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Research Article

Restricted Algebras on Inverse Semigroups—Part II: Positive Definite Functions

Massoud Amini[1] and Alireza Medghalchi[2]

[1] *Department of Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O.Box Tehran 14115-134, Iran*
[2] *Faculty of Mathematical Sciences and Computer, Tarbiat Moalem University, 50 Taleghani Avenue, Tehran 15614, Iran*

Correspondence should be addressed to Massoud Amini, mamini@modares.ac.ir

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The relation between representations and positive definite functions is a key concept in harmonic analysis on topological groups. Recently this relation has been studied on topological groupoids. In this paper, we investigate the concept of restricted positive definite functions and their relation with restricted representations of an inverse semigroup. We also introduce the restricted Fourier and Fourier-Stieltjes algebras of an inverse semigroup and study their relation with the corresponding algebras on the associated groupoid.

1. Introduction

In [1] we introduced the concept of restricted representations for an inverse semigroup S and studied the restricted forms of some important Banach algebras on S. In this paper we continue our study by considering the relation between the restricted positive definite functions and restricted representations. In particular, in Section 2 we prove restricted version of the Godement's characterization of positive definite functions on groups (Theorem 2.9). In Section 3 we study the restricted forms of the Fourier and Fourier-Stieltjes algebras on S. The last section is devoted to the study of the Fourier and Fourier-Stieltjes algebras on the associated groupoid of S, as well as the C^*-algebra of certain related graph groupoids.

An inverse semigroup S is a discrete semigroup such that for each $s \in S$ there is a unique element $s^* \in S$ such that

$$ss^*s = s, \qquad s^*ss^* = s^*. \tag{1.1}$$

The set E of idempotents of S consists of elements of the form ss^*, $s \in S$. Then E is a commutative subsemigroup of S. There is a natural order $\leq$ on E defined by $e \leq f$ if and only if $ef = e$. A $*$- representation of S is a pair $\{\pi, \mathscr{H}_\pi\}$ consisting of a (possibly infinite dimensional) Hilbert space $\mathscr{H}_\pi$ and a map $\pi : S \to \mathcal{B}(\mathscr{H}_\pi)$ satisfying

$$\pi(xy) = \pi(x)\pi(y), \quad \pi(x^*) = \pi(x)^* \quad (x, y \in S), \tag{1.2}$$

that is, a $*$-semigroup homomorphism from S into the $*$-semigroup of partial isometries on $\mathscr{H}_\pi$. Let $\Sigma = \Sigma(S)$ be the family of all $*$-representations π of S with

$$\|\pi\| := \sup_{x \in S} \|\pi(x)\| \leq 1. \tag{1.3}$$

For $1 \leq p < \infty$, $\ell^p(S)$ is the Banach space of all complex valued functions f on S satisfying

$$\|f\|_p := \left(\sum_{x \in S} |f(x)|^p \right)^{1/p} < \infty. \tag{1.4}$$

For $p = \infty$, $\ell^\infty(S)$ consists of those f with $\|f\|_\infty := \sup_{x \in S} |f(x)| < \infty$. Recall that $\ell^1(S)$ is a Banach algebra with respect to the product

$$(f * g)(x) = \sum_{\mathrm{st}=x} f(s)g(t) \quad \left(f, g \in \ell^1(S)\right) \tag{1.5}$$

and $\ell^2(S)$ is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{x \in S} f(x)\overline{g(x)} \quad \left(f, g \in \ell^2(S)\right). \tag{1.6}$$

Let also put

$$\check{f}(x) = f(x^*), \quad \tilde{f}(x) = \overline{f(x^*)}, \tag{1.7}$$

for each $f \in \ell^p(S)$ $(1 \leq p \leq \infty)$.

