Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 154636, 10 pages
doi:10.1155/2011/154636

Review Article

Some Notes on Semiabelian Rings

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Received 12 January 2011; Accepted 6 June 2011

Academic Editor: Frank Sommen

It is proved that if a ring R is semiabelian, then so is the skew polynomial ring $R[x;\sigma]$, where σ is an endomorphism of R satisfying $\sigma(e) = e$ for all $e \in E(R)$. Some characterizations and properties of semiabelian rings are studied.

1. Introduction

Throughout the paper, all rings are associative with identities. We always use $N(R)$ and $E(R)$ to denote the set of all nilpotent elements and the set of all idempotent elements of R.

According to [1], a ring R is called *semiabelian* if every idempotent of R is either right semicentral or left semicentral. Clearly, a ring R is semiabelian if and only if either $eR(1-e) = 0$ or $(1-e)Re = 0$ for every $e \in E(R)$, so, abelian rings (i.e., every idempotent of R is central) are semiabelian. But the converse is not true by [1, Example 2.2].

A ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for any a, $b \in R$. It is well known that abelian rings are directly finite. In Theorem 2.7, we show that semiabelian rings are directly finite.

An element e of a ring R is called a left minimal idempotent if $e \in E(R)$ and Re is a minimal left ideal of R. A ring R is called *left min-abel* [2] if every left minimal idempotent element of R is left semicentral. Clearly, abelian rings are left min-abel. In Theorem 2.7, we show that semiabelian rings are left min-abel.

A ring R is called *left idempotent reflexive* if for any $e \in E(R)$ and $a \in R$, $aRe = 0$ implies $eRa = 0$. Theorem 2.5 shows that R is abelian if and only if R is semiabelian and left idempotent reflexive.

In [3], Wang called an element e of a ring R an *op-idempotent* if $e^2 = -e$. Clearly, op-idempotent need not be idempotent. For example, let $R = Z/3Z$. Then $\overline{2} \in R$ is an op-idempotent, while it is not an idempotent. In [4], Chen called an element $e \in R$ *potent* in case there exists some integer $n \geq 2$ such that $e^n = e$. We write $p(e)$ for the smallest number

n of such. Clearly, idempotent is potent, while there exists a potent element which is not idempotent. For example, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in M_2(Z)$ is a potent element, while it is not idempotent. We use $E^o(R)$ and $PE(R)$ to denote the set of all op-idempotent elements and the set of all potent elements of R. In Corollaries 2.2 and 2.3, we observe that every semiabelian ring can be characterized by its op-idempotent and potent elements.

If R is a ring and $\sigma : R \to R$ is a ring endomorphism, let $R[x;\sigma]$ denote the ring of skew polynomials over R; that is all formal polynomials in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$. In [1], Chen showed that R is a semiabelian ring if and only if $R[x]$ is a semiabelian ring. In Theorem 2.13, we show that if R is a semiabelian ring with an endomorphism σ satisfying $\sigma(e) = e$ for all $e \in E(R)$, then $R[x;\sigma]$ is semiabelian.

2. Main Results

It is well known that an idempotent e of a ring R is left semicentral if and only if $1-e$ is right semicentral. Hence we have the following theorem.

Theorem 2.1. *The following conditions are equivalent for a ring R.*

(1) *R is a semiabelian ring.*

(2) *For any $e \in E(R)$, $eR(1-e) \cup (1-e)Re$ is an ideal of R.*

(3) *For any $e \in E(R)$, $eR(1-e) \cup (1-e)Re = eR(1-e) + (1-e)Re$.*

Proof. (1)$\Rightarrow$(2) assume that $e \in E(R)$. Since R is semiabelian, e is either left semicentral or right semicentral. If e is right semicentral, then $eR(1-e) = 0$ and $1-e$ is left semicentral. Thus $R(1-e)ReR = (1-e)R(1-e)ReRe = (1-e)Re$ and $eR(1-e) \cup (1-\mathrm{v}e)Re = (1-e)Re = R(1-e)ReR$ is an ideal of R. Similarly, if e is left semicentral, then $eR(1-e) \cup (1-e)Re = ReR(1-e)R$ is also an ideal of R.

(2)$\Rightarrow$(3) is clear.

