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Research Article

A Construction of Mirror Q-Algebras

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We investigate how to construct mirror Q-algebras of a Q-algebra, and we obtain the necessary conditions for $M(X)$ to be a Q-algebra.

1. Introduction

Imai and Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [1, 2]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. We refer the reader for useful textbooks for BCK/BCI-algebra to [3–5]. Neggers et al. [6] introduced the notion of Q-algebras which is a generalization of $BCK/BCI/BCH$-algebras, obtained several properties, and discussed quadratic Q-algebras. Ahn and Kim [7] introduced the notion of QS-algebras, and Ahn et al. [8] studied positive implicativity in Q-algebras and discussed some relations between $R-(L-)$ maps and positive implicativity. Neggers and Kim introduced the notion of d-algebras which is another useful generalization of BCK-algebras and then investigated several relations between d-algebras and BCK-algebras as well as several other relations between d-algebras and oriented digraphs [9]. After that some further aspects were studied [10–13]. Allen et al. [14] introduced the notion of mirror image of given algebras and obtained some interesting properties: a mirror algebra of a d-algebra is also a d-algebra, and a mirror algebra of an implicative BCK-algebra is a left L-up algebra.

In this paper we introduce the notion of mirror algebras to Q-algebras, and we investigate how to construct mirror Q-algebras from a Q-algebra; and we also obtain the necessary conditions for $M(X)$ to be a Q-algebra.

2. Q-Algebras and Related Algebras

A *Q-algebra* [6] is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

For brevity we also call X a *Q-algebra*. In X we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

Example 2.1 (see [6]). Let $X : \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	0	0	3	2
2	2	0	0	0
3	3	3	3	0

(2.1)

Then $(X, *, 0)$ is a Q-algebra, which is not a $BCK/BCI/BCH$-algebra.

Ahn and Kim [7] introduced the notion of QS-algebras. They showed that the G-part of an associative QS-algebra is a group in which every element is an involution. A Q-algebra X is said to be a *QS-algebra* if it satisfies the following condition:

(IV) $(x * y) * (x * z) = z * y$, for all $x, y, z \in X$.

Proposition 2.2 (see [6]). *If $(X, *, 0)$ is a Q-algebra, then*

(V) $(x * (x * y)) * y = 0$, *for all $x, y \in X$.*

It was proved that every BCH-algebra is a Q-algebra and every Q-algebra satisfying some additional conditions is a BCI-algebra.

Neggers and Kim [15] introduced the notion of B-algebras which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras and which seems to have rather nice properties without being excessively complicated otherwise. And they demonstrated some interesting connections between B-algebras and groups.

Example 2.3. Let $X := \{0, 1, 2, \ldots, \omega\}$ be a set. Define a binary operation "$*$" on X by

$$x * y := \begin{cases} 0, & x \leq y, \\ \omega, & y < x,\ x \neq 0, \\ x, & y < x,\ y = 0. \end{cases} \tag{2.2}$$

Then $(X, *, 0)$ is a Q-algebra, but not a B-algebra, since $(3 * \omega) * 0 = 0,\ 3 * (0 * (0 * \omega)) = 3$.

Example 2.4. Let $X := \{0, 1, \ldots, 5\}$ be a set with the following table:

$$
\begin{array}{c|cccccc}
* & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & 2 & 1 & 3 & 4 & 5 \\
1 & 1 & 0 & 2 & 4 & 5 & 3 \\
2 & 2 & 1 & 0 & 5 & 3 & 4 \\
3 & 3 & 4 & 5 & 0 & 2 & 1 \\
4 & 4 & 5 & 3 & 1 & 0 & 2 \\
5 & 5 & 3 & 4 & 2 & 1 & 0
\end{array}
\tag{2.3}
$$

Then $(X, *, 0)$ is a B-algebra, but not a Q-algebra, since $(5 * 3) * 1 = 1$, $(5 * 1) * 3 = 0$.

