We close this section by noting that, if Ω is a domain of holomorphy in $\mathbb{C}^n$ and if $V = \{z \in \Omega : f(z) = 0\}$ for some holomorphic f on Ω (we call V a *variety*), then $\Omega' = \Omega \setminus V$ is also a domain of holomorphy (if φ is a holomorphic function on Ω that does not analytically continue to a larger domain then φ / f is a holomorphic function on Ω' that does not analytically continue to any larger domain. And it is easy to see that Ω' is an EL^∞ domain; see [14, page 19] for the details).

4. Complications in Dimension n

As we have indicated, the example of Sibony exhibits a domain which is *not* of type HL^∞ (instead it is of type EL^∞). The theorem of Catlin shows that all smoothly bounded, pseudoconvex domains are of type HL^∞.

It of course makes sense to focus this discussion on pseudoconvex domains. If a domain Ω is *not* pseudoconvex, then there will perforce be a larger domain Ω' to which all holomorphic functions (regardless of growth) on Ω analytically continue. So this situation is not interesting.

Thus we see that the domains of interest for us will be pseudoconvex domains that do *not* have smooth boundary. Our first result is as follows.

Proposition 4.1. *Let $D_1, D_2, \ldots, D_n$ be bounded domains in $\mathbb{C}$, each of which is equal to the interior of its closure. Define*

$$\Omega = D_1 \times D_2 \times \cdots \times D_n. \tag{4.1}$$

Then Ω is a domain of type HL^p for any $1 \le p \le \infty$.

Proof. Fix p as indicated. Then, by Proposition 3.3, there is a holomorphic function ψ_j on D_j, for $1 \le j \le n$, such that ψ_j is holomorphic and L^p on D_j and does not analytically continue to any larger domain.

But then

$$\psi(z_1, \ldots, z_n) = \psi_1(z_1) \cdot \psi_2(z_2) \cdots \psi_n(z_n) \tag{4.2}$$

is holomorphic and L^p on Ω and does not analytically continue to any larger domain. $\square$

Proposition 4.2. *Let $\Omega \subseteq \mathbb{C}^n$ be bounded and convex. Let $1 \le p \le \infty$. Then Ω is a domain of type HL^p.*

Proof. Just imitate the proof of Proposition 3.3. The main point to note is that if $q \notin \overline{\Omega}$ and v is a unit normal vector from $\partial\Omega$ out through q, then the function

$$\psi(z) = \frac{1}{(z - q) \cdot v} \tag{4.3}$$

is holomorphic and bounded on Ω and is singular at q. So the rest of the proof goes through as before. $\square$

In fact more is true.

Proposition 4.3. *Let $\Omega \subseteq \mathbb{C}^n$ be bounded and strongly pseudoconvex with C^2 boundary. Let $1 \le p \le \infty$. Then Ω is a domain of type HL^p.*

Proof. Of course we again endeavor to apply the argument of the proof of Proposition 3.3. It is enough to restrict attention to points q in $^c\Omega$ which are sufficiently close to $\partial\Omega$. If q is such a point, then there is a larger strongly pseudoconvex domain Ω' with C^2 boundary such that

$\overline{\Omega} \subseteq \Omega'$ and $q \in \partial\Omega'$. Now let $L_q(z)$ be the Levi polynomial (see [1]) for Ω' at q. Then there is a neighborhood U of q so that

$$\{z \in U : L_q(z) = 0\} \cap \overline{\Omega'} = \{q\}. \tag{4.4}$$

Thus $f_q(z) = 1/L_q(z)$ is holomorphic on $\Omega' \cap U$ and singular at q. Let φ be a C_c^∞ function that is compactly supported in U and is identically equal to 1 in a small neighborhood of q. We wish to choose a bounded function h so that

$$g(z) = \frac{\varphi(z)}{L_q(z)} + h \tag{4.5}$$

is holomorphic on Ω'. This entails solving the $\overline{\partial}$-problem

$$\overline{\partial}h = -\frac{\overline{\partial}\varphi(z)}{L_q(z)}. \tag{4.6}$$

Of course the data on the right-hand side of this equation is $\overline{\partial}$-closed with bounded coefficients. By work in [15] or [16] we see that a bounded solution h exists.

