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Research Article

On a New Summation Formula for $_2\psi_2$ Basic Bilateral Hypergeometric Series and Its Applications

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We have obtained a new summation formula for $_2\psi_2$ bilateral basic hypergeometric series by the method of parameter augmentation and demonstrated its various uses leading to some development of etafunctions, q-gamma, and q-beta function identities.

1. Introduction

The summation formulae for hypergeometric series form a very interesting and useful component of the theory of (basic) hypergeometric series. The q-binomial theorem of Cauchy [1] is perhaps the first identity in the class of the summation formulae, which can be stated as

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} z^k = \frac{(az)_{\infty}}{(z)_{\infty}}, \quad |z| < 1, \ |q| < 1, \tag{1.1}$$

where

$$(a)_{\infty} := (a;q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^{k}),$$

$$(a)_{k} := (a;q)_{k} := \frac{(a)_{\infty}}{(aq^{k})_{\infty}}, \quad k \text{ is an integer.}$$

$$(1.2)$$

For more details about the *q*-binomial theorem and about the identities which fall in this sequel, one may refer to the book by Gasper and Rahman [2]. Another famous identity in the sequel is the Ramanujan's $_1\psi_1$ summation formula [3]

$$\sum_{k=-\infty}^{\infty} \frac{(a)_k}{(b)_k} z^k = \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty}}{(z)_{\infty} (b)_{\infty} (b/az)_{\infty} (q/a)_{\infty}}, \quad \left| \frac{b}{a} \right| < |z| < 1, \quad |q| < 1.$$
 (1.3)

There are a number of proofs of the $_1\psi_1$ summation formula (1.3) in the literature. For more details, one refers to the book by Berndt [4] and a recent paper of Johnson [5].

In this paper, we derive a new summation formula for $_2\psi_2$ basic bilateral hypergeometric series using the $_1\psi_1$ summation formula (1.3) by the method of parameter augmentation. We then use the formula to derive the q-analogue of Gauss summation formula and to obtain a number of etafunction, q-gamma, and q-beta function identities, which complement the works of Bhargava and Somashekara [6], Bhargava et al. [7], Somashekara and Mamta [8], Srivastava [9], and Bhargava and Adiga [10].

First, we recall that *q*-difference operator and the *q*-shift operator are defined by

$$D_q f(a) = \frac{f(a) - f(aq)}{a}, \quad \zeta(f(a)) = f(aq), \tag{1.4}$$

respectively. In [11], Chen and Liu have constructed an operator θ as

$$\theta = \zeta^{-1} D_q, \tag{1.5}$$

and thereby they defined the operator $E(b\theta)$ as

$$E(b\theta) = \sum_{k=0}^{\infty} \frac{(b\theta)^k q^{k(k-1)/2}}{(q;q)_k}.$$
 (1.6)

Then, we have the following operator identities [12, Theorem 1]:

$$E(b\theta)\{(at;q)_{\infty}\} = (at,bt;q)_{\infty},$$

$$E(b\theta)\{(as,at;q)_{\infty}\} = \frac{(as,at,bs,bt;q)_{\infty}}{(abst/q;q)_{\infty}}, \quad \left|\frac{abst}{q}\right| < 1.$$
(1.7)

Further, the Dedekind etafunction is defined by

$$\eta(\tau) := e^{\pi i \tau / 12} \prod_{k=1}^{\infty} \left(1 - e^{2\pi i k \tau} \right) := q^{1/24} (q; q)_{\infty}, \tag{1.8}$$

where $q = e^{2\pi i \tau}$, and Im $(\tau) > 0$.