(3)$\Rightarrow$(1) assume that $e \in E(R)$. If e is neither left semicentral nor right semicentral, there exist $a, b \in R$ such that $(1-e)ae \neq 0$ and $eb(1-e) \neq 0$. By (3), $(1-e)ae + eb(1-e) \in (1-e)Re \cup eR(1-e)$. If $(1-e)ae + eb(1-e) \in (1-e)Re$, then $eb(1-e) = e((1-e)ae + eb(1-e))(1-e) \in e((1-e)Re)(1-e) = 0$, a contradiction; if $(1-e)ae + eb(1-e) \in eR(1-e)$, then $(1-e)ae = 0$, it is also a contradiction. Hence e is either left semicentral or right semicentral. □

Evidently, R is semiabelian if and only if either $eR(1-e) = 0$ or $(1-e)Re = 0$ for every $e \in E(R)$. On the other hand, an element e of R is op-idempotent if and only if $-e$ is idempotent. Hence, by Theorem 2.1, we have the following corollary.

Corollary 2.2. *The following conditions are equivalent for a ring R.*

(1) *R is a semiabelian ring.*

(2) *For any $e \in E^o(R)$, $eR(1+e) = 0$ or $(1+e)Re = 0$.*

(3) *For any $e \in E^o(R)$, $eR(1+e) \cup (1+e)Re$ is an ideal of R.*

(4) *For any $e \in E^o(R)$, $eR(1+e) \cup (1+e)Re = eR(1+e) + (1+e)Re$.*

Clearly, for any $e \in PE(R)$, $e^{p(e)-1} \in E(R)$, $Re = Re^{p(e)-1}$, and $eR = e^{p(e)-1}R$. Hence, by Theorem 2.1, we have the following corollary.

Corollary 2.3. *The following conditions are equivalent for a ring R.*

(1) *R is a semiabelian ring.*

(2) *For any $e \in PE(R)$, $eR(1-e^{p(e)-1}) = 0$ or $(1-e^{p(e)-1})Re = 0$.*

(3) *For any $e \in PE(R)$, $eR(1-e^{p(e)-1}) \cup (1-e^{p(e)-1})Re$ is an ideal of R.*

(4) *For any $e \in PE(R)$, $eR(1-e^{p(e)-1}) \cup (1-e^{p(e)-1})Re = eR(1-e^{p(e)-1}) + (1-e^{p(e)-1})Re$.*

Using Theorem 2.1, Corollaries 2.2 and 2.3, we have the following corollary.

Corollary 2.4. *Let R be a semiabelian ring. If $e \in E(R)$, $g \in E^o(R)$ and $h \in PE(R)$, then:*

(1) *if $ReR = R$, then $e = 1$,*

(2) *if $RgR = R$, then $g = -1$,*

(3) *if $RhR = R$, then $h^{p(h)-1} = 1$.*

Call a ring R *idempotent reversible* if $gRe = 0$ implies $eRg = 0$ for $e,\ g \in E(R)$. Clearly, abelian rings are left idempotent reflexive, and left idempotent reflexive rings are idempotent reversible. But we do not know that whether idempotent reversible rings must be left idempotent reflexive. It is easy to see that a ring R is left idempotent reflexive if and only if for any $a \in N(R)$, $aRe = 0$ implies $eRa = 0$. (In fact, it is only to show the sufficiency: Let $a \in R$ and $e \in E(R)$ satisfy $aRe = 0$. If $eRa = 0$, then $eba \neq 0$ for some $b \in R$. Since $eba \in N(R)$ and $(eba)Re = 0$, by hypothesis, $eR(eba) = 0$, this implies $eba = ee(eba) = 0$, which is a contradiction. Hence $eRa = 0$, R is a left idempotent reflexive ring.)

Let D be a division ring. Then the 2-by-2 upper triangular matrix ring $\mathrm{UT}_2(D) = \left(\begin{smallmatrix} D & D \\ 0 & D \end{smallmatrix}\right)$ is not idempotent reversible. In fact, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in E(\mathrm{UT}_2(D))$ and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} D & D \\ 0 & D \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = 0$, but $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} D & D \\ 0 & D \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right) \neq 0$. On the other hand, by [1, Example 2.2], $\mathrm{UT}_2(D)$ is a semiabelian ring.

We have the following theorem.

Theorem 2.5. *The following conditions are equivalent for a ring R.*

(1) *R is an abelian ring.*

(2) *R is a semiabelian ring and idempotent reversible ring.*

(3) *R is a semiabelian ring and left idempotent reflexive ring.*

(4) *R is a semiabelian ring and for any $a \in J(R)$, $aRe = 0$ implies $eRa = 0$.*

Proof. (1)$\Rightarrow$(3)$\Rightarrow$(2)$\Rightarrow$(1) and (3)$\Rightarrow$(4) are trivial.

