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Research Article

Characteristic Lightlike Submanifolds of an Indefinite $\mathcal{S}$-Manifold

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We study characteristic r-lightlike submanifolds M tangent to the characteristic vector fields in an indefinite metric $\mathcal{S}$-manifold, and we also discuss the existence of characteristic lightlike submanifolds of an indefinite $\mathcal{S}$-space form under suitable hypotheses: (1) M is totally umbilical or (2) its screen distribution $S(TM)$ is totally umbilical in M.

1. Introduction

In the theory of submanifolds of semi-Riemannian manifolds, it is interesting to study the geometry of lightlike submanifolds due to the fact that the intersection of normal vector bundle and the tangent bundle is nontrivial, making it interesting and remarkably different from the study of nondegenerate submanifolds. In particular, many authors study lightlike submanifolds on indefinite Sasakian manifolds (e.g., [1–4]).

Similar to Riemannian geometry, it is natural that indefinite $\mathcal{S}$-manifolds are generalizations of indefinite Sasakian manifolds. Brunetti and Pastore analyzed some properties of indefinite $\mathcal{S}$-manifolds and gave some characterizations in terms of the Levi-Civita connection and of the characteristic vector fields [5]. After then, they studied the geometry of lightlike hypersurfaces of indefinite $\mathcal{S}$-manifold [6]. As Jin's generalizations of lightlike submanifolds of the Sasakian manifolds with the general codimension [3, 4, 7], Lee and Jin recently extended lightlike hypersurfaces on indefinite $\mathcal{S}$-manifold to lightlike submanifolds with codimension 2 on an indefinite $\mathcal{S}$-manifold, called characteristic half lightlike submanifolds [8]. However, a general notion of characteristic lightlike submanifolds of an indefinite $\mathcal{S}$-manifold have not been introduced as yet.

The objective of this paper is to study characteristic r-lightlike submanifolds M of an indefinite $\mathcal{S}$-manifold $\overline{M}$ subject to the conditions: (1) M is totally umbilcial, or (2) $S(TM)$

is totally umbilcal in M. In Section 2, we begin with some fundamental formulae in the theory of r-lightlike submanifolds. In Section 3, for an indefinite metric $g.f.f$-manifold we consider a lightlike submanifold M tangent to the characteristic vector fields, we recall some basic information about indefinite $\mathcal{S}$-manifolds and deal with the existence of irrotational characteristic submanifolds of an indefinite $\mathcal{S}$-space form. Afterwards, we study characteristic r-lightlike submanifolds of $\overline{M}$ in Sections 4 and 5.

2. Preliminaries

Let (M, g) be an m-dimensional lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the radical distribution $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank r $(1 \leq r \leq \min\{m, n\})$. In general, there exist two complementary nondegenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $\mathrm{Rad}(TM)$ in TM and $TM^{\perp}$, respectively, called the *screen* and *coscreen distributions* on M, such that

$$TM = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S(TM), \qquad TM^{\perp} = \mathrm{Rad}(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right), \tag{2.1}$$

where the symbol $\oplus_{\mathrm{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. We use the same notation for any other vector bundle. We use the following range of indices:

$$i, j, k, \ldots \in \{1, \ldots, r\}, \qquad \alpha, \beta, \gamma, \ldots \in \{r+1, \ldots, n\}. \tag{2.2}$$

Let $\mathrm{tr}(TM)$ and $\mathrm{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$, respectively, and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $\Gamma(\mathrm{ltr}(TM)_{|\mathcal{U}})$ consisting of smooth sections of $S(TM^{\perp})^{\perp}_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\overline{g}(N_i, \xi_j) = \delta_{ij}, \qquad \overline{g}(N_i, N_j) = 0, \tag{2.3}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}(TM))$. Then we have

$$\begin{aligned} T\overline{M} &= TM \oplus \mathrm{tr}(TM) = \{\mathrm{Rad}(TM) \oplus \mathrm{tr}(TM)\} \oplus_{\mathrm{orth}} S(TM) \\ &= \{\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)\} \oplus_{\mathrm{orth}} S(TM) \oplus_{\mathrm{orth}} S\left(TM^{\perp}\right). \end{aligned} \tag{2.4}$$

We say that a lightlike submanifolds $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ are characterized as follows:

(1) *r-lightlike* if $1 \leq r < \min\{m, n\}$;

(2) *coisotropic* if $1 \leq r = n < m$;

(3) *isotropic* if $1 \leq r = m < n$;

(4) *totally lightlike* if $1 \leq r = m = n$.

The above three classes (2)–(4) are particular cases of the class (1) as follows: $S(TM^{\perp}) = \{0\}$, $S(TM) = \{0\}$, and $S(TM) = S(TM^{\perp}) = \{0\}$, respectively. The geometry of r-lightlike submanifolds is more general form than that of the other three type submanifolds. For this reason, in this paper we consider only r-lightlike submanifolds $M \equiv (M, g, S(TM), S(TM^{\perp}))$, with the following local quasiorthonormal field of frames on $\overline{M}$:

$$\{\xi_1, \dots, \xi_r,\ N_1,\ \dots,\ N_r,\ F_{r+1}, \dots, F_m,\ W_{r+1}, \dots, W_n\}, \tag{2.5}$$

where the sets $\{F_{r+1}, \dots, F_m\}$ and $\{W_{r+1}, \dots, W_n\}$ are orthonormal basis of $\Gamma(S(TM))$ and $\Gamma(S(TM^{\perp}))$, respectively.

Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to (2.1). For an r-lightlike submanifold, the local Gauss-Weingartan formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y) N_i + \sum_{\alpha=r+1}^{n} h_{\alpha}^{s}(X, Y) W_{\alpha}, \tag{2.6}$$

$$\overline{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{\alpha=r+1}^{n} \rho_{i\alpha}(X) W_{\alpha}, \tag{2.7}$$

$$\overline{\nabla}_X W_{\alpha} = -A_{W_{\alpha}} X + \sum_{i=1}^{r} \phi_{\alpha i}(X) N_i + \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X) W_{\beta}, \tag{2.8}$$

$$\nabla_X PY = \nabla_X^{*} PY + \sum_{i=1}^{r} h_i^{*}(X, PY) \xi_i, \tag{2.9}$$

$$\nabla_X \xi_i = -A_{\xi_i}^{*} X - \sum_{j=1}^{r} \tau_{ji}(X) \xi_j, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^{*} are induced linear connections on TM and $S(TM)$, respectively, the bilinear forms h_i^{ℓ} and h_{α}^{s} on M are called the *local lightlike* and *screen second fundamental forms* on TM, respectively, h_i^{*} are called the *local radical second fundamental forms* on $S(TM)$. A_{N_i}, $A_{\xi_i}^{*}$, and $A_{W_{\alpha}}$ are linear operators on $\Gamma(TM)$ and τ_{ij}, $\rho_{i\alpha}$, $\phi_{\alpha i}$, and $\sigma_{\alpha\beta}$ are 1-forms on TM. Since $\overline{\nabla}$ is torsion-free, ∇ is also torsion-free and both h_i^{ℓ} and h_{α}^{s} are symmetric. From the fact $h_i^{\ell}(X, Y) = \overline{g}(\overline{\nabla}_X Y, \xi_i)$, we know that h_i^{ℓ} are independent of the choice of a screen distribution. We say that

$$h(X, Y) = \sum_{i=1}^{r} h_i^{\ell}(X, Y) N_i + \sum_{\alpha=r+1}^{n} h_{\alpha}^{s}(X, Y) W_{\alpha} \tag{2.11}$$

is the *second fundamental tensor* of M.

The induced connection ∇ on TM is not metric and satisfies

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^{r} \left\{ h_i^{\ell}(X, Y) \eta_i(Z) + h_i^{\ell}(X, Z)\ \eta_i(Y) \right\}, \tag{2.12}$$

for all $X, Y \in \Gamma(TM)$, where η_is are the 1-forms such that

$$\eta_i(X) = \overline{g}(X, N_i), \quad \forall X \in \Gamma(TM). \tag{2.13}$$

But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g\left(A_{\xi_i}^* X, Y\right) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y), \tag{2.14}$$

$$h_i^\ell(X,PY) = g\left(A_{\xi_i}^* X, PY\right), \qquad \overline{g}\left(A_{\xi_i}^* X, N_j\right) = 0, \tag{2.15}$$

$$\epsilon_\alpha h_\alpha^s(X,Y) = g(A_{W_\alpha}X, Y) - \sum_{i=1}^{r} \phi_{\alpha i}(X)\eta_i(Y), \tag{2.16}$$

$$\epsilon_\alpha h_\alpha^s(X,PY) = g(A_{W_\alpha}X, PY), \qquad \overline{g}(A_{W_\alpha}X, N_i) = \epsilon_\alpha \rho_{i\alpha}(X), \tag{2.17}$$

$$h_i^*(X,PY) = g(A_{N_i}X, PY), \qquad \eta_j(A_{N_i}X) + \eta_i\left(A_{N_j}X\right) = 0, \tag{2.18}$$

where $X, Y \in \Gamma(TM)$ and ϵ_α is the sign of W_α but it is ± 1 related to the causal character of W_α. From (2.18), we know that each A_{N_i} is shape operator related to the local second fundamental form h_i^* on $S(TM)$. Replacing Y by ξ_j in (2.14), we have

$$h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0, \tag{2.19}$$

for all $X \in \Gamma(TM)$. It follows

$$h_i^\ell(X,\xi_i) = 0, \qquad h_i^\ell(\xi_j,\xi_k) = 0. \tag{2.20}$$

Also, replacing X by ξ_j in (2.14) and using (2.20), we have

$$h_i^\ell(X,\xi_j) = g\left(X, A_{\xi_i}^*\xi_j\right), \qquad A_{\xi_i}^*\xi_j + A_{\xi_j}^*\xi_i = 0, \qquad A_{\xi_i}^*\xi_i = 0. \tag{2.21}$$

For an r-lightlike submanifold, replace Y by ξ_i in (2.16), we have

$$h_\alpha^s(X,\xi_i) = -\epsilon_\alpha \phi_{\alpha i}(X), \quad \forall X \in \Gamma(TM). \tag{2.22}$$

