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Research Article

Hybrid Proximal-Point Methods for Zeros of Maximal Monotone Operators, Variational Inequalities and Mixed Equilibrium Problems

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We prove strong and weak convergence theorems of modified hybrid proximal-point algorithms for finding a common element of the zero point of a maximal monotone operator, the set of solutions of equilibrium problems, and the set of solution of the variational inequality operators of an inverse strongly monotone in a Banach space under different conditions. Moreover, applications to complementarity problems are given. Our results modify and improve the recently announced ones by Li and Song (2008) and many authors.

1. Introduction

Let E be a Banach space with norm $\|\cdot\|$, C a nonempty closed convex subset of E, let E^* denote the dual of E and $<\cdot,\cdot>$ is the pairing between E and E^*.

Consider the problem of finding

$$v \in E \quad \text{such that } 0 \in T(v), \tag{1.1}$$

where T is an operator from E into E^*. Such $v \in E$ is called a *zero point* of T. When T is a maximal monotone operator, a well-known method for solving (1.1) in a Hilbert space H is the *proximal point algorithm* $x_1 = x \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, 3, \ldots, \tag{1.2}$$

where $\{r_n\} \subset (0,\infty)$ and $J_{r_n} = (I + r_n T)^{-1}$, then Rockafellar [1] proved that the sequence $\{x_n\}$ converges weakly to an element of $T^{-1}(0)$.

In 2000, Kamimura and Takahashi [2] proved the following strong convergence theorem in Hilbert spaces, by the following algorithm:

$$x_{n+1} = \alpha_n x + (1-\alpha_n) J_{r_n} x_n, \quad n = 1,2,3,\ldots, \tag{1.3}$$

where $J_r = (I + rT)^{-1} J$, then the sequence $\{x_n\}$ converges strongly to $P_{T^{-1}0}(x)$, where $P_{T^{-1}0}$ is the projection from H onto $T^{-1}(0)$. These results were extended to more general Banach spaces see [3, 4].

In 2004, Kohsaka and Takahashi [4] introduced the following iterative sequence for a maximal monotone operator T in a smooth and uniformly convex Banach space: $x_1 = x \in E$ and

$$x_{n+1} = J^{-1}(\alpha_n Jx + (1-\alpha_n) J(J_{r_n} x_n)), \quad n = 1,2,3,\ldots, \tag{1.4}$$

where J is the duality mapping from E into E^* and $J_r = (I + rT)^{-1} J$.

Recently, Li and Song [5] proved a strong convergence theorem in a Banach space, by the following algorithm: $x_1 = x \in E$ and

$$\begin{aligned} y_n &= J^{-1}\left(\beta_n J(x_n) + (1-\beta_n) J(J_{r_n} x_n)\right), \\ x_{n+1} &= J^{-1}\left(\alpha_n J x_1 + (1-\alpha_n) J(y_n)\right), \end{aligned} \tag{1.5}$$

with the coefficient sequences $\{\alpha_n\}$, $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$, and $\lim_{n\to\infty} r_n = \infty$. Where J is the duality mapping from E into E^* and $J_r = (I + rT)^{-1} J$. Then, they proved that the sequence $\{x_n\}$ converges strongly to $\Pi_C x$, where Π_C is the generalized projection from E onto C.

Let C be a nonempty closed convex subset of E, and let A be a *monotone* operator of C into E^*. The *variational inequality problem* is to find a point $x^* \in C$ such that

$$\langle v - x^*, Ax^* \rangle \geq 0, \quad \forall v \in C. \tag{1.6}$$

The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in E$ satisfying $0 = Au$, and so on. An operator A of C into E^* is said to be *inverse-strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \tag{1.7}$$

for all $x, y \in C$. In such a case, A is said to be *α-inverse-strongly monotone*. If an operator A of C into E^* is α-inverse-strongly monotone, then A is *Lipschitz continuous*, that is, $\|Ax - Ay\| \leq (1/\alpha)\|x - y\|$ for all $x, y \in C$.

In a Hilbert space H, Iiduka et al. [6] proved that the sequence $\{x_n\}$ defined by: $x_1 = x \in C$ and

$$x_{n+1} = P_C(x_n - \lambda_n A x_n), \tag{1.8}$$

where P_C is the metric projection of H onto C and $\{\lambda_n\}$ is a sequence of positive real numbers, converges weakly to some element of $\mathrm{VI}(C, A)$.

In 2008, Iiduka and Takahashi [7] introduced the folowing iterative scheme for finding a solution of the variational inequality problem for an inverse-strongly monotone operator A in a Banach space $x_1 = x \in C$ and

$$x_{n+1} = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \tag{1.9}$$

for every $n = 1, 2, 3, \ldots$, where Π_C is the generalized metric projection from E onto C, J is the duality mapping from E into E^* and $\{\lambda_n\}$ is a sequence of positive real numbers. They proved that the sequence $\{x_n\}$ generated by (1.9) converges weakly to some element of $\mathrm{VI}(C, A)$.

Let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ and $\varphi : C \to \mathbb{R}$ a real-valued function. The *mixed equilibrium problem*, denoted by $\mathrm{MEP}(\Theta, \varphi)$, is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.10}$$

If $\varphi \equiv 0$, the problem (1.10) reduces into the *equilibrium problem for* Θ, denoted by $\mathrm{EP}(\Theta)$, is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{1.11}$$

If $\Theta \equiv 0$, the problem (1.10) reduces into the *minimize problem*, denoted by $\mathrm{Argmin}(\varphi)$, is to find $x \in C$ such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.12}$$

The above formulation (1.11) was shown in [8] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems, and Nash equilibria in noncooperative games. In addition, there are several other problems, for example, the complementarity problem, fixed point problem, and optimization problem, which can also be written in the form of an $\mathrm{EP}(\Theta)$. In other words, the $\mathrm{EP}(\Theta)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, and so forth. In the last two decades, many papers have appeared in the literature on the existence of solutions of $\mathrm{EP}(\Theta)$; see, for example, [8–11] and references therein. Some solution methods have been proposed to solve the $\mathrm{EP}(\Theta)$; see, for example, [9, 11–21] and references therein. In 2005, Combettes and Hirstoaga [12] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(\Theta)$ is nonempty and they also proved a strong convergence theorem.

Recall, a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \tag{1.13}$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. If C is bounded closed convex and S is a nonexpansive mapping of C into itself, then $F(S)$ is nonempty (see [22]). A mapping S is said to be *quasi-nonexpansive* if $F(S) \neq \emptyset$ and $\|Sx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(S)$. It is easy to see that if S is nonexpansive with $F(S) \neq \emptyset$, then it is quasi-nonexpansive. We write $x_n \to x(x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges (weakly, resp.) to x. Let E be a real Banach space with norm $\|\cdot\|$ and let J be the *normalized duality mapping* from E into 2^{E^*} given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\|x^*\|, \|x\| = \|x^*\|\}, \tag{1.14}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

Let C be a closed convex subset of E, and let S be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of S [23] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$. The set of asymptotic fixed points of S will be denoted by $\widetilde{F(S)}$. A mapping S from C into itself is said to be *relatively nonexpansive* [24–26] if $\widetilde{F(S)} = F(S)$ and $\phi(p, Sx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(S)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [27, 28]. S is said to be ϕ-nonexpansive, if $\phi(Sx, Sy) \leq \phi(x, y)$ for $x, y \in C$. S is said to be *relatively quasi-nonexpansive* if $F(S) \neq \emptyset$ and $\phi(p, Sx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(S)$.

In 2009, Takahashi and Zembayashi [29] introduced the following shrinking projection method of closed relatively nonexpansive mappings as follows:

$$\begin{gathered}
x_0 = x \in C, \qquad C_0 = C, \\
y_n = J^{-1}(\alpha_n J(x_n) + (1 - \alpha_n) JS(x_n)), \\
u_n \in C \quad \text{such that } \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\
C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x,
\end{gathered} \tag{1.15}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty}\alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then, they proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap \mathrm{EP}(\Theta)} x$.

In 2009, Qin et al. [30] modified the Halpern-type iteration algorithm for closed quasi-ϕ-nonexpansive mappings (or relatively quasi-nonexpansive) defined by

$$
\begin{gathered}
x_0 \in E \text{ chosen arbitrarily}, \\
C_1 = C, \\
x_1 = \Pi_{C_1} x_0, \\
y_n = J^{-1}(\alpha_n J(x_1) + (1-\alpha_n) JT(x_n)), \\
C_{n+1} = \left\{ z \in C_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1.
\end{gathered}
\tag{1.16}
$$

Then, they proved that under appropriate control conditions the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_1$.