Now let $e \in E(R)$. If R is semiabelian, then e is either right semicentral or left semicentral. If e is right semicentral, then $(1-e)ReR(1-e) = 0$. Since $R(1-e)Re \subseteq N(R) \cap J(R)$, (4) implies $(1-e)R(1-e)Re = 0$. Hence $(1-e)Re = 0$. This shows that e is central; if e is left semicentral, then $1-e$ is right semicentral. Hence $1-e$ and so e is also central. Thus (4)$\Rightarrow$(1) holds. □

Since semiprime rings are left idempotent reflexive, we have the following corollary by Theorem 2.5.

Corollary 2.6. *Semiprime semiabelian rings are abelian.*

Theorem 2.7. *Let R be a semiabelian ring and $e \in E(R)$. Then,*

(1) $eR(1-e)Re = (1-e)ReR(1-e) = 0$,

(2) *If $a \in R$ and $ae = 0$, then $Rera \subseteq N(R)$ for all $r \in R$,*

(3) $eR(1-e) \subseteq J(R)$.

Proof. (1) Since e is right semicentral if and only if $eR(1-e) = 0$ and e is left semicentral if and only if $(1-e)Re = 0$, (1) is evident by hypothesis.

(2) Since $a = a(1-e)$, $(ReRa)^2 = ReRaReRa = ReRa(1-e)ReRa = 0$ by (1). Hence $ReRa \subseteq N(R)$, so for any $r \in R$, $Rera \subseteq N(R)$.

(3) Since $(1-e)e = 0$, by (2), $Rer(1-e) \subseteq N(R)$ for all $r \in R$. This implies $Rer(1-e) \subseteq J(R)$ for all $r \in R$. Hence $eR(1-e) \subseteq J(R)$. □

Theorem 2.8. *Let R be a semiabelian ring. Then,*

(1) *R is directly finite,*

(2) *R is left min-abel.*

Proof. (1) Assume that $ab = 1$. Let $e = ba$. Then $e \in E(R)$ and $eb = b$. By Theorem 2.7(3), $b(1-e) = eb(1-e) \in J(R)$. Hence $1-e = ab(1-e) \in J(R)$, which implies $1 = e = ba$.

(2) Let $0 \neq e \in E(R)$ and Re be a minimal left ideal of R. Then $(1-e)Re \neq 0$ and $R(1-e)Re = Re$. Since R is a semiabelian ring, by Theorem 2.7(3), $(1-e)Re \subseteq J(R)$. This implies $e \in J(R)$, that is, $e = 0$ which is a contradiction. Hence $(1-e)Re = 0$, so e is left semicentral. Hence R is a left min-abel ring. □

For a ring R, a proper left ideal P of R is prime if $aRb \subseteq P$ implies that $a \in P$ or $b \in P$. Let $\mathrm{Spec}_l(R)$ be the set of all prime left ideals of R. In [5], it has been shown that if R is not a left quasiduo ring, then $\mathrm{Spec}_l(R)$ is a space with the weakly Zariski topology but not with the Zariski topology.

Let R be a ring. Then the set $\mathrm{Max}_l(R)$ of all maximal left ideals of R is a compact T_1-space by [6, Lemma 2.1]. Recall that a topological space is said to be zero dimensional if it has a base consisting of clopen sets. Where a clopen set in a topological space is a set which is both open and closed.

Now, for a left ideal I of a ring R, let $\alpha(I) = \{P \in \mathrm{Spec}_l(R) \mid I \nsubseteqq P\}$ and $\beta(I) = \mathrm{Spec}_l(R) \setminus \alpha(I)$. If $I = Ra$ for some $a \in R$, then we write $\alpha_l(a)$ and $\beta_l(a)$ for $\alpha(Ra)$ and $\beta(Ra)$.

For any left ideal I of R, we let $U_l(I) = \mathrm{Max}_l(R) \cap \alpha(I)$, $V_l(I) = \mathrm{Max}_l(R) \cap \beta(I)$ and let $\xi = \{U_l(\sum_{1\leq i\leq n} \sum_{1\leq j_1<j_2<\cdots<j_i\leq n} (-1)^{i-1} e_{j_1} e_{j_1} \cdots e_{j_i}) \mid e_{j_i} \in E(R),\ i = 1, 2, \ldots, n,\ n \in Z^+\}$.

A ring R is called left topologically boolean, or a left *tb*-ring [7] for short, if for every pair of distinct maximal left ideals of R there is an idempotent in exactly one of them.

A ring R is called clean [8] if every element of R is the sum of a unit and an idempotent.

The following theorems generalize [6, Lemmas 2.2 and 2.3].