Jackson [13] defined the q-analogue of the gamma function by

$$\Gamma_q(x) = \frac{(q;q)_{\infty}}{(q^x;q)_{\infty}} (1-q)^{1-x}, \quad 0 < q < 1.$$
(1.9)

In his paper on the q-gamma and q-beta function, Askey [14] has obtained q-analogues of several classical results about the gamma function. Further, he has given the definition for q-beta function as

$$B_q(x,y) = (1-q) \sum_{n=0}^{\infty} q^{nx} \frac{(q^{n+1})_{\infty}}{(q^{x+y})_{\infty}}.$$
 (1.10)

In fact, he has shown that

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. (1.11)$$

In Section 2, we prove our main result. In Section 3, we deduce the well-known *q*-analogue of the Gauss summation formula and some etafunction, *q*-gamma, and *q*-beta function identities.

2. Main Result

Theorem 2.1. *If* 0 < |z| < 1, |q| < 1, then

$$\sum_{k=-\infty}^{\infty} \frac{(a)_k (bc/azq)_k}{(b)_k (c)_k} z^k = \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty} (c/a)_{\infty} (bc/azq)_{\infty}}{(z)_{\infty} (b)_{\infty} (c)_{\infty} (b/az)_{\infty} (c/az)_{\infty} (q/a)_{\infty}}.$$
 (2.1)

Proof. Ramanujan's $_1\psi_1$ summation formula (1.3) can be written as

$$\sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} z^k + \sum_{k=1}^{\infty} \frac{(q/b)_k}{(q/a)_k} \left(\frac{b}{az}\right)^k = \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty}}{(z)_{\infty} (b)_{\infty} (b/az)_{\infty} (q/a)_{\infty}}.$$
 (2.2)

This is the same as

$$\sum_{k=0}^{\infty} (a)_k z^k \left\{ \left(bq^k \right)_{\infty} \left(\frac{b}{az} \right)_{\infty} \right\} + \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{\left(q/a \right)_k} \left(\frac{1}{az} \right)^k \left\{ \left(bq^{-k} \right)_{\infty} \left(\frac{b}{az} \right)_{\infty} \right\} \\
= \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty}}{(z)_{\infty} (q/a)_{\infty}} \left\{ \left(\frac{b}{a} \right)_{\infty} \right\}.$$
(2.3)

On applying $E(c\theta)$ to both sides with respect to b, we obtain

$$\sum_{k=0}^{\infty} (a)_{k} z^{k} \left\{ \frac{(bq^{k})_{\infty} (b/az)_{\infty} (cq^{k})_{\infty} (c/az)_{\infty}}{(bcq^{k}/azq)_{\infty}} \right\} \\
+ \sum_{k=1}^{\infty} \frac{(-1)^{k} q^{k(k+1)/2}}{(q/a)_{k}} \left(\frac{1}{az} \right)^{k} \left\{ \frac{(bq^{-k})_{\infty} (b/az)_{\infty} (cq^{-k})_{\infty} (c/az)_{\infty}}{(bcq^{-k}/azq)_{\infty}} \right\} \\
= \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty}}{(z)_{\infty} (q/a)_{\infty}} \left\{ \left(\frac{b}{a} \right)_{\infty} \left(\frac{c}{a} \right)_{\infty} \right\}. \tag{2.4}$$

Multiplying (2.4) throughout by $\{(bc/azq)_{\infty}/(b)_{\infty}(c)_{\infty}\}$, we obtain

$$\sum_{k=0}^{\infty} \frac{(a)_{k} (bc/azq)_{k}}{(b)_{k} (c)_{k}} z^{k} + \sum_{k=1}^{\infty} \frac{(a)_{-k} (bc/azq)_{-k}}{(b)_{-k} (c)_{-k}} z^{-k}$$

$$= \frac{(az)_{\infty} (q)_{\infty} (q/az)_{\infty} (b/a)_{\infty} (c/a)_{\infty} (bc/azq)_{\infty}}{(z)_{\infty} (b)_{\infty} (c)_{\infty} (b/az)_{\infty} (c/az)_{\infty} (q/a)_{\infty}},$$
(2.5)

which yields (2.1).

3. Some Applications of the Main Identity

The following identity is the well-known *q*-analogue of the Gauss summation formula.