Note 1. Using (2.14) and the fact that h_i^ℓ are symmetric, we have

$$g\left(A_{\xi_i}^* X, Y\right) - g\left(X, A_{\xi_i}^* Y\right) = \sum_{k=1}^{r}\left\{h_k^\ell(X,\xi_i)\eta_k(Y) - h_k^\ell(Y,\xi_i)\eta_k(X)\right\}. \tag{2.23}$$

From this, (2.20) and (2.21), we show that *$A_{\xi_i}^*$ are self-adjoint on $\Gamma(TM)$ with respect to g if and only if $h_i^\ell(X,\xi_j) = 0$ for all $X \in \Gamma(TM)$, i and j if and only if $A_{\xi_i}^*\xi_j = 0$ for all i, j*. We call self-adjoint $A^*_{\xi_i}$ the *lightlike shape operators* of M. It follows from the above equivalence and (2.10) that the radical distribution $\mathrm{Rad}(TM)$ of a lightlike submanifold M, with the lightlike shape operators $A^*_{\xi_i}$, is always an integrable distribution.

3. Characteristic Lightlike Submanifolds

A manifold $\overline{M}$ is called a *globally framed f-manifold (or $g.f.f$-manifold)* if it is endowed with a nonnull $(1,1)$-tensor field $\overline{\phi}$ of constant rank, such that $\ker\overline{\phi}$ is parallelizable, that is, there exist global vector fields $\overline{\xi_\alpha}$, $\alpha \in \{1,\dots,k\}$, with their dual 1-forms $\overline{\eta}^\alpha$, satisfying $\overline{\phi}^2 = -I + \overline{\eta}^\alpha \otimes \overline{\xi_\alpha}$ and $\overline{\eta}^\alpha(\overline{\xi_\beta}) = \delta^\alpha_\beta$.

The $g.f.f$-manifold $(\overline{M}^{2n+r}, \overline{\phi}, \overline{\xi_\alpha}, \overline{\eta}^\alpha)$, $\alpha \in \{1,\dots,k\}$, is said to be an indefinite metric $g.f.f$-manifold if $\overline{g}$ is a semi-Riemannian metric, with index ν, $0 < \nu < 2n+k$, satisfying the following compatibility condtion

$$\overline{g}\left(\overline{\phi}X, \overline{\phi}Y\right) = \overline{g}(X,Y) - \sum_{\alpha=1}^{r} \epsilon_\alpha \overline{\eta}^\alpha(X)\overline{\eta}^\alpha(Y), \tag{3.1}$$

for any $X, Y \in \Gamma(T\overline{M})$, being $\epsilon_\alpha = \pm 1$ according to whether $\overline{\xi}_\alpha$ is spacelike or timelike. Then, for any $\alpha \in \{1,\dots,k\}$, one has $\overline{\eta}^\alpha(X) = \epsilon_\alpha \overline{g}(X, \overline{\xi}_\alpha)$. An indefinite metric $g.f.f$-manifold is called an *indefinite $\mathcal{S}$-manifold* if it is normal and $d\overline{\eta}^\alpha = \Phi$, for any $\alpha \in \{1,\dots,k\}$, where $\Phi(X,Y) = \overline{g}(X, \overline{\phi}Y)$ for any $X, Y \in \Gamma(T\overline{M})$. The normality condition is expressed by the vanishing of the tensor field $N = N_{\overline{\phi}} + 2d\overline{\eta}^\alpha \otimes \overline{\xi}_\alpha$, $N_{\overline{\phi}}$ being the Nijenhuis torsion of $\overline{\phi}$. Furthermore, as proved in [5], the Levi-Civita connection of an indefinite $\mathcal{S}$-manifold satisfies:

$$\left(\overline{\nabla}_X \overline{\phi}\right)Y = \overline{g}\left(\overline{\phi}X, \overline{\phi}Y\right)\overline{\xi} + \overline{\eta}(Y)\overline{\phi}^2(X), \tag{3.2}$$

where $\overline{\xi} = \sum_{\alpha=1}^{k} \overline{\xi}_\alpha$ and $\overline{\eta} = \sum_{\alpha=1}^{k} \epsilon_\alpha \overline{\eta}^\alpha$. We recall that $\overline{\nabla}_X \overline{\xi}_\alpha = -\epsilon_\alpha \overline{\phi} X$ and $\ker\overline{\phi}$ is an integrable flat distribution since $\overline{\nabla}_{\overline{\xi}_\alpha} \overline{\xi}_\beta = 0$ (more details in [5]).

Following the notations in [9], we adopt the curvature tensor R, and thus we have $R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, and $R(X,Y,Z,W) = g(R(Z,W,Y),X)$, for any X, Y, $Z, W \in \Gamma(TM)$.