Lemma 2.9. *Let R be a semiabelian ring and $e_i,\ e,\ f \in E(R),\ i = 1, 2, \dots, n$. Then,*

(1) *if N is a maximal left ideal of R and $e \notin N$, then $1 - e \in N$,*

(2) $U_l(e) \cap U_l(f) = U_l(fe)$,

(3) $U_l(e) \cup U_l(f) = U_l(e + f - ef) = U_l(ef)$,

(4) $U_l(e) = V_l(1 - e)$,

(5) $\bigcap_{i=1}^{n} U_l(e_i) = U_l(e_1 e_2 \cdots e_n)$,

(6) $\bigcup_{i=1}^{n} U_l(e_i) = U_l(\sum_{1\leq i\leq n} \sum_{1\leq j_1<j_2<\cdots<j_i\leq n} (-1)^{i-1} e_{j_1} e_{j_1} \dots e_{j_i})$,

(7) $U_l(\sum_{1\leq i\leq n} \sum_{1\leq j_1<j_2<\cdots<j_i\leq n} (-1)^{i-1} e_{j_1} e_{j_1} \cdots e_{j_i}) = V_l(1 - \sum_{1\leq i\leq n} \sum_{1\leq j_1<j_2<\cdots<j_i\leq n} (-1)^{i-1} e_{j_1} e_{j_1} \dots e_{j_i})$.

In particular, every set in ξ is clopen.

Proof. (1) Since $e \notin N$, $Re + N = R$. Let $1 = be + n$ for some $b \in R$ and $n \in N$. Since $eR(1 - e)Re = 0$ by Theorem 2.7(1), $eR(1 - e) = eR(1 - e)n \subseteq N$. Since N is a prime left ideal and $e \notin N$, $1 - e \in N$.

(2) Let $P \in U_l(e) \cap U_l(f)$. Then $e \notin P$ and $f \notin P$. By (1), we have $1 - e, 1 - f \in P$. Hence $1 - e - f + ef = (1 - e)(1 - f) \in P$. Clearly, $ef \notin P$, so $P \in U_l(ef)$. This shows $U_l(e) \cap U_l(f) \subseteq U_l(ef)$. Conversely, if $Q \in U_l(ef)$, then $ef \notin Q$. Since Q is a left ideal, $f \notin Q$. Hence $1 - f \in Q$ by (1). If $e \in Q$, then $1 - e - f + ef = (1 - e)(1 - f) \in Q$ implies $ef \in Q$, which is a contradiction. Hence $e \notin Q$, so $Q \in U_l(e) \cap U_l(f)$. Therefore $U_l(ef) \subseteq U_l(e) \cap U_l(f)$. Thus $U_l(e) \cap U_l(f) = U_l(ef)$. Similarly, we have $U_l(e) \cap U_l(f) = U_l(fe)$.

(3) and (4) They are also straightforward to prove.

By induction on n, we can show (5), (6) and (7).

Thus every set in ξ is clopen. □

Theorem 2.10. *Let R be a semiabelian clean ring. Then R is a left tb-ring.*

Proof. Suppose that M and N are distinct maximal left ideals of R. Let $a \in M \setminus N$. Then $Ra + N = R$ and $1 - xa \in N$ for some $x \in R$. Clearly, $xa \in M \setminus N$. Since R is clean, there exist an idempotent $e \in E(R)$ and a unit u in R such that $xa = e + u$. If $e \in M$, then $u = xa - e \in M$ from which it follows that $R = M$, a contradiction. Thus $e \notin M$. If $e \notin N$, then $1 - e \in N$ by Lemma 2.9 (1) and hence $u = (1 - e) + (xa - 1) \in N$. It follows that $N = R$ which is also not possible. We thus have that e is an idempotent belonging to N only. □

Theorem 2.11. *Let R be a semiabelian ring. If R is a left tb-ring, then ξ forms a base for the weakly Zariski topology on $\mathrm{Max}_l(R)$. In particular, $\mathrm{Max}_l(R)$ is a compact, zero-dimensional Hausdorff space.*

Proof. Similar to the proof of [6, Proposition 2.5], we can complete the proof. □

A ring R is called von Neumann regular if $a \in aRa$ for all $a \in R$ and R is said to be unit-regular if for any $a \in R$, $a = aua$ for some $u \in U(R)$. A ring R is called strongly regular if $a \in a^2R$ for all $a \in R$. Clearly, strongly regular $\Rightarrow$ unit-regular $\Rightarrow$ von Neumann regular. Since von Neumann regular rings are semiprime, it follows that von Neumann regular rings are left idempotent reflexive. And it is well known that R is strongly regular if and only if R is von Neumann regular and *abelian*. In view of Theorem 2.5, we have the following corollary.