Corollary 3.1 (see [15]). *If* |q| < 1, $|\gamma/\alpha\beta| < 1$, *then*

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(q)_k (\gamma)_k} \left(\frac{\gamma}{\alpha \beta}\right)^k = \frac{(\gamma/\alpha)_{\infty} (\gamma/\beta)_{\infty}}{(\gamma)_{\infty} (\gamma/\alpha \beta)_{\infty}}.$$
(3.1)

Proof. Putting $a = \alpha$, b = q, $c = \gamma$, and $z = \gamma/\alpha\beta$ in (2.1), we obtain (3.1).

Corollary 3.2. *If* |q| < 1, then

$$\sum_{k=-\infty}^{\infty} \frac{(q^2; q^4)_k}{(1 - q^{4k+1})(q^3; q^4)_k} q^k = q^{-1/8} \frac{\eta^7(2\tau)}{\eta^3(\tau)\eta^2(4\tau)},\tag{3.2}$$

$$\sum_{k=-\infty}^{\infty} \frac{\left(-q^2; q^2\right)_k}{\left(1+q^{2k+1}\right)^2 \left(-q; q^2\right)_k} q^k = 2q^{-3/4} \frac{\eta^6(4\tau)}{\eta^3(2\tau)},\tag{3.3}$$

$$\sum_{k=-\infty}^{\infty} \frac{\left(-q;q^2\right)_k \left(-q^2;q^2\right)_k}{\left(1-q^{2k+1}\right)^2 \left(q;q^2\right)_n^2} q^k = 2q^{-3/4} \frac{\eta^8(4\tau)}{\eta^7(2\tau)},\tag{3.4}$$

$$\sum_{k=-\infty}^{\infty} \frac{(1+q)(-q;q^2)_k (-q^2;q^2)_k}{(1-q^{2k+2})(q^2;q^2)_k^2} q^k = \frac{\eta(2\tau)}{\eta^2(\tau)},$$
(3.5)

$$\sum_{k=-\infty}^{\infty} \frac{\left(q^2; q^6\right)_k}{\left(1 - q^{6k+1}\right) \left(q^5; q^6\right)_k} q^{3k} = q^{-1/4} \frac{\eta^4(2\tau)}{\eta^2(\tau)},\tag{3.6}$$

$$\sum_{k=-\infty}^{\infty} \frac{\left(1+q^2\right)^2 \left(-q^2; q^2\right)_{k-1} \left(-q^2; q^2\right)_{k+1}}{\left(1-q^{2k+2}\right)^2 \left(q^2; q^2\right)_k^2} q^k = \frac{\eta^2 (4\tau)}{\eta^4 (2\tau)}.$$
(3.7)

Proof. Putting $a = z = q^{1/4}$, $b = q^{3/4}$, $c = q^{5/4}$, and then changing q to q^4 in (2.1), we obtain

$$\sum_{k=-\infty}^{\infty} \frac{(q; q^4)_k (q^2; q^4)_k}{(q; q^4)_{k+1} (q^3; q^4)_k} q^k = \frac{(q^2; q^4)_{\infty}^4 (q^4; q^4)_{\infty}^2}{(q; q^4)_{\infty}^3 (q^3; q^4)_{\infty}^3}.$$
 (3.8)

Simplifying the right hand side and then using (1.8), we obtain (3.2).

Similarly, putting $a = -q^{1/2}$, $z = q^{1/2}$, $b = -q^{3/2}$, $c = -q^{3/2}$, and then changing q to q^2 , we obtain (3.3). Putting $a = -q^{1/2}$, $z = q^{1/2}$, $b = c = q^{3/2}$, and then changing q to q^2 , we obtain (3.4). Putting $a = -q^{1/2}$, $z = q^{1/2}$, and then changing $z = q^{1/2}$, we obtain (3.6). Finally, putting $z = q^{1/2}$, $z = q^{1/2}$, and then changing $z = q^{1/2}$, we obtain (3.7).