This gives us a function g that is (i) holomorphic on Ω' and (ii) singular at q. This is just what we need, for points q in $^c\Omega$ that are close to $\partial\Omega$, in order to imitate the proof of Proposition 3.3. That completes our argument. See also [4, Theorem 3.6] for a similar result with a somewhat different proof in the case $p = \infty$. □

For finite type domains we can prove the following result. Let $\Omega \subseteq \mathbb{C}^2$ be given by $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$. Recall that a point q in the boundary of a domain Ω is said to be of *finite geometric type* m in the sense of Kohn if there is a nonsingular, one-dimensional analytic variety $\varphi : D \to \mathbb{C}^2$ with $\varphi(0) = q$ and

$$|\rho(\varphi(\zeta))| \le C \cdot |\varphi(\zeta) - q|^m, \tag{4.7}$$

and so that there is no other nonsingular, one-dimensional analytic variety satisfying a similar inequality with m replaced by $m + 1$. These ideas are discussed in detail in [1, Chapter 10].

It is known that the geometric definition of finite type given in the last paragraph is equivalent to a more analytic one in terms of commutators of vector fields. Namely, let

$$L = \frac{\partial\rho}{\partial z_1}\frac{\partial}{\partial z_2} - \frac{\partial\rho}{\partial z_2}\frac{\partial}{\partial z_1} \tag{4.8}$$

be a complex tangential vector field to $\partial\Omega$ and $\overline{L}$ its conjugate. A *first-order commutator* is a Lie bracket of the form $[L, \overline{L}]$. A *second-order commutator* is a Lie bracket of the form $[L, M]$ or $[\overline{L}, M]$, where M is a first-order commutator, and so forth. We say that a point $q \in \partial\Omega$ is

of *analytic type m* if all the commutators $\mathcal{L}$ up to and including order $m - 1$ have the property that

$$\mathcal{L}(\rho)[q] = 0, \tag{4.9}$$

but there is a commutator $\mathcal{L}'$ of order m such that

$$\mathcal{L}'(\rho)[q] \neq 0. \tag{4.10}$$

It is a result of Kohn [17] and Bloom and Graham [18] that, when $\Omega \subseteq \mathbb{C}^2$, a point $q \in \partial\Omega$ is of geometric finite type if and only if it is of analytic finite type. Details of these matters may be found in [1].

Now it is easy to see that the notion of analytic finite type varies semi-continuously with smooth variation of ρ. In particular, if each point of $\partial\Omega$ is of some finite type, then the type of the point will vary semi-continously. So there is an upper bound M for all types of points in $\partial\Omega$. In this circumstance we say that Ω is a domain of finite type (at most) M.

As a result of these considerations, one has the following lemma.

Lemma 4.4. *Let* $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ *be a domain of finite type M. Then there are domains* Ω' *of finite type so that* $\Omega' \supset \overline{\Omega}$. *In particular, if* ψ *is a smooth, negative function with* $\|\psi\|_{C^{M+1}}$ *sufficiently small and* $\rho' = \rho + \psi$ *then* $\Omega' \equiv \{z \in \mathbb{C}^2 : \rho'(z) < 0\}$ *will contain* $\overline{\Omega}$ *and be of finite type.*

Now we have the following proposition.

Proposition 4.5. *Let* $\Omega \subseteq \mathbb{C}^2$ *be smoothly bounded and of finite type m. Let* $1 \leq p \leq \infty$. *Then* Ω *is a domain of type* HL^p.

Proof. The argument is similar to that for the last few propositions. If $q \notin \overline{\Omega}$ and is sufficiently close to $\partial\Omega$, then we may use the last lemma and the discussion preceding that to construct a finite type domain $\Omega' \supset \overline{\Omega}$ and with $q \in \partial\Omega'$. Now the theorem of Bedford and Fornæss [19] gives us a peaking function f_q for the point q on the domain Ω'. That is to say,

(i) f_q is continuous on $\overline{\Omega'}$;

(ii) f_q is holomorphic on Ω';

(iii) $|f_q(z)| \leq 1$ for all $z \in \overline{\Omega'}$;

(iv) $f_q(q) = 1$;

(v) $|f_q(z)| < 1$ for all $z \in \overline{\Omega'} \setminus \{q\}$.

Then the function $g_q(z) = 1/[1 - f_q(z)]$ is holomorphic on Ω and singular at q.

The rest of the argument is completed as in the proof of the last proposition. □

We note that the Kohn-Nirenberg domain [20] shows that, even on a finite type domain in $\mathbb{C}^2$, we cannot hope for a holomorphic separating function like L_q in the strongly pseudoconvex case. But the peak function of Bedford-Fornæss suffices for our purposes.