An indefinite $\mathcal{S}$-manifold $(\overline{M}, \overline{\phi}, \overline{\xi}_\alpha, \overline{\eta}^\alpha)$ is called an *indefinite $\mathcal{S}$-space form*, denoted by $\overline{M}(c)$, if it has the constant $\overline{\phi}$-sectional curvature c [5]. The curvature tensor $\overline{R}$ of this space form $\overline{M}(c)$ is given by

$$\begin{aligned}
4\overline{R}(X,Y,Z,W) = {}& -(c+3\epsilon)\left\{\overline{g}\left(\overline{\phi}Y, \overline{\phi}Z\right)\overline{g}\left(\overline{\phi}X, \overline{\phi}W\right) - \overline{g}\left(\overline{\phi}X, \overline{\phi}Z\right)\overline{g}\left(\overline{\phi}Y, \overline{\phi}W\right)\right\} \\
& - (c-\epsilon)\{\Phi(W,X)\Phi(Z,Y) - \Phi(Z,X)\Phi(W,Y) + 2\Phi(X,Y)\Phi(W,Z)\} \\
& - 4\left\{\overline{\eta}(W)\overline{\eta}(X)\overline{g}\left(\overline{\phi}Z, \overline{\phi}Y\right) - \overline{\eta}(W)\overline{\eta}(Y)\overline{g}\left(\overline{\phi}Z, \overline{\phi}X\right)\right. \\
& \left. + \overline{\eta}(Y)\overline{\eta}(Z)\overline{g}\left(\overline{\phi}W, \overline{\phi}X\right) - \overline{\eta}(Z)\overline{\eta}(X)\overline{g}\left(\overline{\phi}W, \overline{\phi}Y\right)\right\},
\end{aligned} \tag{3.3}$$

for any vector fields $X, Y, Z, W \in \Gamma(T\overline{M})$.

Note 2. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^* = TM/\mathrm{Rad}(TM)$ considered by Kupeli [10]. Thus all screen distributions $S(TM)$ are mutually isomorphic. For this reason, we newly define generic lightlike submanifolds of $\overline{M}$ as follows.

Definition 3.1. Let M be a r-lightlike submanifold of $\overline{M}$ such that all the characteristic vector fields $\overline{\xi}_\alpha$ are tangent to M. A screen distribution $S(TM)$ is said to be *characteristic* if $\ker\overline{\phi} \subset S(TM)$ and $\overline{\phi}(S(TM)^{\perp}) \subset \Gamma(S(TM))$.

Definition 3.2. A r-lightlike submanifold M of $\overline{M}$ is said to be *characteristic* if $\ker\overline{\phi} \subset TM$ and a characteristic screen distribution $(S(TM))$ is chosen.

Proposition 3.3 (see [6]). *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite $\mathcal{S}$-manifold $(\overline{M}, \overline{\phi}, \overline{\xi}_\alpha, \overline{\eta}^\alpha, \overline{g})$ such that the characteristic vector fields are tangent to M. Then there exists a screen distribution such that $\ker\overline{\phi} \subset TM$ and $\overline{\phi}(E) \subset \Gamma(S(TM))$, where E is a nonzero section of $\mathrm{Rad}(TM)$.*

Proposition 3.4 (see [8]). *Let $(M, g, S(TM))$ be a 1-lightlike submanifold of codimension 2 of an indefinite $\mathcal{S}$-manifold $(\overline{M}, \overline{\phi}, \overline{\xi}_\alpha, \overline{\eta}^\alpha, \overline{g})$. Then M is a characteristic lightlike submanifold of $\overline{M}$.*

Definition 3.5. A lightlike submanifold M is said to be *irrotational* [10] if $\overline{\nabla}_X\xi_i \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi_i \in \Gamma(\mathrm{Rad}(TM))$ for all i.

Note 3. For an r-lightlike M, the above definition is equivalent to

$$h_j^\ell(X, \xi_i) = 0, \quad h_\alpha^s(X, \xi_i) = \phi_{\alpha i}(X) = 0, \quad \forall X \in \Gamma(TM). \tag{3.4}$$

The extrinsic geometry of lightlike hypersurfaces depends on a choice of screen distribution, or equivalently, normalization. Since the screen distribution is not uniquely determined, a well-defined concept of $\mathcal{S}$-manifold is not possible for an arbitrary lightlike submanifold of a semi-Riemannian manifold, then one must look for a class of normalization for which the induced Riemannian curvature has the desired symmetries. Let (M, g) be a semi-Riemannian manifold, $p \in M$. $F \otimes^4 T_p^* M$ is said to be an *algebraic curvature tensor* [11] on T_pM if it satisfies the following symmetries:

$$\begin{gathered} F(X, Y, Z, W) = -F(Y, X, Z, W) = F(Z, W, X, Y), \\ F(X, Y, Z, W) + F(Y, Z, X, W) + F(Z, X, Y, W) = 0. \end{gathered} \tag{3.5}$$

Definition 3.6. A screen distribution $S(TM)$ is said to be *admissible* if the associated induced Riemannian curvature is an algebraic curvature tensor.

Theorem 3.7. *Let $(M, g, S(TM))$ be an irrotational generic characteristic lightlike submanifold of an indefinite $\mathcal{S}$-space form $(\overline{M}(c), \overline{\phi}, \overline{\xi}_\alpha, \overline{\eta}^\alpha, \overline{g})$ with an admissible screen distribution $S(TM)$. Then one has $c = \epsilon$.*