Recently, Ceng et al. [31] proved the following strong convergence theorem for finding a common element of the set of solutions for an equilibrium and the set of a zero point for a maximal monotone operator T in a Banach space E

$$
\begin{gathered}
y_n = J^{-1}\left(\alpha_n J(x_0) + (1-\alpha_n)\left(\beta_n J x_n + \left(1-\beta_n\right) J J_{r_n}(x_n)\right)\right), \\
H_n = \left\{ z \in C : \phi\left(z, T_{r_n} y_n\right) \le \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n) \right\}, \\
W_n = \{ z \in C : \langle x_n - z, J x_0 - J x_n \rangle \ge 0 \}, \\
x_{n+1} = \Pi_{H_n \cap W_n} x_0.
\end{gathered}
\tag{1.17}
$$

Then, the sequence $\{x_n\}$ converges strongly to $\Pi_{T^{-1}0 \cap \mathrm{EP}(\Theta)} x_0$, where $\Pi_{T^{-1}0 \cap \mathrm{EP}(\Theta)}$ is the generalized projection of E onto $T^{-1}0 \cap \mathrm{EP}(\Theta)$.

In this paper, motivated and inspired by Li and Song [5], Iiduka and Takahashi [7], Takahashi and Zembayashi [29], Ceng et al. [31] and Qin et al. [30], we introduce the following new hybrid proximal-point algorithms defined by $x_1 = x \in C$:

$$
\begin{gathered}
w_n = \Pi_C J^{-1}(J x_n - \lambda_n A x_n), \\
z_n = J^{-1}\left(\beta_n J(x_n) + \left(1-\beta_n\right) J(J_{r_n} w_n)\right), \\
y_n = J^{-1}(\alpha_n J(x_1) + (1-\alpha_n) J(z_n)), \\
u_n \in C \quad \text{such that } \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \ge 0, \quad \forall y \in C, \\
C_{n+1} = \left\{ z \in C_n : \phi(z, u_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x
\end{gathered}
\tag{1.18}
$$

and

$$
\begin{aligned}
u_n \in C \quad \text{such that } \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\
z_n = \Pi_C J^{-1}(Ju_n - \lambda_n A u_n), \\
y_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n} z_n)), \\
x_{n+1} = \Pi_C J^{-1}(\alpha_n J(x_1) + (1-\alpha_n)J(y_n)).
\end{aligned}
\tag{1.19}
$$

Under appropriate conditions, we will prove that the sequence $\{x_n\}$ generated by algorithms (1.18) and (1.19) converges strongly to the point $\Pi_{VI(C,A)\cap T^{-1}(0)\cap MEP(\Theta,\varphi)}x$ and converges weakly to the point $\lim_{n\to\infty}\Pi_{VI(C,A)\cap T^{-1}(0)\cap MEP(\Theta,\varphi)}x_n$, respectively. The results presented in this paper extend and improve the corresponding ones announced by Li and Song [5] and many authors in the literature.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then, the Banach space E is said to be *smooth* provided

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$. The *modulus of convexity* of E is the function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\right\}. \tag{2.2}$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in [0,2]$; see [32, 33] for more details. Observe that every p-uniform convex is uniformly convex. One should note that no Banach space is p-uniform convex for $1 < p < 2$. It is well known that a Hilbert space is 2-uniformly convex and uniformly smooth. For each $p > 1$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\right\}, \tag{2.3}$$

for all $x \in E$. In particular, $J = J_2$ is called *the normalized duality mapping*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

We know the following (see [34]):

(1) if E is smooth, then J is single-valued,

(2) if E is strictly convex, then J is one-to-one and $\langle x-y, x^*-y^*\rangle > 0$ holds for all $(x,x^*),(y,y^*) \in J$ with $x \neq y$,

(3) if E is reflexive, then J is surjective,

(4) if E is uniformly convex, then it is reflexive,

(5) if E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E.

The duality J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* [35] if $x_n \rightharpoonup x$ implies $Jx_n \rightharpoonup^* Jx$, where $\rightharpoonup^*$ implies the weak* convergence.

Lemma 2.1 (see [36, 37]). *If E be a 2-uniformly convex Banach space. Then, for all $x, y \in E$ one has*

$$\|x-y\| \leq \frac{2}{c^2}\|Jx-Jy\|, \tag{2.4}$$

where J is the normalized duality mapping of E and $0 < c \leq 1$.

The best constant $1/c$ in Lemma is called the 2-uniformly convex constant of E; see [32].

Lemma 2.2 (see [36, 38]). *If E a p-uniformly convex Banach space and let p be a given real number with $p \geq 2$. Then, for all $x, y \in E, J_x \in J_p(x)$ and $J_y \in J_p(y)$*

$$\langle x-y, Jx-Jy\rangle \geq \frac{c^p}{2^{p-2}p}\|x-y\|^p, \tag{2.5}$$

where J_p is the generalized duality mapping of E and $1/c$ is the p-uniformly convexity constant of E.

Lemma 2.3 (see Xu [37]). *Let E be a uniformly convex Banach space. Then, for each $r > 0$, there exists a strictly increasing, continuous, and convex function $K : [0,\infty) \to [0,\infty)$ such that $K(0) = 0$ and*

$$\|\lambda x + (1-\lambda y)\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)K(\|x-y\|), \tag{2.6}$$

for all $x, y \in \{z \in E : \|z\| \leq r\}$ and $\lambda \in [0,1]$.

Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty closed convex subset of E. Throughout this paper, we denote by ϕ the function defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for } x, y \in E. \tag{2.7}$$

Following Alber [39], the *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x) \tag{2.8}$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J. It is obvious from the definition of function ϕ that (see [39])

$$\left(\|y\| - \|x\|\right)^2 \leq \phi(y, x) \leq \left(\|y\| + \|x\|\right)^2, \quad \forall x, y \in E. \tag{2.9}$$

If E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$.

If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2.9), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [34, 40] for more details.

Lemma 2.4 (see Kamimura and Takahashi [3]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.5 (see Alber [39]). *Let C be a nonempty, closed, convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\left\langle x_0 - y, Jx - Jx_0 \right\rangle \geq 0, \quad \forall y \in C. \tag{2.10}$$

Lemma 2.6 (see Alber [39]). *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C. \tag{2.11}$$

Let E be a strictly convex, smooth, and reflexive Banach space, let J be the duality mapping from E into E^*. Then, J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E. Define a function $V : E \times E^* \to \mathbb{R}$ as follows (see [4]):

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \tag{2.12}$$

for all $x \in E, x \in E$ and $x^* \in E^*$. Then, it is obvious that $V(x, x^*) = \phi(x, J^{-1}(x^*))$ and $V(x, J(y)) = \phi(x, y)$.

Lemma 2.7 (see Kohsaka and Takahashi [4, Lemma 3.2]). *Let E be a strictly convex, smooth, and reflexive Banach space, and let V be as in* (2.12). *Then,*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^* \right\rangle \leq V(x, x^* + y^*), \tag{2.13}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Let E be a reflexive, strictly convex, and smooth Banach space. Let C be a closed convex subset of E. Because $\phi(x, y)$ is strictly convex and coercive in the first variable, we know that the minimization problem $\inf_{y\in C}\phi(x, y)$ has a unique solution. The operator $\Pi_C x := \arg\min_{y\in C}\phi(x, y)$ is said to be the generalized projection of x on C.

A set-valued mapping $T : E \to E^*$ with domain $D(T) = \{x \in E : T(x) \neq \emptyset\}$ and range $R(T) = \{x^* \in E^* : x^* \in T(x), x \in D(T)\}$ is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $x^* \in T(x)$, $y^* \in T(y)$. We denote the set $\{s \in E : 0 \in Tx\}$ by $T^{-1}0$. T is *maximal* monotone if its graph $G(T)$ is not properly contained in the graph of any other monotone operator. If T is maximal monotone, then the solution set $T^{-1}0$ is closed and convex.

Let E be a reflexive, strictly convex, and smooth Banach space, it is known that T is a maximal monotone if and only if $R(J + rT) = E^*$ for all $r > 0$.

Define the *resolvent* of T by $J_r x = x_r$. In other words, $J_r = (J + rT)^{-1}J$ for all $r > 0$. J_r is a single-valued mapping from E to $D(T)$. Also, $T^{-1}(0) = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of all fixed points of J_r. Define, for $r > 0$, the *Yosida approximation* of T by $A_r = (J - JJ_r)/r$. We know that $A_r x \in T(J_r x)$ for all $r > 0$ and $x \in E$.

Lemma 2.8 (see Kohsaka and Takahashi [4, Lemma 3.1]). *Let E be a smooth, strictly convex, and reflexive Banach space, $T \subset E \times E^*$ a maximal monotone operator with $T^{-1}0 \neq \emptyset$, $r > 0$ and $J_r = (J + rT)^{-1}J$. Then,*

$$\phi(x, J_r y) + \phi(J_r y, y) \leq \phi(x, y), \tag{2.14}$$

for all $x \in T^{-1}0$ and $y \in E$.