Proposition 4.6. *Let* $\Omega \subseteq \mathbb{C}^n$ *be a bounded analytic polyhedron. Certainly* Ω *is then a domain of holomorphy. We have that* Ω *is a domain of type* HL^p *for* $1 \leq p \leq \infty$.

Proof. We know by the standard definition (see [1]) that

$$\Omega = \left\{ z \in \mathbb{C}^n : \left| f_1(z) \right| < 1, \left| f_2(z) \right| < 1, \ldots, \left| f_k(z) \right| < 1 \right\} \tag{4.11}$$

for some holomorphic functions f_j. Now if $q \notin \overline{\Omega}$, then there is some complex constant λ with $|\lambda| > 1$ and some j so that $f_j(q) = \lambda$. That being the case, the function

$$\psi(z) = \frac{1}{\lambda - f_j(z)} \tag{4.12}$$

is a function that is bounded and holomorphic on Ω but singular at q. Now the proof can be completed as in the previous propositions. $\qquad\square$

Proposition 4.7. *Let $\Omega \subseteq \mathbb{C}^n$ be a complete circular domain. Assume that Ω is pseudoconvex. Then Ω is a domain of type HL^p, $1 \leq p \leq \infty$.*

Proof. Let q be a point that does not lie in $\overline{\Omega}$. Let q^* be the nearest point to q in the boundary of Ω, and let ν be the unit outward normal vector at q^*. Set

$$f_q(z) = (z - q) \cdot \nu. \tag{4.13}$$

Then f_q is holomorphic, and we claim that the zero set $\mathcal{Z}_q$ of f_q does not intersect $\overline{\Omega}$. Suppose to the contrary that it does.

Let x be a point that lies in both $\overline{\Omega}$ and in $\mathcal{Z}_q$. Of course any point that can be obtained by rotating the coordinates of x will also lie in $\overline{\Omega}$. One such choice of rotations will give a point that lies on the ray from the origin out to q. But that rotated point will be further from the origin than q itself (by the Pythagorean theorem). Since it lies in $\overline{\Omega}$, then so does q (because the domain is complete circular). That is a contradiction. Therefore x does not exist and $\overline{\Omega}$ and the zero set of f_q are disjoint.

As a result, the function $g_q \equiv 1/f_q$ is holomorphic and bounded on Ω and singular at q. The proof may now be completed as in the preceding propositions. $\qquad\square$

The next result points in the general direction that any reasonable pseudoconvex domain will be of type HL^p for $1 \leq p \leq \infty$.

Proposition 4.8. *Let Ω be a bounded, pseudoconvex domain with a Stein neighborhood basis. (Here a Stein neighborhood basis for Ω is a decreasing collection of pseudoconvex domains Ω_j such that $\cap_j \Omega_j = \overline{\Omega}$; see [21] for further details in this matter.) Then Ω is a domain of type HL^p for $1 \leq p \leq \infty$.*

Remark 4.9. Of course a domain with Stein neighborhood basis can have rough boundary. So this proposition says something new and with content.

Proof of the proposition. Let $\epsilon > 0$. By definition of Stein neighborhood basis, there is a pseudoconvex domain $\widetilde{\Omega}$ so that $\widetilde{\Omega} \supseteq \overline{\Omega}$. Therefore (see [1, Chapter 3]) there is a smoothly bounded, strongly pseudoconvex domain $\widetilde{\widetilde{\Omega}}$ so that $\widetilde{\Omega} \supseteq \widetilde{\widetilde{\Omega}} \supset \overline{\Omega}$. Let $P \in \partial\widetilde{\widetilde{\Omega}}$. Then we may imitate the construction in the proof of Proposition 3.3 to find a function that is holomorphic

and bounded on Ω, extends past the boundary of Ω, but is singular at P. Now the rest of the argument—elementary functional analysis—is just as in the proof of Proposition 3.3. □

The interest of Propositions 4.5, 4.6, and 4.7 is that the domains constructed there have only Lipschitz boundary. We know for certain (thanks to Catlin and Hakim/Sibony) that pseudoconvex domains with smooth boundary are of type HL^∞. And there are domains with rough boundary, such as the Sibony domain, that are of type EL^∞. So the last two propositions give examples of domains with rough boundary which are of type HL^∞.