Proof. Denote by $\overline{R}$ and R the curvature tensors of $\overline{\nabla}$ and ∇, respectively. Using the local Gauss-Weingarten formulas for M, we obtain

$$\begin{aligned}
\overline{R}(X,Y)Z = R(X,Y)Z &+ \sum_{i=1}^{r}\left\{h_i^{\ell}(X,Z)A_{N_i}Y - h_i^{\ell}(Y,Z)A_{N_i}X\right\} \\
&+ \sum_{\alpha=r_1}^{n}\{h_\alpha^s(X,Z)A_{W_\alpha}Y - h_\alpha^s(Y,Z)A_{W_\alpha}X\} \\
&+ \sum_{i=1}^{r}\Bigg\{\left(\nabla_X h_i^{\ell}\right)(Y,Z) - \left(\nabla_Y h_i^{\ell}\right)(X,Z) \\
&\qquad + \sum_{j=1}^{r}\left[\tau_{ji}(X)h_j^{\ell}(Y,Z) - \tau_{ji}(Y)h_j^{\ell}(X,Z)\right] \\
&\qquad + \sum_{\alpha=r+1}^{n}\left[\phi_{\alpha i}(X)h_\alpha^s(Y,Z) - \phi_{\alpha i}(Y)h_\alpha^s(X,Z)\right]\Bigg\}N_i \\
&+ \sum_{\alpha=r+1}^{n}\Bigg\{(\nabla_X h_\alpha^s)(Y,Z) - (\nabla_Y h_\alpha^s)(X,Z) \\
&\qquad + \sum_{i=1}^{r}\left[\rho_{i\alpha}(X)h_i^{\ell}(Y,Z) - \rho_{i\alpha}(Y)h_\alpha^s(X,Z)\right] \\
&\qquad + \sum_{\beta=r+1}^{n}\left[\sigma_{\beta\alpha}(X)h_\beta^s(Y,Z) - \sigma_{\beta\alpha}(Y)h_\beta^s(X,Z)\right]\Bigg\}W_\alpha,
\end{aligned} \tag{3.6}$$

for all $X, Y, Z \in \Gamma(TM)$. Replace Z by ξ_k in (3.6) and use (2.10), (2.15), (2.17), and (3.4), we have

$$\begin{aligned}
\overline{R}(X,Y)\xi_k = R(X,Y)\xi_k &+ \sum_{i=1}^{r}\left\{g\left(A_{\xi_i}^*Y, A_{\xi_k}^*X\right) - g\left(A_{\xi_i}^*X, A_{\xi_k}^*Y\right)\right\}N_i \\
&+ \sum_{\alpha=r+1}^{n}\epsilon_\alpha\left\{g\left(A_{W_\alpha}Y, A_{\xi_k}^*X\right) - g\left(A_{W_\alpha}X, A_{\xi_k}^*Y\right)\right\}W_\alpha.
\end{aligned} \tag{3.7}$$

Using (3.7), the fact $R(X,Y)Z \in \Gamma(TM)$ for $X, Y, Z \in \Gamma(TM)$, and a screen distribution $S(TM)$ is admissible, we get

$$\begin{aligned}
\overline{g}\left(\overline{R}(X,Y)Z, \xi_k\right) &= -\overline{g}\left(\overline{R}(X,Y)\xi_k, Z\right) \\
&= -g(R(X,Y)\xi_k, Z) + \sum_{i=1}^{r}\left\{g\left(A_{\xi_i}^*X, A_{\xi_k}^*Y\right) - g\left(A_{\xi_i}^*Y, A_{\xi_k}^*X\right)\right\}\eta_i(Z)
\end{aligned}$$

$$= g(R(X,Y)Z,\xi_k) + \sum_{i=1}^{r}\left\{g\left(A^*_{\xi_i}X, A^*_{\xi_k}Y\right) - g\left(A^*_{\xi_i}Y, A^*_{\xi_k}X\right)\right\}\eta_i(Z)$$

$$= \sum_{i=1}^{r}\left\{g\left(A^*_{\xi_i}X, A^*_{\xi_k}Y\right) - g\left(A^*_{\xi_i}Y, A^*_{\xi_k}X\right)\right\}\eta_i(Z), \quad \forall X,Y,Z \in \Gamma(TM). \tag{3.8}$$

On the other hand, since $\overline{\eta}(\xi_\alpha) = 0$ and $\overline{g}(\overline{\phi}\xi_\alpha, \overline{\phi}X) = 0$ for any $X \in \Gamma(TM)$, $\overline{M}(c)$ is an indefinite $\mathcal{S}$-space form implies the Riemannian curvature $\overline{R}$ in (3.3) is given by

$$\begin{aligned}4\overline{R}(X,Y,Z,\xi_\alpha) &= (c-\epsilon)\{\Phi(\xi_\alpha,X)\Phi(Z,Y) - \Phi(Z,X)\Phi(\xi_\alpha,Y) + 2\Phi(X,Y)\Phi(\xi_\alpha,Z)\} \\ &= -(c-\epsilon)\left\{\overline{g}\left(\overline{\phi}\xi_\alpha, X\right)\Phi(Z,Y) - \Phi(Z,X)\overline{g}\left(\overline{\phi}\xi_\alpha, Y\right) + 2\Phi(X,Y)\overline{g}\left(\overline{\phi}\xi_\alpha, Z\right)\right\},\end{aligned} \tag{3.9}$$

for any $X, Y, Z, \in \Gamma(TM)$. So, replacing X, Y, Z by PX, ξ, PZ in (3.9), we find

$$\begin{aligned}4\overline{R}(X,Y,Z,\xi_\alpha) &= -(c-\epsilon)\left\{-\overline{g}\left(\overline{\phi}\xi_\alpha, PX\right)\overline{g}\left(PZ, \overline{\phi}\xi_\alpha\right) - 2\overline{g}\left(X, \overline{\phi}\xi_\alpha\right)\overline{g}\left(\overline{\phi}\xi_\alpha, Z\right)\right\} \\ &= 3(c-\epsilon)\overline{g}\left(\overline{\phi}\xi_\alpha, PX\right)\overline{g}\left(\overline{\phi}\xi_\alpha, PZ\right).\end{aligned} \tag{3.10}$$