Let A be an inverse-strongly monotone mapping of C into E^* which is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0, 1]$ into E^*, defined by $F(t) = A(tx + (1 - t)y)$, is continuous with respect to the weak* topology of E^*. We define by $N_C(v)$ the *normal cone* for C at a point $v \in C$, that is,

$$N_C(v) = \left\{x^* \in E^* : \langle v - y, x^*\rangle \geq 0,\ \forall y \in C\right\}. \tag{2.15}$$

Theorem 2.9 (see Rockafellar [1]). *Let C be a nonempty, closed, convex subset of a Banach space E and A a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{2.16}$$

Then, T is maximal monotone and $T^{-1}0 = VI(C, A)$.

Lemma 2.10 (see Tan and Xu [41]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} = a_n + b_n, \quad \forall n \geq 0. \tag{2.17}$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

For solving the mixed equilibrium problem, let us assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ and $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$,

(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$,

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x, y), \tag{2.18}$$

(A4) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex and lower semicontinuous.

Motivated by Blum and Oettli [8], Takahashi and Zembayashi [29, Lemma 2.7] obtained the following lemmas.

Lemma 2.11 (see [29, Lemma 2.7]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $r > 0$, and let $x \in E$. Then, there exists $z \in C$ such that*

$$\Theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C. \tag{2.19}$$

Lemma 2.12 (see Takahashi and Zembayashi [29]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E and let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For all $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \left\{ z \in C : \Theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\}, \tag{2.20}$$

for all $x \in E$. Then, the followings hold:

(1) *T_r is single-valued,*

(2) *T_r is a firmly nonexpansive-type mapping, that is, for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle, \tag{2.21}$$

(3) *$F(T_r) = EP(\Theta)$,*

(4) *$EP(\Theta)$ is closed and convex.*

Lemma 2.13 (see Takahashi and Zembayashi [29]). *Let C be a closed, convex subset of a smooth, strictly convex, and reflexive Banach space E, let Θ a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.22}$$

Lemma 2.14. *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $\varphi : C \to \mathbb{R}$ is convex and lower semicontinuous and Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in E$, then there exists $u \in C$ such that*

$$\Theta(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle. \tag{2.23}$$

Define a mapping $K_r : E \to C$ as follows:

$$K_r(x) = \left\{ u \in C : \Theta(u, y) + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0,\ \forall y \in C \right\} \tag{2.24}$$

for all $x \in E$. Then, the followings hold:

(1) *K_r is single-valued,*

(2) *K_r is firmly nonexpansive, that is, for all $x, y \in E$, $\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$,*

(3) *$F(K_r) = MEP(\Theta, \varphi)$,*

(4) *$MEP(\Theta, \varphi)$ is closed and convex.*

Proof. Define a bifunction $F : C \times C \to \mathbb{R}$ as follows:

$$F(u, y) = \Theta(u, y) + \varphi(y) - \varphi(u), \quad \forall u, y \in C. \tag{2.25}$$

It is easily seen that F satisfies (A1)–(A4). Therefore, K_r in Lemma 2.14 can be obtained from Lemma 2.12 immediately. □

3. Strong Convergence Theorem

In this section, we prove a strong convergence theorem for finding a common element of the zero point of a maximal monotone operator, the set of solutions of equilibrium problems, and the set of solution of the variational inequality operators of an inverse strongly monotone in a Banach space by using the shrinking hybrid projection method.

Theorem 3.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, and let $T : E \to E^*$ be a maximal monotone operator. Let $J_r = (J + rT)^{-1}J$ for $r > 0$ and let A be an α-inverse-strongly monotone* *operator of C into E^* with $F := VI(C, A) \cap T^{-1}(0) \cap MEP(\Theta, \varphi) \neq \emptyset$ and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in F$. Let $\{x_n\}$ be a sequence generated by $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$,*

$$\begin{aligned}
&w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\
&z_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n) J(J_{r_n} w_n)), \\
&y_n = J^{-1}(\alpha_n J(x_1) + (1-\alpha_n) J(z_n)), \\
&u_n \in C \quad \text{such that } \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y\ C, \\
&C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\
&x_{n+1} = \Pi_{C_{n+1}} x_0,
\end{aligned} \tag{3.1}$$

for $n \in \mathbb{N}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$ satisfying $\lim_{n\to\infty}\alpha_n = 0, \limsup_{n\to\infty}\beta_n < 1, \liminf_{n\to\infty} r_n > 0$, and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, $1/c$ is the 2-uniformly convexity constant of E. Then, the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. We first show that $\{x_n\}$ is bounded. Put $v_n = J^{-1}(Jx_n - \lambda_n Ax_n)$, let $p \in F := \mathrm{VI}(C, A) \cap T^{-1}(0) \cap \mathrm{MEP}(\Theta, \varphi)$, and let $\{K_{r_n}\}$ be a sequence of mapping define as Lemma 2.14 and $u_n = K_{r_n} y_n$. By (3.1) and Lemma 2.7, the convexity of the function V in the second variable, we have

$$\begin{aligned}
\phi(p, w_n) &= \phi(p, \Pi_C v_n) \\
&\leq \phi(p, v_n) = \phi\left(p, J^{-1}(Jx_n - \lambda_n Ax_n)\right) \\
&\leq V(p, Jx_n - \lambda_n Ax_n + \lambda_n Ax_n) - 2\left\langle J^{-1}(Jx_n - \lambda_n Ax_n) - p, \lambda_n Ax_n\right\rangle \\
&= V(p, Jx_n) - 2\lambda_n\langle v_n - p, Ax_n\rangle \\
&= \phi(p, x_n) - 2\lambda_n\langle x_n - p, Ax_n\rangle + 2\langle v_n - x_n, -\lambda_n Ax_n\rangle.
\end{aligned} \tag{3.2}$$

Since $p \in \mathrm{VI}(A, C)$ and A is α-inverse-strongly monotone, we have

$$\begin{aligned}
-2\lambda_n\langle x_n - p, Ax_n\rangle &= -2\lambda_n\langle x_n - p, Ax_n - Ap\rangle - 2\lambda_n\langle x_n - p, Ap\rangle \\
&\leq -2\alpha\lambda_n\|Ax_n - Ap\|^2,
\end{aligned} \tag{3.3}$$

and by Lemma 2.1, we obtain

$$\begin{aligned}
2\langle v_n - x_n, -\lambda_n A x_n\rangle &= 2\left\langle J^{-1}(Jx_n - \lambda_n A x_n) - x_n, -\lambda_n A x_n \right\rangle \\
&\le 2\left\| J^{-1}(Jx_n - \lambda_n A x_n) - x_n \right\| \|\lambda_n A x_n\| \\
&\le \frac{4}{c^2}\|Jx_n - \lambda_n A x_n - Jx_n\| \|\lambda_n A x_n\| \\
&= \frac{4}{c^2}\lambda_n^2 \|Ax_n\|^2 \le \frac{4}{c^2}\lambda_n^2 \left\|Ax_n - Ap\right\|^2.
\end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we get

$$\begin{aligned}
\phi(p, w_n) &\le \phi(p, x_n) - 2\alpha\lambda_n \left\|Ax_n - Ap\right\|^2 + \frac{4}{c^2}\lambda_n^2 \left\|Ax_n - Ap\right\|^2 \\
&\le \phi(p, x_n) + 2\lambda_n\left(\frac{2}{c^2}\lambda_n - \alpha\right)\left\|Ax_n - Ap\right\|^2 \\
&\le \phi(p, x_n).
\end{aligned} \tag{3.5}$$

By Lemmas 2.7, 2.8 and (3.5), we have

$$\begin{aligned}
\phi(p, z_n) &= \phi\Big(p, J^{-1}(\beta_n J(x_n) + (1-\beta_n) J(J_{r_n} w_n))\Big) \\
&= V(p, \beta_n J(x_n) + (1-\beta_n) J(J_{r_n} w_n)) \\
&\le \beta_n V(p, J(x_n)) + (1-\beta_n) V(p, J(J_{r_n} w_n)) \\
&= \beta_n \phi(p, x_n) + (1-\beta_n)\phi(p, J_{r_n} w_n) \\
&\le \beta_n \phi(p, x_n) + (1-\beta_n)(\phi(p, w_n) - \phi(J_{r_n} w_n, w_n)) \\
&\le \beta_n \phi(p, x_n) + (1-\beta_n)\phi(p, w_n) \\
&\le \beta_n \phi(p, x_n) + (1-\beta_n)\phi(p, x_n) \\
&= \phi(p, x_n).
\end{aligned} \tag{3.6}$$

It follows that

$$\begin{aligned}
\phi(p, y_n) &= \phi\Big(p, J^{-1}(\alpha_n J(x_1) + (1-\alpha_n) J(z_n))\Big) \\
&= V(p, \alpha_n J(x_1) + (1-\alpha_n) J(z_n)) \le \alpha_n V(p, J(x_1)) + (1-\alpha_n) V(p, J(z_n)) \\
&= \alpha_n \phi(p, x_1) + (1-\alpha_n)\phi(p, z_n) \le \alpha_n \phi(p, x_1) + (1-\alpha_n)\phi(p, x_n).
\end{aligned} \tag{3.7}$$

From (3.1) and (3.7), we obtain

$$\phi(p,u_n) = \phi(p,K_{r_n}y_n) \le \phi(p,y_n) \le \phi(p,x_1) + (1-\alpha_n)\phi(p,x_n). \tag{3.8}$$

So, we have $p \in C_{n+1}$. This implies that $F \subset C_n$, for all $n \in \mathbb{N}$.