5. Other Properties of HL^p and EL^p Domains

In [4] an example is given which shows that the increasing union of HL^∞ domains need not be HL^∞. Indeed, it is well known (see [22]) that *any* domain of holomorphy is the increasing union of analytic polyhedra (see Proposition 3.3). Of course an analytic polyhedron is HL^p for $1 \leq p \leq \infty$, but the Sibony domain (which is certainly the union of analytic polyhedra) described above is pseudoconvex and not HL^∞. Berg in addition shows (see his Theorem 1.15) that the decreasing intersection of HL^∞ domains is HL^∞.

Now we describe some other related examples. Again see [4] for cognate ideas.

Example 5.1. There is a decreasing sequence $\Omega_1 \supseteq \Omega_2 \supseteq \cdots$ of EL^∞ domains such that the intersection domain $\Omega_0 \equiv \cap_j \Omega_j$ is not EL^∞.

To see this, we follow the construction of [2, page 206]. Let $\{a_j\}$ be a sequence in the unit disc D with no interior accumulation point and such that every boundary point of D is the nontangential limit of some subsequence. Let λ_j be a summable sequence of positive real numbers. Define, for $\epsilon > 0$ and $z \in D$,

$$\varphi^\epsilon(z) = \sum_j \epsilon \lambda_j \log \left| \frac{z - a_j}{2} \right|. \tag{5.1}$$

Then certainly φ^ϵ is subharmonic and negative on D. Further note that the functions φ^ϵ increase pointwise to the identically 0 function as $\epsilon \to 0^+$. Now set

$$V_0^\epsilon(z) = \exp(\varphi^\epsilon(z)). \tag{5.2}$$

Then V_0^ϵ is also subharmonic, $0 \leq V_0^\epsilon < 1$. The function takes the value 0 only at the points a_j.

Finally define the domains

$$M^\epsilon(D, V_0^\epsilon) = \left\{ (z, w) \in \mathbb{C}^2 : z \in D, \ w \in \mathbb{C}, \ |w| < \exp(-V_0^\epsilon(z)) \right\}. \tag{5.3}$$

Each $M^\epsilon(D, V_0^\epsilon)$ is pseudoconvex. And the argument of Sibony shows that it is a domain of type EL^∞. But notice that the function $\exp(-V_0^\epsilon(z))$ decreases pointwise to the function that is identically equal to $1/e$ as $\epsilon \to 0^+$. Hence the domains $M^\epsilon(D, V_0^\epsilon)$ decrease to the bidisc $\{(z, w) : z \in D, \ |w| < 1/e\}$. And the latter is a domain of type HL^∞.

So we have produced a decreasing sequence of EL^∞ domains whose intersection is HL^∞.

We now give a separate proof, which has independent interest, of the contrapositive of Proposition 4.8.

Proposition 5.2. *If Ω is a bounded domain of type EL^p, $1 \leq p \leq \infty$ then Ω does not have a Stein neighborhood basis.*

Proof. Suppose that every holomorphic L^p function on Ω analytically continues to a larger domain $\widehat{\Omega}$. Seeking a contradiction, we assume that Ω has a Stein neighborhood basis. Choose a pseudoconvex domain $U \supseteq \overline{\Omega}$ so that $\widehat{\Omega} \setminus U$ is nonempty.

Now there is some holomorphic function g on U that does not analytically continue to any larger open domain. Therefore the restriction of g to Ω is a holomorphic L^p function $\widetilde{g}$ on Ω that analytically continues to U but no further. This contradicts the fact that $\widetilde{g}$ must analytically continue to $\widehat{\Omega}$. We conclude that Ω cannot have a Stein neighborhood basis.   □

We close with the following useful property of EL^∞ domains.

Proposition 5.3. *Let Ω be a bounded, EL^∞ domain in $\mathbb{C}^n$, so that any bounded, holomorphic function f on Ω analytically continues to some bounded, holomorphic function $\widehat{f}$ on some $\widehat{\Omega}$. Let f be a bounded, holomorphic function on Ω so that $|f|$ is bounded from 0 by some $\eta > 0$. Then $\widehat{f}$ will be nonvanishing.*

Proof. Of course $g = 1/f$ makes sense on Ω and is holomorphic and bounded; so it analytically continues to some bounded, holomorphic function $\widehat{g}$ on $\widehat{\Omega}$. But of course $1 \equiv f \cdot g$ analytically continues to the identically 1 function on $\widehat{\Omega}$. So we see that $\widehat{f} \cdot \widehat{g} \equiv 1$ on $\widehat{\Omega}$. We conclude then that $\widehat{f}$ cannot vanish.   □

6. Relationship with the $\overline{\partial}$-Problem

In the paper [5], Sibony exhibits a smoothly bounded, pseudoconvex domain on which the equation

$$\overline{\partial} u = f, \tag{6.1}$$

for f a $\overline{\partial}$-closed $(0,1)$ form with bounded coefficents, has no bounded solution u. This is important information for function theory, and also for the theory of partial differential equations.