Corollary 3.3. *If* 0 < q < 1, 0 < x, y < 1, and 0 < x + y < 1, then

$$B_q^3(x,y) = \frac{\Gamma_q(1-x+y)\Gamma_q(x-y)(1-q)^2}{(1-q^y)^2} \sum_{k=-\infty}^{\infty} \frac{(q^{1-x})_k (q^{x+y})_k}{(q^{1+y})_k^2} q^{ky}.$$
 (3.9)

Proof. Putting $a = q^{1-x}$, $z = q^y$, and $b = c = q^{1+y}$ in (2.1), we get

$$\sum_{k=-\infty}^{\infty} \frac{(q^{1-x})_k (q^{x+y})_k}{(q^{1+y})_k^2} q^{ky} = \frac{(q^{1-x+y})_{\infty} (q)_{\infty} (q^{x-y})_{\infty}}{(q^y)_{\infty} (q^{1+y})_{\infty} (q^{1+y})_{\infty}} \left(\frac{(q^{x+y})_{\infty}}{(q^x)_{\infty}}\right)^3.$$
(3.10)

On using (1.9), (1.10), and (1.11), we obtain (3.9).

Corollary 3.4. *If* 0 < q < 1, 0 < x, y < 1, and 1 < x + y < 2, then

$$B_q^2(x,y) = \frac{\Gamma_q(y-x)\Gamma_q(x-y+1)\Gamma_q(x+y-1)}{\Gamma_q(y)\Gamma_q(1+x)} \sum_{k=-\infty}^{\infty} \frac{(q^{-x})_k (q^{x+y-1})_k}{(q^y)_k^2} q^{ky}.$$
 (3.11)

Proof. Putting $a = q^{-x}$, $z = b = c = q^y$, in (2.1), and then using (1.9), (1.10), and (1.11), we obtain (3.11).

Corollary 3.5. *If* 0 < x, y < 1, and 0 < x + y < 1, then

$$B^{3}(x,y) = \frac{\Gamma(1-x+y)\Gamma(x-y)}{y^{2}} \left[\sum_{k=0}^{\infty} \frac{(1-x)_{k}(x+y)_{k}}{(1+y)_{k}^{2}} + \sum_{k=1}^{\infty} \frac{(-y)_{k}^{2}}{(x)_{k}(1-x-y)_{k}} \right].$$
(3.12)

Proof. Letting $q \rightarrow 1$ in (3.9), we obtain (3.12).

Corollary 3.6. *If* 0 < x, y < 1, and 1 < x + y < 2, then

$$B^{2}(x,y) = \frac{\Gamma(y-x)\Gamma(x-y+1) \Gamma(x+y-1)}{\Gamma(y)\Gamma(1+x)} \times \left[\sum_{k=0}^{\infty} \frac{(-x)_{k}(x+y-1)_{k}}{(y)_{k}^{2}} + \sum_{k=1}^{\infty} \frac{(1-y)_{k}^{2}}{(1+x)_{k}(2-x-y)_{k}} \right].$$
(3.13)

Proof. Letting $q \rightarrow 1$ in (3.11), we obtain (3.13).

Corollary 3.7. *If* 0 < q < 1, 0 < x, y < 1, and 0 < x + y < 1, then

$$B_q(x,y) = \Gamma_q(y)\Gamma_q(1-y)\sum_{k=0}^{\infty} \frac{(q^{1-x-y})_k (q^y)_k}{(q)_k^2} q^{kx}.$$
 (3.14)

Proof. Putting $a = q^{1-x-y}$, $z = q^x$, and b = c = q in (2.1), we obtain (3.14).

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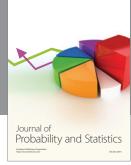
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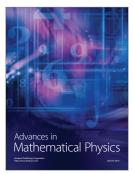




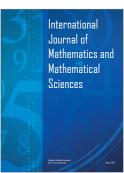


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