Given $x, y \in S$, the restricted product of x, y is xy if $x^*x = yy^*$ and undefined, otherwise. The set S with its restricted product forms a groupoid S_a, called the associated groupoid of S [2]. If we adjoin a zero element 0 to this groupoid and put $0^* = 0$, we get an inverse semigroup S_r with the multiplication rule

$$x \cdot y = \begin{cases} xy, & \text{if } x^*x = yy^* \\ 0, & \text{otherwise} \end{cases} \quad (x, y \in S \cup \{0\}), \tag{1.8}$$

which is called the restricted semigroup of S. A restricted representation $\{\pi, \mathscr{H}_\pi\}$ of S is a map $\pi : S \to \mathcal{B}(\mathscr{H}_\pi)$ such that $\pi(x^*) = \pi(x)^* (x \in S)$ and

$$\pi(x)\pi(y) = \begin{cases} \pi(xy), & \text{if } x^*x = yy^*, \\ 0, & \text{otherwise,} \end{cases} \quad (x, y \in S). \tag{1.9}$$

Let $\Sigma_r = \Sigma_r(S)$ be the family of all restricted representations π of S with $\|\pi\| = \sup_{x\in S}\|\pi(x)\| \le 1$. It is not hard to guess that $\Sigma_r(S)$ should be related to $\Sigma(S_r)$. Let $\Sigma_0(S_r)$ be the set of all $\pi \in \Sigma(S_r)$ with $\pi(0) = 0$. Note that $\Sigma_0(S_r)$ contains all cyclic representations of S_r. Now it is clear that, via a canonical identification, $\Sigma_r(S) = \Sigma_0(S_r)$. Two basic examples of restricted representations are the restricted left and right regular representations λ_r and ρ_r of S [1]. For each $\xi, \eta \in \ell^1(S)$ put

$$(\xi \cdot \eta)(x) = \sum_{x^*x = yy^*} \xi(xy)\eta(y^*) \quad (x \in S), \tag{1.10}$$

then $(\ell^1(S), \cdot, \sim)$ is a semisimple Banach $*$-algebra [1] which is denoted by $\ell^1_r(S)$ and is called the restricted semigroup algebra of S.

All over this paper, S denotes a unital inverse semigroup with identity 1. E denotes the set of idempotents of S which consists of elements of the form ss^*, $s \in S$. $\Sigma = \Sigma(S)$ is the family of all $*$-representations π of S with

$$\|\pi\| := \sup_{x\in S}\|\pi(x)\| \le 1. \tag{1.11}$$

2. Restricted Positive Definite Functions

A bounded complex valued function $u : S \to \mathbb{C}$ is called positive definite if for all positive integers n and all $c_1, \dots, c_n \in \mathbb{C}$, and $x_1, \dots, x_n \in S$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c}_i c_j u(x_i^* x_j) \ge 0, \tag{2.1}$$

and it is called restricted positive definite if for all positive integers n and all $c_1, \dots, c_n \in \mathbb{C}$, and $x_1, \dots, x_n \in S$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c}_i c_j (\lambda_r(x_i)u)(x_j) \ge 0. \tag{2.2}$$

We denote the set of all positive definite and restricted positive definite functions on S by $P(S)$ and $P_r(S)$, respectively. The two concepts coincide for (discrete) groups.

It is natural to expect a relation between $P_r(S)$ and $P(S_r)$. Before checking this, note that S_r is hardly ever unital. This is important, as the positive definite functions in nonunital case should be treated with extra care [3]. Let us take any inverse semigroup T with possibly no unit. Of course, one can always adjoin a unit 1 to T with $1^* = 1$ to get a unital inverse semigroup $T^1 = T \cup \{1\}$ (if T happened to have a unit we put $T^1 = T$). However, positive definite functions on T do not necessarily extend to positive definite functions on T^1. Following [3], we consider the subset $P_e(T)$ of extendible positive definite functions on T which are those $u \in P(T)$ such that $u = \tilde{u}$, and there exists a constant $c > 0$ such that for all $n \geq 1$, $x_1, \ldots, x_n \in T$, and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\left| \sum_{i=1}^{n} c_i u(x_i) \right|^2 \leq c \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c}_i c_j u\left(x_i^* x_j\right). \tag{2.3}$$