Then, using (3.3), (3.8), and (3.9), we get

$$\begin{aligned}&4\sum_{i=1}^{r}\left\{g\left(A^*_{\xi_i}X, A^*_{\xi_k}Y\right) - g\left(A^*_{\xi_i}Y, A^*_{\xi_k}X\right)\right\}\eta_i(Z) \\ &\quad = -3(c-\epsilon)\overline{g}\left(\overline{\phi}\xi_\alpha, PX\right)\overline{g}\left(\overline{\phi}\xi_\alpha, PZ\right), \quad \forall X,Y,Z \in \Gamma(TM).\end{aligned} \tag{3.11}$$

Choosing $X = Z = \overline{\phi}N_\alpha \in \Gamma(S(TM))$, we obtain $c = \epsilon$. □

Corollary 3.8. *There exist no irrotational characteristic r-lightlike submanifolds $(M, g, S(TM))$ of an indefinite $\mathcal{S}$-space form $(\overline{M}(c), \overline{\phi}, \overline{\xi}_\alpha, \overline{\eta}^\alpha, \overline{g})$ with $c \neq \epsilon$ such that the screen distribution $S(TM)$ is admissible.*

4. Totally Umbilical Characteristic Lightlike Submanifolds

Definition 4.1. An r-lightlike submanifold M of $\overline{M}$ is said to be *totally umbilical* [1] if there is a smooth vector field $\mathscr{H} \in \Gamma(\mathrm{tr}(TM))$ such that

$$h(X,Y) = \mathscr{H}g(X,PY), \tag{4.1}$$

for all $X, Y \in \Gamma(TM)$. In case $\mathscr{H} = 0$, we say that M is *totally geodesic*.

It is easy to see that M is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions A_i and B_α such that

$$h_i^\ell(X,Y) = A_i g(X,Y), \qquad h_\alpha^s(X,Y) = B_\alpha g(X,Y), \tag{4.2}$$

for any $X, Y \in \Gamma(TM)$. From (4.2) we show that any totally umbilical r-lightlike submanifold of $\overline{M}$ is irrotational. Thus, by Theorem 3.7, we have the following.

Theorem 4.2. *Let $(M, g, S(TM))$ be a totally umbilical characteristic r-lightlike submanifold of an indefinite $\mathcal{S}$-space form $(\overline{M}(c), \overline{\phi}, \overline{\xi_\alpha}, \overline{\eta}^\alpha, \overline{g})$. Then one has $c = \epsilon$.*

Theorem 4.3. *Let $(M, g, S(TM))$ be a totally umbilical characteristic r-lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\overline{M}, \overline{\phi}, \overline{\xi_\alpha}, \overline{\eta}^\alpha, \overline{g})$. Then M is totally geodesic.*

Proof. Apply $\overline{\nabla}_X$ to $\overline{g}(\overline{\phi}\xi_i, W_\alpha) = 0$ with $X \in \Gamma(TM)$, for all i and α, and use (2.8), (2.10), (2.15), (2.17), (2.22), and (3.2), we have

$$h_i^\ell\left(X, \overline{\phi}W_\alpha\right) = \epsilon_\alpha h_\alpha^s\left(X, \overline{\phi}\xi_i\right), \quad \forall X \in \Gamma(TM). \tag{4.3}$$

Assume that M is totally umbilical. Then we have

$$A_i g\left(X, \overline{\phi}W_\alpha\right) = \epsilon_\alpha B_\alpha g\left(X, \overline{\phi}\xi_i\right), \quad \forall X \in \Gamma(TM). \tag{4.4}$$

Replace X by $\overline{\phi}N_i$ and X by $\overline{\phi}W_\alpha$ by turns, we get $A_i = 0$ for all i and $B_\alpha = 0$ for all α. Thus we show that $\mathcal{H} = \sum_{i=1}^r A_i N_i + \sum_{\alpha=r+1}^n B_\alpha W_\alpha = 0$ and M is totally geodesic. □

Corollary 4.4 (see [1]). *Let $(M, g, S(TM))$ be a totally umbilical characteristic r-lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\overline{M}, \overline{\phi}, \overline{\xi_\alpha}, \overline{\eta}^\alpha, \overline{g})$. Then there exists a unique torsion-free metric connection ∇ on M induced by the connection $\overline{\nabla}$ on $\overline{M}$.*

Proof. From (4.2) and Theorem 4.3, we have $h_i^\ell(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$ and i. Thus, using (2.12), we obtain our assertion. □

5. Totally Umbilical Screen Distributions

Definition 5.1. A screen distribution $S(TM)$ of M is said to be *totally umbilical* [1] in M if, for each locally second fundamental form h_i^*, there exist smooth functions C_i on any coordinate neighborhood $\mathcal{U}$ in M such that

$$h_i^*(X, PY) = C_i g(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{5.1}$$

In case $C_i = 0$ for all i, we say that $S(TM)$ is *totally geodesic* in M.