Corollary 2.12. *The following conditions are equivalent for a ring R.*

(1) *R is strongly regular.*

(2) *R is unit-regular and semiabelian.*

(3) *R is von Neumann regular and semiabelian.*

Following [9], a ring R is called left NPP if for any $a \in N(R)$, Ra is projective left R-module, and R is said to be n-regular if for any $a \in N(R)$, $a \in aRa$. A ring R is said to be reduced if $a^2 = 0$ implies $a = 0$ for each $a \in R$, or equivalently, $N(R) = 0$. Obviously, reduced rings are n-regular and *abelian*, and n-regular rings are left NPP and semiprime. Using Theorem 2.5, the following theorem gives some new characterization of reduced rings in terms of semiabelian rings.

Theorem 2.13. *The following conditions are equivalent for a ring R.*

(1) *R is reduced.*

(2) *R is n-regular and semiabelian.*

(3) *R is left NPP, semiprime, and semiabelian.*

Proof. (1)⇒(2)⇒(3) are trivial.

(3)⇒(1) let $a \in R$ such that $a^2 = 0$. Since R is left NPP, $l(a) = Re$, $e \in E(R)$. Hence $ea = 0$ and $a = ae$ because $a \in l(a)$. Since R is semiabelian and $aRa = (1-e)aeR(1-e)a \subseteq (1-e)ReR(1-e)a$, $aRa = 0$ by Theorem 2.7. Since R is semiprime, $a = 0$, which shows that R is reduced. □

If R is a ring and $\sigma : R \to R$ is a ring endomorphism, let $R[x;\sigma]$ denote the ring of skew polynomials over R; that is all formal polynomials in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$. Note that if $R(\sigma)$ is the (R,R)-bimodule defined by ${}_RR(\sigma)={}_RR$ and $m \circ r = m\sigma(r)$, for all $m \in R(\sigma)$ and $r \in R$, then $R[x;\sigma]/(x^2) \cong R \propto R(\sigma)$.

Theorem 2.14. *Let R be a semiabelian ring. If σ is a ring endomorphism of R satisfying $\sigma(e) = e$ for all $e \in E(R)$. Then $R[x;\sigma]$ is semiabelian.*

Proof. Let $f(x) = e_0 + e_1x + \cdots + e_nx^n \in E(R[x;\sigma])$. Then

$$\begin{aligned}
&e_0^2 = e_0,\\
&e_1 = e_0e_1 + e_1\sigma(e_0),\\
&e_2 = e_0e_2 + e_1\sigma(e_1) + e_2\sigma^2(e_0),\\
&\quad\vdots\\
&e_n = e_0e_n + e_1\sigma(e_{n-1}) + e_2\sigma^2(e_{n-2}) + \cdots + e_{n-1}\sigma^{n-1}(e_1) + e_n\sigma^n(e_0).
\end{aligned} \tag{2.1}$$

Since $e_0 \in E(R)$, $\sigma(e_0) = e_0$ by hypothesis. Hence we have the following equations:

$$\begin{aligned}
e_1 &= e_0e_1 + e_1e_0,\\
e_2 &= e_0e_2 + e_1\sigma(e_1) + e_2e_0,\\
&\vdots\\
e_n &= e_0e_n + e_1\sigma(e_{n-1}) + e_2\sigma^2(e_{n-2}) + \cdots + e_{n-1}\sigma^{n-1}(e_1) + e_ne_0.
\end{aligned} \tag{2.2}$$

If e_0 is right semicentral, then $e_0e_1 = e_0e_1 + e_0e_1e_0 = e_0e_1 + e_0e_1$, which implies $e_0e_1 = 0$. Hence $e_1 = e_1e_0$.

Assume that $e_0e_i = 0$ and $e_i = e_ie_0$ for $i = 1, 2, \ldots, n-1$. Then

$$e_0e_n = e_0e_n + e_0e_ne_0 = e_0e_n + e_0e_n, \tag{2.3}$$

so

$$\begin{aligned}
e_0e_n &= 0,\\
e_n &= e_1\sigma(e_{n-1}) + e_2\sigma^2(e_{n-2}) + \cdots + e_{n-1}\sigma^{n-1}(e_1) + e_ne_0\\
&= e_1e_0\sigma(e_{n-1}) + e_2e_0\sigma^2(e_{n-2}) + \cdots + e_{n-1}e_0\sigma^{n-1}(e_1) + e_ne_0\\
&= e_1\sigma(e_0e_{n-1}) + e_2\sigma^2(e_0e_{n-2}) + \cdots + e_{n-1}\sigma^{n-1}(e_0e_1) + e_ne_0\\
&= e_ne_0.
\end{aligned} \tag{2.4}$$

Hence $f(x)e_0 = e_0 + e_1\sigma(e_0)x + \cdots + e_n\sigma^n(e_0)x^n = e_0 + e_1x + \cdots + e_nx^n = f(x)$ and $e_0f(x) = e_0$.