From Lemma 2.5 and $x_n = \Pi_{C_n}x_0$, we have

$$\begin{gathered}\langle x_n - z, Jx_0 - Jx_n\rangle \ge 0, \quad \forall z \in C_n, \\ \langle x_n - p, Jx_0 - Jx_n\rangle \ge 0, \quad \forall p \in F.\end{gathered} \tag{3.9}$$

From Lemma 2.6, one has

$$\phi(x_n,x_0) = \phi(\Pi_{C_n}x_0,x_0) \le \phi(p,x_0) - \phi(p,x_n) \le \phi(p,x_0), \tag{3.10}$$

for all $p \in F \subset C_n$ and $n \ge 1$. Then, the sequence $\{\phi(x_n,x_0)\}$ is bounded. Since $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n,x_0) \le \phi(x_{n+1},x_0), \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Therefore, $\{\phi(x_n,x_0)\}$ is nondecreasing. Hence, the limit of $\{\phi(x_n,x_0)\}$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m}x_0 \in C_n$ for any positive integer $m \ge n$. It follows that

$$\phi(x_m,x_n) = \phi(x_m,\Pi_{C_n}x_0) \le \phi(x_m,x_0) - \phi(\Pi_{C_n}x_0,x_0) = \phi(x_m,x_0) - \phi(x_n,x_0). \tag{3.12}$$

Letting $m,n \to \infty$ in (3.12), we get $\phi(x_m,x_n) \to 0$. It follows from Lemma 2.4, that $\|x_m - x_n\| \to 0$ as $m,n \to \infty$, that is, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, we can assume that $x_n \to u \in C$, as $n \to \infty$. Since

$$\phi(x_{n+1},x_n) = \phi(x_{n+1},\Pi_{C_n}x_0) \le \phi(x_{n+1},x_0) - \phi(\Pi_{C_n}x_0,x_0) = \phi(x_{n+1},x_0) - \phi(x_n,x_0), \tag{3.13}$$

for all $n \in \mathbb{N}$, we also have $\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0$. From Lemma 2.4, we get $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$ and by definition of C_{n+1}, we have

$$\phi(x_{n+1},u_n) \le \alpha_n\phi(x_{n+1},x_1) + (1-\alpha_n)\phi(x_{n+1},x_n). \tag{3.14}$$

Noticing the conditions $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0$, we obtain

$$\lim_{n\to\infty}\phi(x_{n+1},u_n) = 0. \tag{3.15}$$

From again Lemma 2.4,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\|x_{n+1} - u_n\| = 0. \tag{3.16}$$

So, by the triangle inequality, we get

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.17}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|Jx_n - Ju_n\| = 0. \tag{3.18}$$

On the other hand, we observe that

$$\begin{aligned}\phi(p,x_n) - \phi(p,u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n\rangle \\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\|.\end{aligned} \tag{3.19}$$

It follows that

$$\phi(p,x_n) - \phi(p,u_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.20}$$

From (3.1), (3.5), (3.6), (3.7), and (3.8), we have

$$\begin{aligned}\phi(p,u_n) \leq \phi(p,y_n) &\leq \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,z_n) \\ &\leq \alpha_n\phi(p,x_1) + (1-\alpha_n)\left[\beta_n\phi(p,x_n) + (1-\beta_n)\left(\phi(p,w_n) - \phi(J_{r_n}w_n, w_n)\right)\right] \\ &\leq \alpha_n\phi(p,x_1) + (1-\alpha_n)\left[\beta_n\phi(p,x_n) + (1-\beta_n)\left(\phi(p,x_n) - \phi(J_{r_n}w_n, w_n)\right)\right] \\ &\leq \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,x_n) - (1-\alpha_n)(1-\beta_n)\phi(J_{r_n}w_n, w_n)\end{aligned} \tag{3.21}$$

and then

$$(1-\alpha_n)(1-\beta_n)\phi(J_{r_n}w_n, w_n) \leq \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,x_n) - \phi(p,u_n). \tag{3.22}$$

From conditions $\lim_{n\to\infty}\alpha_n = 0$, $\limsup_{n\to\infty}\beta_n < 1$ and (3.20), we obtain

$$\lim_{n\to\infty}\phi(J_{r_n}w_n, w_n) = 0. \tag{3.23}$$

By again Lemma 2.4, we have $\lim_{n\to\infty}\|J_{r_n}w_n - w_n\| = 0$.

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty}\|J(J_{r_n}w_n) - J(w_n)\| = 0. \tag{3.24}$$

Applying (3.5) and (3.6), we observe that

$$\begin{aligned}\phi(p,u_n) &\le \phi(p,y_n) \le \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,z_n)\\ &\le \alpha_n\phi(p,x_1) + (1-\alpha_n)\left[\beta_n\phi(p,x_n) + (1-\beta_n)\phi(p,w_n)\right] \le \alpha_n\phi(p,x_1)\\ &\quad + (1-\alpha_n)\left[\beta_n\phi(p,x_n) + (1-\beta_n)\left[\phi(p,x_n) - 2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Ax_n - Ap\|^2\right]\right]\\ &\le \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,x_n) - (1-\alpha_n)(1-\beta_n)2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Ax_n - Ap\|^2\end{aligned} \tag{3.25}$$

and, hence,

$$2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Ax_n - Ap\|^2 \le \frac{1}{(1-\alpha_n)(1-\beta_n)}\left(\alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,x_n) - \phi(p,u_n)\right), \tag{3.26}$$

for all $n \in \mathbb{N}$. Since $0 < a \le \lambda_n \le b < c^2\alpha/2$, $\lim_{n\to\infty}\alpha_n = 0$, $\limsup_{n\to\infty}\beta_n < 1$ and (3.20), we have

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{3.27}$$

From Lemmas 2.6, 2.7, and (3.4), we get

$$\begin{aligned}\phi(x_n,w_n) = \phi(x_n,\Pi_C v_n) \le \phi(x_n,v_n) &= \phi\left(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)\right) = V(x_n, Jx_n - \lambda_n Ax_n)\\ &\le V(x_n, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n)\\ &\quad - 2\left\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\right\rangle\\ &= \phi(x_n,x_n) + 2\langle v_n - x_n, -\lambda_n Ax_n\rangle\\ &= 2\langle v_n - x_n, -\lambda_n Ax_n\rangle \le \frac{4\lambda_n^2}{c^2}\|Ax_n - Ap\|^2.\end{aligned} \tag{3.28}$$

From Lemma 2.4 and (3.27), we have

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{3.29}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty}\|J(x_n) - J(w_n)\| = 0. \tag{3.30}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u \in E$. Since $x_n - w_n \to 0$, then we get $w_{n_i} \rightharpoonup u$ as $i \to \infty$.

Now, we claim that $u \in F$. First, we show that $u \in T^{-1}0$. Indeed, since $\liminf_{n\to\infty} r_n > 0$, it follows from (3.24) that

$$\lim_{n\to\infty} \|A_{r_n} w_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|Jw_n - J(J_{r_n} w_n)\| = 0. \tag{3.31}$$

If $(z, z^*) \in T$, then it holds from the monotonicity of T that

$$\left\langle z - w_{n_i}, z^* - A_{r_{n_i}} w_{n_i} \right\rangle \geq 0, \tag{3.32}$$

for all $i \in \mathbb{N}$. Letting $i \to \infty$, we get $\langle z - u, z^* \rangle \geq 0$. Then, the maximality of T implies $u \in T^{-1}0$.