Due to (2.18) and (5.1), we know that $S(TM)$ is totally umbilical in M if and only if each shape operators A_{N_i} of $S(TM)$ satisfies

$$g(A_{N_i}X, PY) = C_i g(X, PY), \quad \forall X, Y \in \Gamma(TM), \tag{5.2}$$

for some smooth functions C_i on $\mathcal{U} \subseteq M$.

In general, $S(TM)$ is not necessarily integrable. The following result gives equivalent conditions for the integrability of a screen $S(TM)$.

Theorem 5.2 (see [1]). *Let M be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the following assertions are equivalent:*

(1) *$S(TM)$ is integrable,*

(2) *h_i^* is symmetric on $\Gamma(S(TM))$, for each i,*

(3) *A_{N_i} is self-adjoint on $\Gamma(S(TM))$ with respect to g, for each i.*

We know that, from (5.2), each shape operator A_{N_i} is self-adjoint on $\Gamma(S(TM))$ with respect to g, which further follows from that above theorem that any totally umbilical screen distribution $S(TM)$ of M is integrable.

Theorem 5.3. *Let $(M, g, S(TM))$ be a characteristic r-lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\overline{M}, \overline{\phi}, \overline{\xi_\alpha}, \overline{\eta}^\alpha, \overline{g})$. If $S(TM)$ is totally umbilical in M, then $S(TM)$ is totally geodesic in M.*

Proof. Apply the operator $\overline{\nabla}_X$ to $\overline{g}(\overline{\phi}N_k, N_j) = 0$ for some k, j such that $k \neq j$, and use (2.7) and (2.18)

$$h_k^*\left(X, \overline{\phi}N_j\right) = h_j^*\left(X, \overline{\phi}N_k\right), \quad \forall X \in \Gamma(TM). \tag{5.3}$$

Assume that $S(TM)$ is totally umbilical in M. Then we have

$$C_k g\left(X, \overline{\phi}N_j\right) = C_j g\left(X, \overline{\phi}N_k\right), \quad \forall X \in \Gamma(TM). \tag{5.4}$$

Replacing X by $\overline{\phi}\xi_j$ in (5.4) and taking $(k, j) = (1,2), (2,3), \ldots, (r-1, r)$ and $(r, 1)$ by turns and use the above method, we have $C_i = 0$ for all $i \in \{1, \ldots, r\}$. Thus we have our assertion. □

Theorem 5.4. *Let $(M, g, S(TM))$ be a characteristic r-lightlike submanifold of an indefinite $\mathcal{S}$-manifold $(\overline{M}, \overline{\phi}, \overline{\xi_\alpha}, \overline{\eta}^\alpha, \overline{g})$ such that $S(TM)$ is totally umbilical in M. Then M is not irrotational.*

Proof. Apply the operator $\overline{\nabla}_X$ to $\overline{g}(\overline{\phi}\xi_i, N_j) = 0$ for all i and j, and use (2.6), (2.10), (2.15), (2.18), and Theorem 5.3, we have

$$h_i^\ell\left(X, \overline{\phi}N_j\right) = h_j^*\left(X, \overline{\phi}\xi_i\right), \quad \forall X \in \Gamma(TM). \tag{5.5}$$

Since $S(TM)$ is totally umbilical, by Theorem 5.3, we have that $S(TM)$ is totally geodesic. Then, by (5.5), we have

$$h_i^\ell\left(X, \overline{\phi}N_j\right) = 0, \quad \forall X \in \Gamma(TM). \tag{5.6}$$

Apply the operator $\overline{\nabla}_X$ to $\overline{g}(\overline{\xi_\alpha}, \xi_i) = 0$ and use (2.6), (2.10), (2.15), we have

$$h_i^\ell\left(X, \overline{\xi_\alpha}\right) = -\epsilon_\alpha g\left(X, \overline{\phi}\xi_i\right), \quad \forall X \in \Gamma(TM). \tag{5.7}$$

Replace X by $\overline{\phi}N_i$ in this equation and (5.6), we have

$$0 = h_i^\ell\left(\overline{\phi}N_i, \overline{\xi_\alpha}\right) = -g\left(\overline{\phi}N_i, \overline{\phi}\xi_i\right) = -1. \tag{5.8}$$

It is a contradiction. Thus M is not irrotational. □

Since any totally umbilical r-lightlike submanifold of $\overline{M}$ is irrotational, by Theorem 5.4,
we have the following result.

Corollary 5.5. *There exist no totally umbilical characteristic r-lightlike submanifolds $(M, g, S(TM))$ of an indefinite $\mathcal{S}$-manifold $(\overline{M}, \overline{\phi}, \overline{\xi_\alpha}, \overline{\eta}^\alpha, \overline{g})$ equipped with a totally umbilical screen distribution $S(TM)$ in M.*

References

[1] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic, Dodrecht, The Netherlands, 1996.

[2] K. L. Duggal and B. Sahin, "Lightlike submanifolds of indefinite Sasakian manifolds," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 57585, 21 pages, 2007.