For any $g(x) = b_0 + b_1x + \cdots + b_mx^m \in R[x;\sigma]$, we have $e_0g(x)e_0 = \sum_{0\le i\le m} e_0b_i\sigma^i(e_0)x^i = \sum_{0\le i\le m} e_0b_ie_0x^i = \sum_{0\le i\le m} e_0b_ix^i = e_0g(x)$. Thus $f(x)g(x)f(x) = f(x)e_0g(x)f(x) = f(x)e_0g(x)e_0f(x) = f(x)e_0g(x)e_0 = f(x)e_0g(x) = f(x)g(x)$, which implies $f(x)$ is right semicentral in $R[x;\sigma]$. Similarly, if e_0 is left semicentral in R, then we can show that $f(x)$ is left semicentral in $R[x;\sigma]$. Hence $R[x;\sigma]$ is a semiabelian ring. □

Corollary 2.15. *Let R be a semiabelian ring. If σ is a ring endomorphism of R satisfying $\sigma(e) = e$ for all $e \in E(R)$. Then $R[x;\sigma]/(x^2)$ is semiabelian.*

Proof. Since every element $f(x)$ of $R[x;\sigma]/(x^2)$ can be written $f(x) = a_0 + a_1x$ with $x^2 = 0$, by the same proof as Theorem 2.14, we can complete the proof. □

Corollary 2.16. *Let R be a semiabelian ring. If σ is a ring endomorphism of R satisfying $\sigma(e) = e$ for all $e \in E(R)$. Then $R \propto R(\sigma)$ is semiabelian.*

Corollary 2.17. *Let D be a division ring with an endomorphism σ, then $D[x;\sigma]/(x^2)$ is semiabelian.*

A ring R is called left $MC2$ [2] if for any $a \in R$ and $e \in ME_l(R)$, $aRe = 0$ implies $eRa = 0$. Clearly, left idempotent reflexive rings are left $MC2$. We do not know whether idempotent reversible rings are left $MC2$. But we know that there exists a left $MC2$ ring which is not idempotent reversible. In fact, there exists a semiabelian ring R which is not abelian (see the example above Theorem 2.5), by [1, Corollary 2.4], $R[x]$ is a semiabelian ring which is not abelian. Hence, by Theorem 2.5, $R[x]$ is not idempotent reversible. But $R[x]$ is a left $MC2$ ring.

The authors in [10, Theorem 4.1] showed that if R is a left $MC2$ ring containing an injective maximal left ideal, then R is a left self-injective ring. And [11, Proposition 5] showed that if R is a left idempotent reflexive ring containing an injective maximal left ideal, then R is a left self-injective ring.

Proposition 2.18. *Let R be an idempotent reversible ring. If R contains an injective maximal left ideal, then R is a left self-injective ring.*

Proof. Let M be an injective maximal left ideal of R. Then $R = M \oplus N$ for some minimal left ideal N of R. Hence we have $M = Re$ and $N = R(1-e)$ for some $e^2 = e \in R$. If $MN = 0$, then we have $eR(1-e) = 0$. Since R is idempotent reversible, $(1-e)Re = 0$. So e is central. Now let L be any proper essential left ideal of R and $f : L \to N$ any non-zero left R-homomorphism. Then $L/U \cong N$, where $U = ker f$ is a maximal submodule of L. Now $L = U \oplus V$, where $V \cong N = R(1-e)$ is a minimal left ideal of R. Since e is central, $V = R(1-e)$. For any $z \in L$, let $z = x + y$, where $x \in U, y \in V$. Then $f(z) = f(x) + f(y) = f(y)$. Since $y = y(1-e) = (1-e)y$, $f(z) = f(y) = f(y(1-e)) = yf(1-e)$. Since $x(1-e) = (1-e)x \in V \cap U = 0$, $xf(1-e) = f(x(1-e)) = f(0) = 0$. Thus $f(z) = yf(1-e) = yf(1-e) + xf(1-e) = (y+x)f(1-e) = zf(1-e)$. Hence ${}_RN$ is injective. If $MN \neq 0$, by the proof of [10, Proposition 5], we have that ${}_RN$ is injective. Hence $R = M \oplus N$ is left self-injective. □

Recall that a ring R is left pp if every principal left ideal of R is projective. As an application of Proposition 2.18, we have the following result.