Next, we show that $u \in \mathrm{VI}(C, A)$. Let $B \subset E \times E^*$ be an operator as follows:

$$Bv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{3.33}$$

By Theorem 2.9, B is maximal monotone and $B^{-1}0 = \mathrm{VI}(A, C)$. Let $(v, w) \in G(B)$. Since $w \in Bv = Av + N_C(v)$, we get $w - Av \in N_C(v)$. From $w_n \in C$, we have

$$\langle v - w_n, w - Av \rangle \geq 0. \tag{3.34}$$

On the other hand, since $w_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$, then by Lemma 2.5, we have

$$\langle v - w_n, Jw_n - (Jx_n - \lambda_n Ax_n) \rangle \geq 0. \tag{3.35}$$

Thus,

$$\left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n \right\rangle \leq 0. \tag{3.36}$$

It follows from (3.34) and (3.36) that

$$\begin{aligned}\langle v - w_n, w\rangle &\geq \langle v - w_n, Av\rangle \geq \langle v - w_n, Av\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n} - Ax_n\right\rangle \\ &= \langle v - w_n, Av - Ax_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n}\right\rangle \\ &= \langle v - w_n, Av - Aw_n\rangle + \langle v - w_n, Aw_n - Ax_n\rangle + \left\langle v - w_n, \frac{Jx_n - Jw_n}{\lambda_n}\right\rangle \\ &\geq -\|v - w_n\|\frac{\|w_n - x_n\|}{\alpha} - \|v - w_n\|\frac{\|Jx_n - Jw_n\|}{a} \\ &\geq -M\left(\frac{\|w_n - x_n\|}{\alpha} + \frac{\|Jx_n - Jw_n\|}{a}\right),\end{aligned} \tag{3.37}$$

where $M = \sup_{n\geq 1}\{\|v - w_n\|\}$. From (3.29) and (3.30), we obtain $\langle v - u, w\rangle \geq 0$. By the maximality of B, we have $u \in B^{-1}0$ and, hence, $u \in \mathrm{VI}(C, A)$.

Next, we show that $u \in \mathrm{MEP}(\Theta, \varphi)$. Since $u_n = K_{r_n}y_n$. From Lemmas 2.13 and 2.14, we have

$$\phi(u_n, y_n) = \phi(K_{r_n}y_n, y_n) \leq \phi(u, y_n) - \phi(u, K_{r_n}y_n) \leq \phi(u, x_n) - \phi(u, u_n). \tag{3.38}$$

Similarly by (3.20),

$$\lim_{n\to\infty} \phi(u_n, y_n) = 0, \tag{3.39}$$

and so

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.40}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0. \tag{3.41}$$

From (3.1) and (A2), we also have

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq \Theta(y, u_n), \quad \forall y \in C. \tag{3.42}$$

Hence,

$$\varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{Ju_{n_i} - Jy_{n_i}}{r_{n_i}}\right\rangle \geq \Theta(y, u_{n_i}), \quad \forall y \in C. \tag{3.43}$$

From $\|x_n - u_n\| \to 0$, $\|x_n - w_n\| \to 0$, we get $u_{n_i} \rightharpoonup u$. Since $(Ju_{n_i} - Jy_{n_i}/r_{n_i}) \to 0$, it follows by (A4) and the weak, lower semicontinuous of φ that

$$\Theta(y,u) + \varphi(u) - \varphi(y) \leq 0, \quad \forall y \in C. \tag{3.44}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)u$. Since $y \in C$ and $u \in C$, we have $y_t \in C$ and hence $\Theta(y_t, u) + \varphi(u) - \varphi(y_t) \leq 0$. So, from (A1), (A4), and the convexity of φ, we have

$$\begin{aligned}0 = \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) &\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, u) + t\varphi(y) + (1-t)\varphi(y) - \varphi(y_t)\\ &\leq t\left(\Theta(y_t, y) + \varphi(y) - \varphi(y_t)\right).\end{aligned} \tag{3.45}$$

Dividing by t, we get $\Theta(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0$. From (A3) and the weakly lower semicontinuity of φ, we have $\Theta(u, y) + \varphi(y) - \varphi(u) \geq 0$ for all $y \in C$ implies $u \in \mathrm{MEP}(\Theta, \varphi)$. Hence, $u \in F := \mathrm{VI}(C, A) \cap T^{-1}(0) \cap \mathrm{MEP}(\Theta, \varphi)$.

Finally, we show that $u = \Pi_F x$. Indeed, from $x_n = \Pi_{C_n} x$ and Lemma 2.5, we have

$$\langle Jx - Jx_n, x_n - z\rangle \geq 0, \quad \forall z \in C_n. \tag{3.46}$$

Since $F \subset C_n$, we also have

$$\langle Jx - Jx_n, x_n - p\rangle \geq 0, \quad \forall p \in F. \tag{3.47}$$

Taking limit $n \to \infty$, we have

$$\langle Jx - Ju, u - p\rangle \geq 0, \quad \forall p \in F. \tag{3.48}$$

By again Lemma 2.5, we can conclude that $u = \Pi_F x_0$. This completes the proof. □

Corollary 3.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space, let C be a nonempty, closed, convex subset of E. Let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, and let $T : E \to E^*$ be a maximal monotone operator. Let $J_r = (J + rT)^{-1}J$ for $r > 0$ with $F := T^{-1}(0) \cap MEP(\Theta, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in E$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$,*

$$\begin{gathered}z_n = J^{-1}\left(\beta_n J(x_n) + (1-\beta_n) J(J_{r_n} x_n)\right),\\ y_n = J^{-1}(\alpha_n J(x_1) + (1-\alpha_n) J(z_n)),\\ u_n \in C \quad \text{such that } \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C,\\ C_{n+1} = \left\{z \in C_n : \phi(z, u_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\right\},\\ x_{n+1} = \Pi_{C_{n+1}} x_0,\end{gathered} \tag{3.49}$$

for $n \in \mathbb{N}$, *where* Π_C *is the generalized projection from* E *onto* C, J *is the duality mapping on* E. *The coefficient sequence* $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$ *satisfying* $\lim_{n\to\infty}\alpha_n = 0$, $\limsup_{n\to\infty}\beta_n < 1$ *and* $\liminf_{n\to\infty} r_n > 0$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

Proof. In Theorem 3.1 if $A \equiv 0$, then (3.1) reduced to (3.49). □

4. Weak Convergence Theorem

In this section, we first prove the following strong convergence theorem by using the idea of Plubtieng and Sriprad [42].

Theorem 4.1. *Let* E *be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping* J *is weak sequentially continuous. Let* $T : E \to E^*$ *be a maximal monotone operator and let* $J_r = (J + rT)^{-1}J$ *for* $r > 0$. *Let* C *be a nonempty, closed, convex subset of* E *such that* $D(T) \subset C \subset J^{-1}(\bigcap_{r>0} R(J + rT))$, *let* Θ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)–(A4), let* $\varphi : C \to \mathbb{R}$ *be a lower semicontinuous and convex function, and let* A *be an* α*-inverse-strongly monotone operator of* C *into* E^* *with* $F := VI(C, A) \cap T^{-1}(0) \cap MEP(\Theta, \varphi) \neq \emptyset$ *and* $\|Ay\| \leq \|Ay - Au\|$ *for all* $y \in C$ *and* $u \in F$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 = x \in C$ *and*

$$\begin{aligned} u_n &= K_{r_n} x_n, \\ z_n &= \Pi_C J^{-1}(Ju_n - \lambda_n A u_n), \\ y_n &= J^{-1}(\beta_n J(x_n) + (1-\beta_n) J(J_{r_n} z_n)), \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n J(x_1) + (1-\alpha_n) J(y_n)), \end{aligned} \tag{4.1}$$

for $n \in \mathbb{N} \cup \{0\}$, *where* Π_C *is the generalized projection from* E *onto* C, J *is the duality mapping on* E. *The coefficient sequence* $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ *satisfying* $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\limsup_{n\to\infty}\beta_n < 1$ $\liminf_{n\to\infty} r_n > 0$ *and* $\{\lambda_n\} \subset [a,b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$, $1/c$ *is the 2-uniformly convexity constant of* E. *Then, the sequence* $\{\Pi_F x_n\}$ *converges strongly to an element of* F, *which is a unique element* $v \in F$ *such that*

$$\lim_{n\to\infty} \phi(v, x_n) = \min_{y\in F} \lim_{n\to\infty} \phi(y, x_n), \tag{4.2}$$

where Π_F *is the generalized projection from* C *onto* F.