[3] D. H. Jin, "Geometry of lightlike hypersurfaces of an indefinite Sasakian manifold," *Indian Journal of Pure and Applied Mathematics*, vol. 41, no. 4, pp. 569–581, 2010.

[4] D. H. Jin, "Geometry of half lightlike submanifolds of an indefinite Sasakian manifold," submitted to *Bulletin of the Korean Mathematical Society*.

[5] L. Brunetti and A. M. Pastore, "Curvature of a class of indefinite globally framed f-manifolds," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 51(99), no. 3, pp. 183–204, 2008.

[6] L. Brunetti and A. M. Pastore, "Lightlike hypersurfaces in indefinite $\mathcal{S}$-manifolds," *Differential Geometry*, vol. 12, pp. 18–40, 2010.

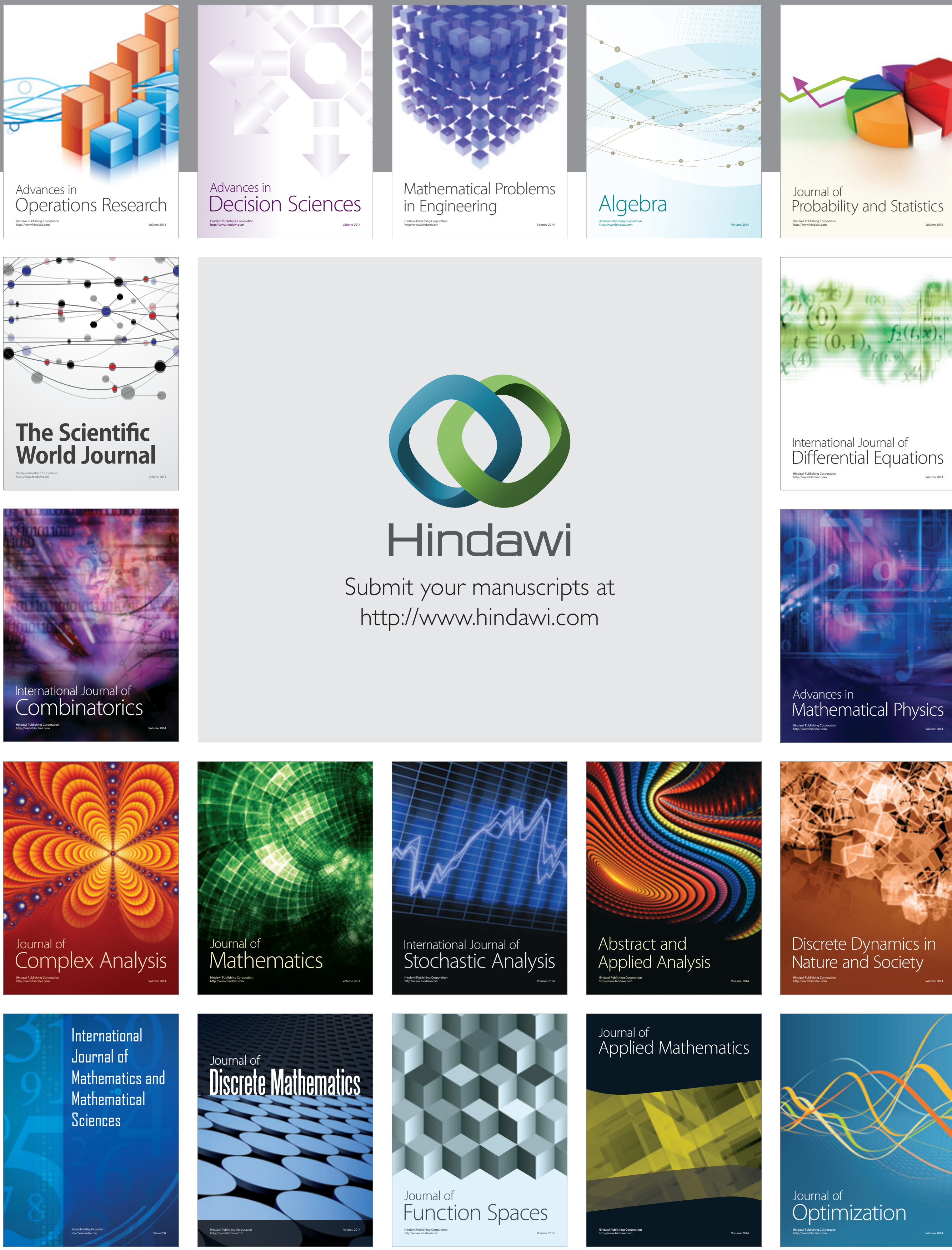
[7] D. H. Jin, "Generic lightlike submanifolds of an indefinite Sasakian manifold".

[8] J. W. Lee and D. H. Jin, "Half lightlike submanifolds in indefinite $\mathcal{S}$-manifolds," to appear in *Taiwanese Journal of Mathematics*.

[9] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 1-2, Interscience Publishers, New York, NY, USA, 1963.

[10] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Kluwer Academic, Dodrecht, The Netherlands, 1996.

[11] C. Atindogbe and K. L. Duggal, "Pseudo-Jacobi operators and Osserman lightlike hypersurfaces," *Kodai Mathematical Journal*, vol. 32, no. 1, pp. 91–108, 2009.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