Corollary 2.19. *The following conditions are equivalent for a ring R.*

(1) *R is a von Neumann regular left self-injective ring with $Soc({}_RR) \neq 0$.*

(2) *R is an idempotent reversible left pp ring containing an injective maximal left ideal.*

Proof. (1)⇒(2) is trivial.

(2)⇒(1) by Proposition 2.18, R is a left self-injective ring. Hence, by [12, Theorem 1.2], R is left $C2$, so, R is von Neumann regular because R is left pp. Also we have $\mathrm{Soc}({}_RR) \neq 0$ since there is an injective maximal left ideal. □

By [13], a ring R is said to be left HI if R is left hereditary containing an injective maximal left ideal. Osofsky [14] proved that left self-injective left hereditary ring is semisimple Artinian. We can generalize the result as follows.

Corollary 2.20. *The following conditions are equivalent for a ring R.*

(1) *R is a semisimple Artinian ring.*

(2) *R is an idempotent reversible left HI ring.*

According to [8], an element $x \in R$ is called exchange if there exists $e \in E(R)$ such that $e \in xR$ and $1 - e \in (1-x)R$, and x is said to be clean if $x = e + u$ where $e \in E(R)$ and $u \in U(R)$. By [8], clean elements are exchange and the converse holds when R is an abelian ring. A ring R is called exchange (clean) ring if every element of R is an exchange (clean) element.

Proposition 2.21. *Let R be a semiabelian ring. If $x \in R$ is an exchange element, then x is a clean element.*

Proof. Since x is an exchange element, there exists $e \in E(R)$ such that $e \in xR$ and $1 - e \in (1 - x)R$. Let $e = xy$ and $1 - e = (1 - x)z$ where $y = ye, z = z(1 - e) \in R$. Then $(x - (1 - e))(y - z) = xy - xz - (1 - e)y + (1 - e)z = xy + (1 - x)z - (1 - e)y - ez = e + 1 - e - (1 - e)y - ez = 1 - (1 - e)y - ez$. Since R is a semiabelian ring, e is either left semicentral or right semicentral. If e is left semicentral, then $(1 - e)y = (1 - e)ye = 0$ and $(ezR)^2 = ezRezR = ez(1 - e)RezR \subseteq eR(1 - e)ReR = 0$ by Theorem 2.7(1). Hence $ez \in J(R)$. Similarly, if e is right semicentral, then $ez = ez(1 - e) = 0$ and $(1 - e)y \in J(R)$. This implies $1 - (1 - e)y - ez \in U(R)$, so $(x - (1 - e))(y - z) \in U(R)$. Since R is a semiabelian ring, by Theorem 2.8, R is a directly finite ring. Hence $x - (1 - e) \in U(R)$, which implies x is a clean element. □

Corollary 2.22. *If R is a semiabelian exchange ring, then R is a clean ring.*

Theorem 2.23. *Let R be a semiabelian ring and $a, b \in R$. If $ab = 0$, then $aE(R)b \subseteq J(R)$.*

Proof. Let $ab = 0$ and $e \in E(R)$. Since R is a semiabelian ring, either e is left semicentral or e is right semicentral. If e is left semicentral, then $(Raeb)^2 = RabRaeb = 0$. If e is right semicentral, then $(Raeb)^2 = RaebRab = 0$. Hence $Raeb \subseteq J(R)$ for each $e \in E(R)$, which implies $aE(R)b \subseteq J(R)$. □

Corollary 2.24. *Let R be an abelian ring and $a, b \in R$. If $ab = 0$, then $aE(R)b \subseteq J(R)$.*

The converse of Corollary 2.24 is not true, in general.

Example 2.25. let F be a field, and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Evidently, $E(R) = \bigcup_{x\in F}\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}\}$, $J(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in R$ and $AB = 0$. Then $a_1a_2 = c_1c_2 = 0$. Since $A\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}B = \begin{pmatrix} 0 & a_1b_2+a_1xc_2 \\ 0 & 0 \end{pmatrix} \in J(R)$ and $A\begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}B = \begin{pmatrix} 0 & a_1xc_2+b_1c_2 \\ 0 & 0 \end{pmatrix} \in J(R)$. Hence $AE(R)B \subseteq J(R)$, but R is not an abelian ring.

A ring R is called *EIFP* if $a, b \in R$, $ab = 0$ implies $aE(R)b \subseteq J(R)$. Clearly, semiabelian rings are *EIFP* by Theorem 2.23. But the converse of Theorem 2.23 is not true, in general.