It is natural to speculate that there is some relation between those domains on which the $\overline{\partial}$-equation satisfies uniform estimates and those domains which are of type HL^∞. In that vein, we offer the following result.

Proposition 6.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain which is of finite type m and so that the $\overline{\partial}$-equation $\overline{\partial} u = f$ satisfies uniform estimates on Ω. That is to say, there is a universal constant $C > 0$ so that, given a $\overline{\partial}$-closed $(0,1)$ form f with bounded coefficients, there is a solution u to the equation $\overline{\partial} u = f$ with*

$$\|u\|_{L^\infty} \leq C \cdot \|f\|_{L^\infty}. \tag{6.2}$$

Then Ω is a domain of type HL^∞.

Remark 6.2. It is important to notice in this last proposition that the domain Ω need not have C^∞ boundary. For type 2, it suffices for the boundary to be C^2. For type $m \geq 2$, it suffices for the boundary to be C^m.

It is known that strongly pseudoconvex domains [15], finite type domains in $\mathbb{C}^2$[23], and the polydisc [24] all satisfy uniform estimates for the $\bar{\partial}$-problem.

Proof of the proposition. It is known (see, e.g., [25, 26] or [27, 28]) that the strongly pseudoconvex points in $\partial\Omega$ form an open, dense set. Let $q \in \partial\Omega$ be such a point and let $\epsilon > 0$. Let ν be the unit outward normal vector to $\partial\Omega$ at q and set $q' = q + \epsilon\nu$. If ϵ is small then there is a "bumped domain" Ω' with these properties.

(i) There is a small neighborhood U of q so that $U \cap \partial\Omega$ consists only of strongly pseudoconvex points.

(ii) $\partial\Omega \setminus U = \partial\Omega' \setminus U$.

(iii) $\partial\Omega' \cap U$ is strongly pseudoconvex and lies outside Ω.

(iv) $\text{dist}_{\text{Euclid}}(q, \partial\Omega') > 0$.

(v) $q' \in \partial\Omega'$.

We exhibit the situation in Figure 1.

Now let $L_{q'}$ be the Levi polynomial for $\partial\Omega'$ at q'. Let $\varphi \in C_c^\infty(U)$ be identically equal to 1 in a small neighborhood of q'.

We do not know *a priori* that the $\bar{\partial}$-problem satisfies uniform estimates on the domain Ω'. But we may apply the construction of Beatrous and Range [29] to see that this is in fact the case (we thank Frank Beatrous and R. Michael Range for helpful remarks regarding this device). In detail, suppose that f is a $\bar{\partial}$ closed form on Ω'. Solve $\bar{\partial}u = f$ on Ω with uniform estimates. Let χ be a cutoff function which is 0 on U and identically 1 in the complement of a slightly larger strongly pseudoconvex neighborhood of q. Let $u_0 = \chi u$, extended as zero across the perturbed part of the boundary. Let $f_0 = f - \bar{\partial}u_0$, which is defined and bounded on Ω' and vanishes in a neighborhood of $\partial\Omega \setminus U$. We can therefore solve $\bar{\partial}v = f_0$ in Ω', with uniform estimates, by [29], Theorem 1.1. The solution in Ω' to the original equation is then $u_0 + v$. And that solution is bounded.

Now we use this last result to solve the equation

$$\bar{\partial}u = \left(\bar{\partial}\varphi\right) \cdot \frac{1}{L_{q'}} \tag{6.3}$$

on Ω'. The data on the righthand side is $\bar{\partial}$-closed and has bounded coefficients. So there is a bounded solution u by our hypothesis.