Proof. Put $v_n = J^{-1}(Ju_n - \lambda_n A u_n)$. Let $p \in F := \mathrm{VI}(C, A) \cap T^{-1}(0) \cap \mathrm{MEP}(\Theta, \varphi)$, by Lemma 2.14 and nonexpansiveness of K_r, we have

$$\phi(p, u_n) = \phi(p, K_{r_n} x_n) \leq \phi(p, x_n). \tag{4.3}$$

By (4.1) and Lemma 2.7, the convexity of the function V in the second variable, we obtain

$$\begin{aligned}\phi(p,z_n) &= \phi(p,\Pi_C v_n) \le \phi(p,v_n) = \phi\left(p, J^{-1}(Ju_n - \lambda_n A u_n)\right)\\&\le V(p, Ju_n - \lambda_n A u_n + \lambda_n A u_n) - 2\left\langle J^{-1}(Ju_n - \lambda_n A u_n) - p, \lambda_n A u_n\right\rangle\\&= V(p, Jxu_n) - 2\lambda_n\langle v_n - p, Au_n\rangle\\&= \phi(p,u_n) - 2\lambda_n\langle u_n - p, Au_n\rangle + 2\langle v_n - u_n, -\lambda_n A u_n\rangle.\end{aligned} \tag{4.4}$$

Since $p \in \mathrm{VI}(A,C)$ and A is α-inverse-strongly monotone, we also have

$$-2\lambda_n\langle u_n - p, Au_n\rangle = -2\lambda_n\langle u_n - p, Au_n - Ap\rangle - 2\lambda_n\langle u_n - p, Ap\rangle \le -2\alpha\lambda_n\left\|Au_n - Ap\right\|^2, \tag{4.5}$$

$$\begin{aligned}2\langle v_n - u_n, -\lambda_n A u_n\rangle &= 2\left\langle J^{-1}(Ju_n - \lambda_n A u_n) - x_n, -\lambda_n A u_n\right\rangle\\&\le 2\left\|J^{-1}(Ju_n - \lambda_n A u_n) - x_n\right\|\|\lambda_n A u_n\|\\&\le \frac{4}{c^2}\|Ju_n - \lambda_n A u_n - Ju_n\|\|\lambda_n A u_n\| \le \frac{4}{c^2}\lambda_n^2\left\|Au_n - Ap\right\|^2.\end{aligned} \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4) and (4.3), we get

$$\begin{aligned}\phi(p,z_n) &\le \phi(p,u_n) - 2\alpha\lambda_n\left\|Au_n - Ap\right\|^2 + \frac{4}{c^2}\lambda_n^2\left\|Au_n - Ap\right\|^2\\&\le \phi(p,u_n) - 2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\left\|Au_n - Ap\right\|^2 \le \phi(p,u_n) \le \phi(p,x_n).\end{aligned} \tag{4.7}$$

By Lemmas 2.7, 2.8, (4.7), and using the same argument in Theorem 3.1, (3.6), we obtain

$$\phi(p,y_n) \le \phi(p,x_n), \tag{4.8}$$

and hence by Lemma 2.6 and (4.7), we note that

$$\begin{aligned}\phi(p,x_{n+1}) &= \phi\left(p, J^{-1}(\alpha_n J(x_1) + (1-\alpha_n)J(y_n))\right)\\&= V(p, \alpha_n J(x_1) + (1-\alpha_n)J(y_n)) \le \alpha_n V(p, J(x_1)) + (1-\alpha_n)V(p, J(y_n))\\&= \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,y_n) \le \alpha_n\phi(p,x_1) + (1-\alpha_n)\phi(p,x_n),\end{aligned} \tag{4.9}$$

for all $n \ge 0$. So, from $\sum_{n=0}^{\infty}\alpha_n < \infty$ and Lemma 2.10, we deduce that $\lim_{n\to\infty}\phi(p,x_n)$ exists. This implies that $\{\phi(p,x_n)\}$ is bounded. It implies that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{J_{r_n}z_n\}$ are bounded. Define a function $g : F \to [0,\infty)$ as follows:

$$g(p) = \lim_{n\to\infty}\phi(p,x_n), \quad \forall p \in F. \tag{4.10}$$

Then, by the same argument as in proof of [43, Theorem 3.1], we obtain g is a continuous convex function and if $\|z_n\| \to \infty$, then $g(z_n) \to \infty$. Hence, by [34, Theorem 1.3.11], there exists a point $v \in F$ such that

$$g(v) = \min_{y \in F} g(y)(:= l). \tag{4.11}$$

Put $w_n = \Pi_F x_n$ for all $n \geq 0$. We next prove that $w_n \to v$ as $n \to \infty$. Suppose on the contrary that there exists $\epsilon_0 > 0$ such that, for each $n \in \mathbb{N}$, there is $n' \geq n$ satisfying $\|w_{n'} - v\| \geq \epsilon_0$. Since $v \in F$, we have

$$\phi(w_n, x_n) = \phi(\Pi_F x_n, x_n) \leq \phi(v, \Pi_F x_n) + \phi(\Pi_F x_n, x_n) \leq \phi(v, x_n), \tag{4.12}$$

for all $n \geq 0$. This implies that

$$\limsup_{n \to \infty} \phi(w_n, x_n) \leq \lim_{n \to \infty} \phi(v, x_n) = l. \tag{4.13}$$

Since $(\|v\| - \|\Pi_F x_n\|)^2 \leq \phi(v, w_n) \leq \phi(v, x_n)$ for all $n \geq 0$ and $\{x_n\}$ is bounded, $\{w_n\}$ is bounded. By Lemma 2.3, there exists a stricly increasing, continuous, and convex function $K : [0, \infty) \to [0, \infty)$ such that $K(0) = 0$ and

$$\left\| \frac{w_n + v}{2} \right\|^2 \leq \frac{1}{2}\|w_n\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{4}K(\|w_n - v\|), \tag{4.14}$$

for all $n \geq 0$. Now, choose σ satisfying $0 < \sigma < (1/4)K(\epsilon_0)$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\phi(w_n, x_n) \leq l + \sigma, \qquad \phi(v, x_n) \leq l + \sigma, \tag{4.15}$$

for all $n \geq 0$. Thus, there exists $k \geq n_0$ satisfying the following:

$$\phi(w_k, x_k) \leq l + \sigma, \qquad \phi(v, x_k) \leq l + \sigma, \qquad \|w_k - v\| \geq \epsilon_0. \tag{4.16}$$

From (4.9), (4.14), and (4.16), we obtain

$$\begin{aligned} \phi\left(\frac{w_k + v}{2}, x_{n+k}\right) \leq \phi\left(\frac{w_k + v}{2}, x_k\right) &= \left\| \frac{w_k + v}{2} \right\|^2 - 2\left\langle \frac{w_k + v}{2}, Jx_k \right\rangle + \|x_k\|^2 \\ &\leq \frac{1}{2}\|w_k\|^2 + \frac{1}{2}\|v\|^2 - \frac{1}{4}K(\|w_k - v\|) - \langle w_k + v, Jx_k \rangle + \|x_k\|^2 \\ &= \frac{1}{2}\phi(w_k, x_k) + \frac{1}{2}\phi(v, x_k) - \frac{1}{4}K(\|w_k - v\|) \leq l + \sigma - \frac{1}{4}K(\epsilon_0), \end{aligned} \tag{4.17}$$

for all $n \geq 0$. Hence,

$$l \leq \lim_{n \to \infty} \phi\left(\frac{w_k + v}{2}, x_n\right) = \lim_{n \to \infty} \phi\left(\frac{w_k + v}{2}, x_{n+k}\right) \leq l + \sigma - \frac{1}{4}K(\epsilon_0) < l + \sigma - \sigma = l. \tag{4.18}$$

This is a contradiction. So, $\{w_n\}$ converges strongly to $v \in F := \mathrm{VI}(C, A) \cap T^{-1}(0) \cap \mathrm{MEP}(\Theta, \varphi)$. Consequently, $v \in F$ is the unique element of F such that

$$\lim_{n\to\infty} \phi(v, x_n) = \min_{y\in F} \lim_{n\to\infty} \phi(y, x_n). \tag{4.19}$$

This completes the proof. □

Now, we prove a weak convergence theorem for the algorithm (4.20) below under different condition on data.

Theorem 4.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Let $T : E \to E^*$ be a maximal monotone operator and let $J_r = (J + rT)^{-1}J$ for $r > 0$. Let C be a nonempty closed convex subset of E such that $D(T) \subset C \subset J^{-1}(\bigcap_{r>0} R(J + rT))$, let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, and let A be an α-inverse-strongly monotone operator of C into E^* with $F := VI(C, A) \cap T^{-1}(0) \cap MEP(\Theta, \varphi) \neq \emptyset$ and $\|Ay\| \le \|Ay - Au\|$ for all $y \in C$ and $u \in F$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{aligned} u_n &= K_{r_n} x_n, \\ z_n &= \Pi_C J^{-1}(Ju_n - \lambda_n A u_n), \\ y_n &= J^{-1}(\beta_n J(x_n) + (1-\beta_n) J(J_{r_n} z_n)), \\ x_{n+1} &= \Pi_C J^{-1}(\alpha_n J(x_1) + (1-\alpha_n) J(y_n)), \end{aligned} \tag{4.20}$$

for $n \in \mathbb{N} \cup \{0\}$, where Π_C is the generalized projection from E onto C, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ satisfying $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\limsup_{n\to\infty} \beta_n < 1$ $\liminf_{n\to\infty} r_n > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, $1/c$ is the 2-uniformly convexity constant of E. Then, the sequence $\{x_n\}$ converges weakly to an element v of F, where $v = \lim_{n\to\infty} \Pi_F x_n$.