Example 2.26. Take the ring R in Example 2.25, and let $S = R \oplus R$. Then S is *EIFP*, but not semiabelian. Indeed, take $e_1 = E_{11} + E_{12}$ and $e_2 = E_{12} + E_{22}$ in R, where E_{ij} are matrix units. Then $(e_1; e_2)$ is an idempotent. By a direct computation, $(e_1; e_2)$ is neither left nor right semicentral. Hence $R \oplus R$ is not semiabelian while $R \oplus R$ is *EIFP*.

Acknowledgments

This paper was supported by the Foundation of Natural Science of China (10771182) and the Scientific Research Foundation of Graduate School of Jiangsu Province (CX09B_309Z). The authors would like to thank the referee for his/her helpful suggestions and comments.

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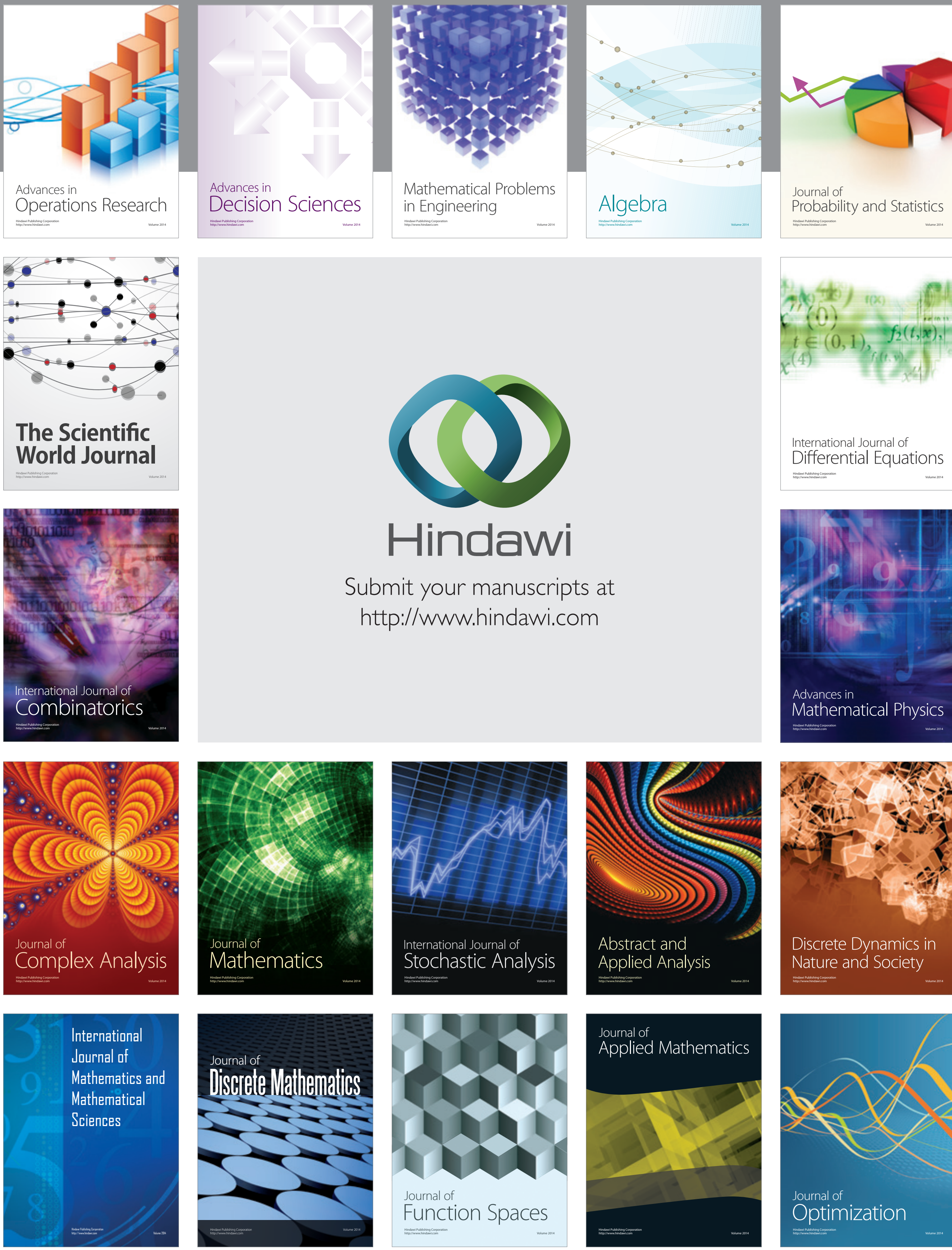
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