If $\tau : \ell^\infty(T) \to \ell^1(T)^*$ is the canonical isomorphism, then τ maps $P_e(T)$ onto the set of extendible positive bounded linear functionals on $\ell^1(T)$ (those which are extendible to a positive bounded linear functional on $\ell^1(T^1)$), and the restriction of τ to $P_e(T)$ is an isometric affine isomorphism of convex cones [3, 1.1]. Also the linear span $B_e(T)$ of $P_e(T)$ is an algebra [3, 3.4] which coincides with the set of coefficient functions of $*$-representations of T [3, 3.2]. If T has a zero element, then so is T^1. In this case, we put $P_0(T) = \{u \in P(T) : u(0) = 0\}$ and $P_{0,e}(T) = P_0(T) \cap P_e(T)$. To each $u \in P_e(T)$, there corresponds a cyclic $*$- representation of $\ell^1(T^1)$ which restricts to a cyclic representation of T (see the proof of [3, 3.2]). Let ω be the direct sum of all cyclic representations of T obtained in this way, then the set of all coefficient functions of ω is the linear span of $P_e(T)$ [3, 3.2]. We call ω the universal representation of T.

All these arrangements are for $T = S_r$, as it is an inverse 0-semigroup which is not unital unless S is a group. We remind the reader that our blanket assumption is that S is a unital inverse semigroup. From now on, we also assume that S has no zero element (see Example 2.2).

Lemma 2.1. *The restriction map $\tau : P_0(S_r) \to P_r(S)$ is an affine isomorphism of convex cones.*

Proof. Let $u \in P(S_r)$. For each $n \geq 1$, $x_1, \ldots, x_n \in S_r$ and $c_1, \ldots, c_n \in \mathbb{C}$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c}_i c_j u\left(x_i^* \cdot x_j\right) = \sum_{x_i x_i^* = x_j x_j^*} \overline{c}_i c_j u\left(x_i^* x_j\right) + u(0) \left(\sum_{x_i x_i^* \neq x_j x_j^*} \overline{c}_i c_j \right), \tag{2.4}$$

in particular if $u \in P_0(S_r)$, then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c}_i c_j u\left(x_i^* \cdot x_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{c}_i c_j \left(\lambda_r(x_i) u\right)\left(x_j\right), \tag{2.5}$$

so τ maps $P_0(S_r)$ into $P_r(S)$.

τ is clearly an injective affine map. Also if $u \in P_r(S)$ and v is extension by zero of u on S_r, then from the above calculation applied to v, $v \in P_0(S_r)$ and $\tau(v) = u$, so τ is surjective. $\square$

It is important to note that the restriction map τ may fail to be surjective when S already has a zero element.

Example 2.2. If $S = [0,1]$ with discrete topology and operations

$$xy = \max\{x,y\}, \quad x^* = x \quad (0 \le x, y \le 1), \tag{2.6}$$

then S is a zero inverse semigroup with identity. Here $S_r = S$, as sets, $P(S) = \{u : u \geq 0, u \text{ is decreasing}\}$ [4], but the constant function $u = 1$ is in $P_r(S)$. This in particular shows that the map τ is not necessarily surjective, if S happens to have a zero element. To show that $1 \in P_r(S)$ note that for each $n \geq 1$, each $c_1, \ldots, c_n \in \mathbb{C}$ and each $x_1, \ldots x_n \in S$, if $y_1, \ldots, y_k$ are distinct elements in $\{x_1, \ldots, x_n\}$, then for $J_l := \{j : 1 \le j \le n, x_j = y_l\}$, we have $J_i = J_l$, whenever $i \in J_l$, for each $1 \le i, l \le k$. Hence

$$\begin{aligned}\sum_{i,j=1}^{n} \overline{c}_i c_j \lambda_r(x_i) 1(x_j) &= \sum_{i=1}^{n} \overline{c}_i \left(\sum_{x_j = x_i} c_j \right) = \sum_{l=1}^{k} \left(\sum_{i \in J_l} \overline{c}_i \left(\sum_{j \in J_i} c_j \right) \right) \\ &= \sum_{l=1}^{k} \left(\sum_{i \in J_l} \overline{c}_i \left(\sum_{j \in J_i} c_j \right) \right) = \sum_{l=1}^{k} \left| \sum_{i \in J_l} c_i \right|^2 \geq 0.\end{aligned} \tag{2.7}$$