Example 2.5. Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x * y := \frac{n(x - y)}{n + y} \tag{2.4}$$

for any $x, y \in X$. Then $(X, *, 0)$ is both a Q-algebra and B-algebra.

If we consider several families of abstract algebras including the well-known BCK-algebras and several larger classes including the class of d-algebras which is a generalization of BCK-algebras, then it is usually difficult and often impossible to obtain a complementation operation and the associated "de Morgan's laws." In the sense of this point of view it is natural to construct a "mirror image" of a given algebra which when adjoined to the original algebra permits a natural complementation to take place. The class of BCK-algebras is not closed under this operation but the class of d-algebras is, thus explaining why it may be better to work with this class rather than the class of BCK-algebras. Allen et al. [14] introduced the notion of mirror algebras of a given algebra.

Let $(X, *, 0)$ be an algebra. Let $M(X) := X \times \{0, 1\}$, and define a binary operation "$*$" on $M(X)$ as follows:

$$
\begin{gathered}
(x, 0) * (y, 0) := (x * y, 0), \\
(x, 1) * (y, 1) := (y * x, 0), \\
(x, 0) * (y, 1) := (x * (x * y), 0), \\
(x, 1) * (y, 0) := \begin{cases} (y, 1) & \text{when } x * y = 0, \\ (x, 1) & \text{when } x * y \neq 0. \end{cases}
\end{gathered}
\tag{2.5}
$$

Then we say that $M(X) := (M(X), *, (0, 0))$ is a *left mirror algebra* of the algebra $(X, *, 0)$. Similarly, if we define

$$(x, *) * (y, 1) := (y * (y * x), 0), \tag{2.6}$$

then $M(X) := (M(X), *, (0, 0))$ is a *right mirror algebra* of the algebra $(X, *, 0)$.

It was shown [14] that the mirror algebra of a d (resp., d-BH)-algebra is also a d (resp., d-BH)-algebra, but the mirror algebra of a BCK-algebra need not be a BCK-algebra.

3. A Construction of Mirror Q-Algebras

In [14] Allen et al. defined (left, right) mirror algebras of an algebra, but it is not known how to construct mirror algebras of any given algebra. In this paper, we investigate a construction of a mirror algebra in Q-algebras.

Let $(X, *, 0)$ be a Q-algebra, and let $M(X) := X \times \{0, 1\}$. Define a binary operation "$\oplus$" on $M(X)$ by

(M1) $(x, 0) \oplus (y, 0) = (x * y, 0)$,

(M2) $(x, 1) \oplus (y, 1) = (y * x, 0)$,

(M3) $(x, 0) \oplus (y, 1) = (\alpha(x, y), 0)$,

(M4) $(x, 1) \oplus (y, 0) = (\beta(x, y), 1)$,

where $\alpha, \beta : X \times X \to X$ are mappings.

Consider condition (I). If we let $x = y$ in (1) and (2), then (I) holds trivially. Consider condition (II). For any $(x, 0) \in M(X)$, we have $(x, 0) \oplus (0, 0) = (x * 0, 0) = (x, 0)$. For any $(x, 1) \in M(X)$, we have $(x, 1) = (x, 1) \oplus (0, 0) = (\beta(x, 0), 1)$, which shows that the required condition is $\beta(x, 0) = x$. Consider condition (III). There are 8 cases to check that condition (III) holds.

Case 1 $((x, 0), (y, 0), (z, 0))$. It holds trivially.

Case 2 $((x, 0), (y, 1), (z, 0))$. Since $((x, 0) \oplus (y, 1)) \oplus (z, 0) = (\alpha(x, y), 0) \oplus (z, 0) = (\alpha(x, y) * z, 0)$ and $((x, 0) \oplus (z, 0)) \oplus (y, 1) = (x * z, 0) \oplus (y, 1) = (\alpha(x * z, y), 0)$, we obtain the requirement that $\alpha(x, y) * z = \alpha(x * z, y)$.

Case 3 $((x, 0), (y, 0), (z, 1))$. It is the same as Case 2.