Proof. By Theorem 4.1, we have $\{x_n\}$ is bounded and so are $\{z_n\}$, $\{J_{r_n} z_n\}$.

From (4.9), we obtain

$$\begin{aligned} \phi(p, x_{n+1}) &\le \alpha_n \phi(p, x_1) + (1-\alpha_n)\phi(p, y_n) \\ &\le \alpha_n \phi(p, x_1) + (1-\alpha_n)\left[\beta_n \phi(p, x_n) + (1-\beta_n)(\phi(p, z_n) - \phi(J_{r_n} z_n, z_n))\right] \\ &\le \alpha_n \phi(p, x_1) + (1-\alpha_n)\left[\beta_n \phi(p, x_n) + (1-\beta_n)(\phi(p, x_n) - \phi(J_{r_n} z_n, z_n))\right] \\ &\le \alpha_n \phi(p, x_1) + (1-\alpha_n\beta_n)\phi(p, x_n) - (1-\alpha_n)(1-\beta_n)\phi(J_{r_n} z_n, z_n), \end{aligned} \tag{4.21}$$

and then

$$(1-\alpha_n)(1-\beta_n)\phi(J_{r_n} z_n, z_n) \le \alpha_n \phi(p, x_1) + (1-\alpha_n\beta_n)\phi(p, x_n) - \phi(p, x_{n+1}). \tag{4.22}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\limsup_{n\to\infty}\beta_n < 1$ and $\{\phi(p, x_n)\}$ exists, then we have

$$\lim_{n\to\infty}\phi(J_{r_n}z_n, z_n) = 0. \tag{4.23}$$

By again Lemma 2.4, we have $\lim_{n\to\infty}\|J_{r_n}z_n - z_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty}\|J(J_{r_n}z_n) - J(z_n)\| = 0. \tag{4.24}$$

Apply (4.7), (4.8), and (4.9), we observe that

$$\begin{aligned}
&\phi(p, x_{n+1})\\
&\quad\le \alpha_n\phi(p, x_1) + (1-\alpha_n)\phi(p, y_n) \le \alpha_n\phi(p, x_1) + (1-\alpha_n)\left[\beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, z_n)\right]\\
&\quad\le \alpha_n\phi(p, x_1) + (1-\alpha_n)\left[\beta_n\phi(p, x_n) + (1-\beta_n)\left[\phi(p, u_n) - 2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Au_n - Ap\|^2\right]\right]\\
&\quad\le \alpha_n\phi(p, x_1) + (1-\alpha_n)\left[\beta_n\phi(p, x_n) + (1-\beta_n)\left[\phi(p, x_n) - 2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Au_n - Ap\|^2\right]\right]\\
&\quad\le \alpha_n\phi(p, x_1) + (1-\alpha_n)\phi(p, x_n) - (1-\alpha_n)(1-\beta_n)2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Au_n - Ap\|^2,
\end{aligned} \tag{4.25}$$

and hence

$$2\lambda_n\left(\alpha - \frac{2}{c^2}\lambda_n\right)\|Au_n - Ap\|^2 \le \frac{1}{(1-\alpha_n)(1-\beta_n)}\left(\alpha_n\phi(p, x_1) + (1-\alpha_n)\phi(p, x_n) - \phi(p, x_{n+1})\right), \tag{4.26}$$

for all $n \in \mathbb{N}$. Since $0 < a \le \lambda_n \le b < c^2\alpha/2$, $\lim_{n\to\infty}\alpha_n = 0$ and $\limsup_{n\to\infty}\beta_n < 1$, we have

$$\lim_{n\to\infty}\|Au_n - Ap\| = 0. \tag{4.27}$$

From Lemmas 2.6, 2.7, and (4.7), we get

$$\begin{aligned}
\phi(u_n, z_n) = \phi(u_n, \Pi_C v_n) \le \phi(u_n, v_n) &= \phi\left(u_n, J^{-1}(Ju_n - \lambda_n Au_n)\right) = V(u_n, Ju_n - \lambda_n Au_n)\\
&\le V(u_n, (Ju_n - \lambda_n Au_n) + \lambda_n Au_n) - 2\left\langle J^{-1}(Ju_n - \lambda_n Au_n) - x_n, \lambda_n Au_n\right\rangle\\
&= \phi(u_n, u_n) + 2\langle v_n - u_n, \lambda_n Au_n\rangle = 2\langle v_n - u_n, \lambda_n Au_n\rangle\\
&\le \frac{4\lambda_n^2}{c^2}\|Au_n - Ap\|^2.
\end{aligned} \tag{4.28}$$

From Lemma 2.4 and (4.27), we have

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{4.29}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|J(u_n) - J(z_n)\| = 0. \tag{4.30}$$

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup u \in C$. It follows that $J_{r_{n_i}} z_{n_i} \rightharpoonup u$ and $u_{n_i} \rightharpoonup u \in C$ as $i \to \infty$.

Now, we claim that $u \in F$. First, we show that $u \in T^{-1}0$. Indeed, since $\liminf_{n\to\infty} r_n > 0$, it follows that

$$\lim_{n\to\infty} \|A_{r_n} z_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|Jz_n - J(J_{r_n} z_n)\| = 0. \tag{4.31}$$

If $(z, z^*) \in T$, then it holds from the monotonicity of T that

$$\left\langle z - J_{r_{n_i}} z_{n_i}, z^* - A_{r_{n_i}} z_{n_i} \right\rangle \geq 0, \tag{4.32}$$

for all $i \in \mathbb{N}$. Letting $i \to \infty$, we get $\langle z-u, z^*\rangle \geq 0$. Then, the maximality of T implies $u \in T^{-1}0$.

Next, we show that $u \in \mathrm{VI}(C, A)$. Let $B \subset E \times E^*$ be an operator as follows:

$$Bv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{4.33}$$

By Theorem 2.9, B is maximal monotone and $B^{-1}0 = \mathrm{VI}(A, C)$. Let $(v, w) \in G(B)$. Since $w \in Bv = Av + N_C(v)$, we get $w - Av \in N_C(v)$. From $z_n \in C$, we have

$$\langle v - z_n, w - Av \rangle \geq 0. \tag{4.34}$$

On the other hand, since $z_n = \Pi_C J^{-1}(Ju_n - \lambda_n Au_n)$. Then, by Lemma 2.5, we have

$$\langle v - z_n, Jw_n - (Ju_n - \lambda_n Au_n) \rangle \geq 0. \tag{4.35}$$

Thus,

$$\left\langle v - z_n, \frac{Ju_n - Jz_n}{\lambda_n} - Au_n \right\rangle \leq 0. \tag{4.36}$$

It follows from (4.34) and (4.36) that

$$\begin{aligned}\langle v - z_n, w\rangle &\geq \langle v - z_n, Av\rangle \geq \langle v - z_n, Av\rangle + \left\langle v - z_n, \frac{Ju_n - Jz_n}{\lambda_n} - Ax_n\right\rangle \\ &= \langle v - z_n, Av - Au_n\rangle + \left\langle v - z_n, \frac{Ju_n - Jz_n}{\lambda_n}\right\rangle \\ &= \langle v - z_n, Av - Az_n\rangle + \langle v - z_n, Az_n - Au_n\rangle + \left\langle v - z_n, \frac{Ju_n - Jz_n}{\lambda_n}\right\rangle \\ &\geq -\|v - z_n\|\frac{\|z_n - u_n\|}{\alpha} - \|v - z_n\|\frac{\|Ju_n - Jz_n\|}{a} \\ &\geq -M\left(\frac{\|z_n - u_n\|}{\alpha} + \frac{\|Ju_n - Jz_n\|}{a}\right),\end{aligned} \tag{4.37}$$

where $M = \sup_{n\geq 1}\{\|v - z_n\|\}$. From (4.29) and (4.30), we obtain $\langle v - u, w\rangle \geq 0$. By the maximality of B, we have $u \in B^{-1}0$ and hence $u \in \mathrm{VI}(C, A)$.