Set

$$h(z) = \varphi(z) \cdot \frac{1}{L_{q'}(z)} - u. \tag{6.4}$$

Then h is holomorphic and bounded on Ω and does not analytically continue past q'. So we may complete the argument just as in the proofs of Proposition 3.3. $\square$
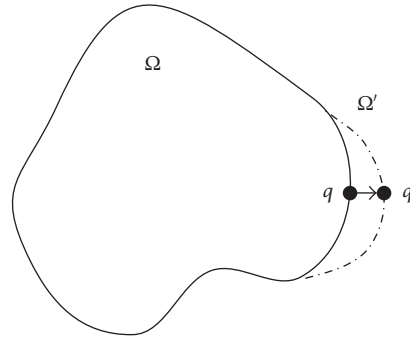
Figure 1: The domains Ω and Ω'.

Corollary of the proof

If Ω is a smoothly bounded domain on which uniform estimates for the $\bar{\partial}$-equation hold, and if Ω' is a domain obtained from Ω by perturbing the strongly pseudoconvex points (so that the perturbed points are also strongly pseudoconvex), then the $\bar{\partial}$-problem on Ω' also satisfies uniform estimates.

We conclude this section by noting that in fact the proof of Theorem 1.1 in [29] goes through verbatim if "strongly pseudoconvex" is replaced by "finite type" in $\mathbb{C}^2$. As a result, in view of the discussion above, we have the following proposition.

Proposition 6.3. *If Ω is a smoothly bounded domain in $\mathbb{C}^2$ on which uniform estimates for the $\bar{\partial}$-equation hold, and if Ω' is a domain obtained from Ω by perturbing the finite type points (so that the perturbed points are also finite type), then the $\bar{\partial}$-problem on Ω' also satisfies uniform estimates.*

7. Peak Points

We have seen peak points and peaking functions put to good use in the proof of Proposition 4.5. Now we will see them in a more general context.

Let Ω be a domain of type EL^∞. So Ω is pseudoconvex, and there is a strictly larger domain $\widehat{\Omega}$ so that every bounded holomorphic function on Ω analytically continues to a bounded holomorphic function $\widehat{f}$ on $\widehat{\Omega}$. Of course the operator $T : f \mapsto \widehat{f}$ is linear. It is one-to-one and onto. It follows from the closed graph theorem that T continuous. Now we have a lemma.

Lemma 7.1. *The operator T has norm 1.*

Proof. Of course the norm of T is at least 1. Suppose that it is actually greater than 1. Then there is an H^∞ function f on Ω so that f has norm 1, and its extension $\widehat{f}$ has norm greater than 1. For k being a positive integer consider $g_k = f^k$. Then the extension of g_k to $\widehat{\Omega}$ is $\widehat{g_k} = (\widehat{f})^k$. As $k \to +\infty$, the norm of $\widehat{g_k}$ tends to $+\infty$ while the norm of g_k remains 1. That is a contradiction. □

Proposition 7.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $q \in \partial\Omega$ be a peak point (see the proof of Proposition 4.5). Let f_q be the peaking function. Then there cannot be a domain $\widehat{\Omega}$ which properly contains Ω so that* (i) *any bounded holomorphic function on Ω analytically continues to $\widehat{\Omega}$ and* (ii)q *lies in the interior of $\widehat{\Omega}$.*

Proof. Suppose to the contrary that there is such a domain $\widehat{\Omega}$. Then the holomorphic function f_q analytically continues to a function $\widehat{f_q}$ on $\widehat{\Omega}$. Of course f_q has H^∞ norm 1. Thus the extended function $\widehat{f_q}$ will also have norm 1. But $\widehat{f_q}(q) = 1$. This contradicts the maximum modulus principle unless $f_q \equiv 1$. But that is impossible by the definition of peak function. □

Remark 7.3. In fact one does not need the full force of q being a peak point in order for this last result to hold. It is sufficient, for instance, for the nontangential limit of f at q to be 1, and the values of f at other points of Ω have modulus smaller than 1.

It may also be noted that, by a result of Basener [30], the set of peak points for a domain is contained in the closure of the strongly pseudoconvex points. This observation is helpful in applying the last proposition.

8. Concluding Remarks

It would have been best if we could have given a characterization of HL^p domains or EL^p domains. Unfortunately such a result is beyond our reach at this time.

We hope that the information gathered here will help to inform the situation and lead, in future work, to increased understanding of this fascinating problem. It is clear that there is a spectrum of domains of holomorphy, and it is in our best interest to understand the elements of this spectrum.

Acknowledgments

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