Case 4 $((x, 0), (y, 1), (z, 1))$. Since $((x, 0) \oplus (y, 1)) \oplus (z, 1) = (\alpha(x, y), 0) \oplus (z, 1) = (\alpha(\alpha(x, y), z), 0)$ and $((x, 0) \oplus (z, 1)) \oplus (y, 1) = (\alpha(x, z), 0) \oplus (y, 1) = (\alpha(\alpha(x, z), y), 0)$, we obtain the requirement that $\alpha(\alpha(x, y), z) = \alpha(\alpha(x, z), y)$.

Case 5 $((x, 1), (y, 0), (z, 0))$. Since $((x, 1) \oplus (y, 0)) \oplus (z, 0) = (\beta(x, y), 1) \oplus (z, 0) = (\beta(\beta(x, y), z), 0)$ and $((x, 1) \oplus (z, 0)) \oplus (y, 0) = (\beta(x, z), 1) \oplus (y, 0) = (\beta(\beta(x, z), y), 0)$, we obtain the requirement that $\beta(\beta(x, y), z) = \beta(\beta(x, z), y)$.

Case 6 $((x, 1), (y, 0), (z, 1))$. Since $((x, 1) \oplus (y, 0)) \oplus (z, 1) = (\beta(x, y), 1) \oplus (z, 1) = (z * \beta(x, y), 0)$ and $((x, 1) \oplus (z, 1)) \oplus (y, 0) = (z * x, 0) \oplus (y, 0) = ((z * x) * y, 0)$, we obtain the requirement that $z * \beta(x, y) = (z * x) * y$.

Case 7 $((x, 1), (y, 1), (z, 0))$. It is the same as Case 6.

Case 8 $((x, 1), (y, 1), (z, 1))$. Since $((x, 1) \oplus (y, 1)) \oplus (z, 1) = (\beta(x, y), 0) \oplus (z, 1) = (\alpha(\beta(x, y), z), 0)$ and $((x, 1) \oplus (z, 1)) \oplus (y, 1) = (\alpha(\beta(x, z), y), 0)$ by exchanging y with z, we obtain the requirement that $\alpha(\beta(x, y), z) = \alpha(\beta(x, z), y)$. If we summarize this discussion, we obtain the following theorem.

Theorem 3.1. *Let $(X, *, 0)$ be a Q-algebra, and let $M(X) := X \times \{0, 1\}$ be a set with a binary operation "$\oplus$" on $M(X)$ with $(M1) \sim (M4)$. Then the necessary conditions for $(M(X), \oplus, (0, 0))$ to be a Q-algebra are the following:*

(i) $\beta(x, 0) = x$,

(ii) $\alpha(\alpha(x, y), z) = \alpha(\alpha(x, z), y)$,

(iii) $\alpha(x, y) * z = \alpha(x * z, y)$,

(iv) $\beta(\beta(x, y), z) = \beta(\beta(x, z), y)$,

(v) $z * \beta(x, y) = (z * x) * y$,

(vi) $\alpha(\beta(x, y), z) = \alpha(\beta(x, z), y)$

for any $x, y, z \in X$.

Remark 3.2. By condition $(M1)$, if we identify $(x, 0) \equiv x$ for any $x \in X$, then X is a subalgebra of $M(X)$. By applying Theorem 3.1, we obtain many (mirror) Q-algebras: $X \subseteq M(X) \subseteq M(M(X)) = M^2(X) \subseteq M^3(X) \subseteq M^4(X) \subseteq \cdots$.

Example 3.3. Let Z be the set of all integers. Then $(Z, -, 0)$ is a Q-algebra where "$-$" is the usual subtraction in Z. If we define mappings $\alpha, \beta : Z \times Z \to Z$ by $\alpha(x, y) = \beta(x, y) = x + y$ for any $x, y \in Z$, then the mirror algebra $(M(Z), \oplus, (0, 0))$ is also a Q-algebra, that is, $(x, 0) \oplus (y, 0) = (x - y, 0)$, $(x, 1) \oplus (y, 1) = (y - x, 0)$, $(x, 0) \oplus (y, 1) = (x + y, 0)$, and $(x, 1) \oplus (y, 0) = (x + y, 1)$.