Next, we show $u \in \mathrm{MEP}(f) = F(K_r)$. From $u_n = K_{r_n}x_n$. It follows from (4.7), (4.8), and (4.9) that

$$\begin{aligned}\phi(p, x_{n+1}) &\leq \alpha_n\phi(p, x_1) + (1-\alpha_n)\phi(p, y_n) \\ &\leq \alpha_n\phi(p, x_1) + (1-\alpha_n)\left[\beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, z_n)\right] \\ &\leq \alpha_n\phi(p, x_1) + (1-\alpha_n)\left[\beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, u_n)\right] \\ &\leq \alpha_n\phi(p, x_1) + (1-\alpha_n)\left[\beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, x_n)\right],\end{aligned} \tag{4.38}$$

or, equivalently,

$$\phi(p, x_{n+1}) - \alpha_n\phi(p, x_1) \leq (1-\alpha_n)\left[\beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, u_n)\right] \leq (1-\alpha_n)\phi(p, x_n), \tag{4.39}$$

with $\lim_{n\to\infty}\alpha_n = 0$ and $\limsup_{n\to\infty}\beta_n < 1$, yield that $\lim_{n\to\infty}\phi(p, u_n) = \lim_{n\to\infty}\phi(p, x_n)$.

From Lemmas 2.13 and 2.14, for $p \in F$,

$$\phi(u_n, x_n) \leq \phi(p, x_n) - \phi(p, u_n). \tag{4.40}$$

This implies that $\lim_{n\to\infty}\phi(u_n, x_n) = 0$. Noticing Lemma 2.4, we get

$$\|u_n - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{4.41}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty}\|Ju_n - Jx_n\| = 0. \tag{4.42}$$

From (4.20) and (A2), we also have

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geq \Theta(y, u_n), \quad \forall y \in C. \tag{4.43}$$

Hence,

$$\varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{Ju_{n_i} - Jx_{n_i}}{r_{n_i}} \right\rangle \geq \Theta(y, u_{n_i}), \quad \forall y \in C. \tag{4.44}$$

From $\|u_n - z_n\| \to 0$, we get $u_{n_i} \rightharpoonup u$. Since $(Ju_{n_i} - Jx_{n_i}/r_{n_i}) \to 0$, it follows by (A4) and the weakly lower semicontinuous of φ that

$$\Theta(y, u) + \varphi(u) - \varphi(y) \leq 0, \quad \forall y \in C. \tag{4.45}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)u$. Since $y \in C$ and $u \in C$, we have $y_t \in C$ and hence $\Theta(y_t, u) + \varphi(u) - \varphi(y_t) \leq 0$. So, from (A1), (A4), and the convexity of φ, we have

$$\begin{aligned} 0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, u) + t\varphi(y) + (1-t)\varphi(y) - \varphi(y_t) \\ &\leq t(\Theta(y_t, y) + \varphi(y) - \varphi(y_t)). \end{aligned} \tag{4.46}$$

Dividing by t, we get $\Theta(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0$. From (A3) and the weakly lower semicontinuity of φ, we have $\Theta(u, y) + \varphi(y) - \varphi(u) \geq 0$ for all $y \in C$ implies $u \in \mathrm{MEP}(\Theta, \varphi)$. Hence, $u \in F := \mathrm{VI}(C, A) \cap T^{-1}(0) \cap \mathrm{MEP}(\Theta, \varphi)$.

By Theorem 4.1, the $\{\Pi_F x_n\}$ converges strongly to a point $v \in F$ which is a unique element of F such that

$$\lim_{n\to\infty} \phi(v, x_n) = \min_{y\in F} \lim_{n\to\infty} \phi(y, x_n). \tag{4.47}$$

By the uniform smoothness of E, we also have $\lim_{n\to\infty}\|J\Pi_F x_{n_i} - Jv\| = 0$.

Finally, we prove $u = v$. From Lemma 2.5 and $u \in F$, we have

$$\langle \Pi_F x_{n_i} - u, Jx_{n_i} - J\Pi_F x_{n_i}\rangle \geq 0. \tag{4.48}$$

Since J is weakly sequentially continuous, $u_{n_i} \rightharpoonup u$ and $u_n - x_n \to 0$. Then,

$$\langle v - u, Ju - Jv\rangle \geq 0. \tag{4.49}$$

On the other hand, since J is monotone, we have

$$\langle v - u, Ju - Jv\rangle \leq 0. \tag{4.50}$$

Hence,

$$\langle v-u, Ju-Jv\rangle = 0. \tag{4.51}$$

Since E is strict convexity, it follows that $u = v$. Therefore, the sequence $\{x_n\}$ converges weakly to $v = \lim_{n\to\infty}\Pi_F x_n$. This completes the proof. □

5. Application to Complementarity Problems

Let C be a nonempty, closed convex cone in E and A an operator of C into E^*. We define its *polar* in E^* to be the set

$$K^* = \{y^* \in E^* : \langle x, y^*\rangle \geq 0,\ \forall x \in C\}. \tag{5.1}$$

Then, the element $u \in C$ is called a solution of the *complementarity problem* if

$$Au \in K^*, \quad \langle u, Au\rangle = 0. \tag{5.2}$$

The set of solutions of the complementarity problem is denoted by $\mathrm{CP}(K, A)$; see [34], for more detial.

Theorem 5.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space and let K be a nonempty closed convex subset of E. Let Θ be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying (A1)–(A4) let $\varphi : K \to \mathbb{R}$ be a lower semicontinuous and convex function, and let $T : E \to E^*$ be a maximal monotone operator. Let $J_r = (J + rT)^{-1}J$ for $r > 0$ and let A be an α-inverse-strongly monotone operator of K into E^* with $F := T^{-1}(0) \cap CP(K, A) \cap MEP(\Theta, \varphi) \neq \emptyset$ and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in K$ and $u \in F$. For an initial point $x_0 \in E$ with $x_1 = \Pi_{C_1}x_0$ and $K_1 = K$,*

$$\begin{aligned}
&w_n = \Pi_K J^{-1}(Jx_n - \lambda_n Ax_n),\\
&z_n = J^{-1}(\beta_n J(x_n) + (1-\beta_n)J(J_{r_n}w_n)),\\
&y_n = J^{-1}(\alpha_n J(x_1) + (1-\alpha_n)J(z_n)),\\
&u_n \in K \quad \text{such that } \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in K,\\
&K_{n+1} = \{z \in K_n :\ \phi(z, u_n) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\},\\
&x_{n+1} = \Pi_{K_{n+1}}x_0,
\end{aligned} \tag{5.3}$$

for $n \in \mathbb{N}$, where Π_K is the generalized projection from E onto K and J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\limsup_{n\to\infty}\beta_n < 1$, $\liminf_{n\to\infty} r_n > 0$ and $\{\lambda_n\} \subset [a,b]$ for some a, b with $0 < a < b < c^2\alpha/2$, $1/c$ is the 2-uniformly convexity constant of E. Then, the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

Proof. As in the proof Lemma 7.1.1 of Takahashi in [44], we have $\mathrm{VI}(C, A) = \mathrm{CP}(K, A)$. So, we obtain the desired result. □

Theorem 5.2. *Let E be a 2-uniformly convex and uniformly smooth Banach space whose duality mapping J is weakly sequentially continuous. Let $T : E \to E^*$ be a maximal monotone operator and let $J_r = (J + rT)^{-1}J$ for $r > 0$. Let K be a nonempty closed convex subset of E such that $D(T) \subset K \subset J^{-1}(\bigcap_{r>0} R(J + rT))$, let Θ be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying (A1)–(A4), let $\varphi : K \to \mathbb{R}$ be a proper lower semicontinuous and convex function, and let A be an α-inverse-strongly monotone operator of K into E^* with $F := CP(K, A) \cap T^{-1}(0) \cap MEP(\Theta, \varphi) \neq \emptyset$ and $\|Ay\| \leq \|Ay - Au\|$ for all $y \in K$ and $u \in F$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in K$ and*

$$\begin{aligned} u_n &= K_{r_n} x_n, \\ z_n &= \Pi_K J^{-1}(Ju_n - \lambda_n A u_n), \\ y_n &= J^{-1}\big(\beta_n J(x_n) + (1 - \beta_n) J(J_{r_n} z_n)\big), \\ x_{n+1} &= \Pi_K J^{-1}\big(\alpha_n J(x_1) + (1 - \alpha_n) J(y_n)\big), \end{aligned} \tag{5.4}$$

for $n \in \mathbb{N} \cup \{0\}$, where Π_K is the generalized projection from E onto K, J is the duality mapping on E. The coefficient sequence $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ satisfying $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\limsup_{n\to\infty} \beta_n < 1$ $\liminf_{n\to\infty} r_n > 0$ and $\{\lambda_n\} \subset [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, $1/c$ is the 2-uniformly convexity constant of E. Then, the sequence $\{x_n\}$ converges weakly to an element v of F, where $v = \lim_{n\to\infty} \Pi_F x_n$.

Proof. It follows by Lemma 7.1.1 of Takahashi in [44], we have $\mathrm{VI}(C, A) = \mathrm{CP}(K, A)$. Hence, Theorem 4.2, $\{x_n\}$ converges weakly to an element v of F, where $v = \lim_{n\to\infty} \Pi_F x_n$. □

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