Example 3.4. Let $X := \{0, 1\}$ be a set with the following table:

$*$	0	1
0	0	0
1	1	0

(3.1)

Then $(X, *, 0)$ is a Q-algebra. Using the same method we obtain its mirror algebra as follows: $M(X) = \{0, \alpha, \beta, \gamma\}$ with the following table:

$\oplus$	0	α	β	γ
0	0	0	0	0
α	α	0	α	0
β	β	β	0	0
γ	γ	β	α	0

(3.2)

where $0 := (0, 0)$, $\alpha := (0, 1)$, $\beta := (1, 0)$, and $\gamma := (1, 1)$. It is easy to see that $(M(X), \oplus, 0)$ is a Q-algebra.

Problems

(1) Find necessary conditions for $M(X)$ to be a QS-algebra if $(X, *, 0)$ is a QS-algebra.

(2) Given a homomorphism $f : X \to Y$ of Q-algebras, construct a homomorphism $\widehat{f} : M(X) \to M(Y)$ of Q-algebras which is an extension of f.

(3) Given Q-algebras X, Y, are the mirror algebras $M(M(X \times Y))$ and $M(X) \times M(Y)$ isomorphic?

References

[1] K. Iséki, "On BCI-algebras," *Mathematics Seminar Notes*, vol. 8, no. 1, pp. 125–130, 1980.
[2] K. Iséki and S. Tanaka, "An introduction to the theory of BCK-algebras," *Mathematica Japonica*, vol. 23, no. 1, pp. 1–26, 1978.
[3] A. Iorgulescu, *Algebras of Logic as BCK Algebras*, Editura ASE, Bucharest, Romania, 2008.
[4] J. Meng and Y. B. Jun, *BCK-Algebras*, Kyung Moon Sa, Seoul, Republic of Korea, 1994.
[5] H. Yisheng, *BCI-Algebras*, Science Press, Beijing, China, 2006.
[6] J. Neggers, S. S. Ahn, and H. S. Kim, "On Q-algebras," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 12, pp. 749–757, 2001.
[7] S. S. Ahn and H. S. Kim, "On QS-algebras," *The Journal of Chungcheong Mathematical Society*, vol. 12, pp. 33–41, 1999.
[8] S. S. Ahn, H. S. Kim, and H. D. Lee, "R-maps and L-maps in Q-algebras," *International Journal of Pure and Applied Mathematics*, vol. 12, no. 4, pp. 419–425, 2004.
[9] J. Neggers and H. S. Kim, "On d-algebras," *Mathematica Slovaca*, vol. 49, no. 1, pp. 19–26, 1999.
[10] P. J. Allen, H. S. Kim, and J. Neggers, "Companion d-algebras," *Mathematica Slovaca*, vol. 57, no. 2, pp. 93–106, 2007.
[11] P. J. Allen, H. S. Kim, and J. Neggers, "Deformations of d/BCK-algebras," to appear in *Bulletin of the Korean Mathematical Society*.
[12] J. Neggers, A. Dvure enskij, and H. S. Kim, "On d-fuzzy functions in d-algebras," *Foundations of Physics*, vol. 30, no. 10, pp. 1807–1816, 2000.
[13] J. Neggers, Y. B. Jun, and H. S. Kim, "On d-ideals in d-algebras," *Mathematica Slovaca*, vol. 49, no. 3, pp. 243–251, 1999.
[14] P. J. Allen, J. Neggers, and H. S. Kim, "L-up and mirror algebras," *Scientiae Mathematicae Japonicae*, vol. 59, no. 3, pp. 605–612, 2004.
[15] J. Neggers and H. S. Kim, "On B-algebras," *Matematichki Vesnik*, vol. 54, no. 1-2, pp. 21–29, 2002.