Notation 1. Let $P_{0,e}(S_r)$ be the set of all extendible elements of $P_0(S_r)$. This is a subcone which is mapped isomorphically onto a subcone $P_{r,e}(S)$ by τ. The elements of $P_{r,e}(S)$ are called extendible restricted positive definite functions on S. These are exactly those $u \in P_r(S)$ such that $u = \tilde{u}$, and there exists a constant $c > 0$ such that for all $n \geq 1$, $x_1, \ldots, x_n \in$ S and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\left| \sum_{i=1}^{n} c_i u(x_i) \right|^2 \le c \sum_{x_i x_i^* = x_j x_j^*} \overline{c}_i c_j u(x_i^* \mathrm{x}_j). \tag{2.8}$$

Proposition 2.3. *There is an affine isomorphism τ of convex cones from $P_{r,e}(S)$ onto*

$$\ell_r^1(S)_+^* \simeq (\mathbb{C}\delta_0)_+^\perp = \left\{ f \in \ell^\infty(S_r)_+ : f(0) = 0 \right\} =: \ell_0^\infty(S_r)_+. \tag{2.9}$$

Proof. The affine isomorphism $\ell_r^1(S)_+^* \simeq \ell_0^\infty(S_r)_+$ is just the restriction of the linear isomorphism of [1, Theorem 4.1] to the corresponding positive cones. Let us denote this by τ_3. In Notation 1 we presented an affine isomorphism τ_2 from $P_{e,e}(S_r)$ onto $P_{r,e}(S)$. Finally [3, 1.1], applied to S_r, gives an affine isomorphism from $P_e(S_r)$ onto $\ell^\infty(S_r)_+$, whose restriction is an affine isomorphism τ_1 from $P_{0,e}(S_r)$ onto $\ell_0^\infty(S_r)_+$. Now the obvious map τ, which makes the diagram

$$\begin{tikzcd} P_{0,e}(S_r) \arrow[r, "\tau_1"] \arrow[d, "\tau_2"'] & \ell_0^\infty(S_r)_+ \arrow[d, "\tau_3"] \\ P_{r,e}(S) \arrow[r, "\tau"'] & \ell_r^1(S)_+^* \end{tikzcd} \tag{2.10}$$

commutative, is the desired affine isomorphism. □

In [5] the authors defined the Fourier algebra of a topological foundation $*$-semigroups (which include all inverse semigroups) and in particular studied positive definite functions on these semigroups. Our aim in this section is to develop a parallel theory for the restricted case and among other results prove the generalization of the Godement's characterization of positive definite functions on groups [6] in our restricted context (Theorem 2.9).

For $F, G \subseteq S$, put

$$F \cdot G = \{st : s \in F, t \in G, s^* s = tt^*\}. \tag{2.11}$$

This is clearly a finite set, when F and G are finite.

Lemma 2.4. *If S is an inverse semigroup and $f, g \in \ell^2(S)$, then* $\operatorname{supp}(f \cdot \widetilde{g}) = (\operatorname{supp} f) \cdot (\operatorname{supp} g)^*$. *In particular, when f and g are of finite supports, then so is $f \cdot \widetilde{g}$.*

Proof. $f \cdot \widetilde{g}(x) = \sum_{x^*x = yy^*} f(xy)\overline{g(y)} \neq 0$ if and only if $xy \in \operatorname{supp}(f)$, for some $y \in \operatorname{supp}(g)$ with $x^*x = yy^*$. This is clearly the case if and only if $x = st^*$, for some $s \in \operatorname{supp}(f)$ and $t \in \operatorname{supp}(g)$ with $s^*s = t^*t$. Hence $\operatorname{supp}(f \cdot \widetilde{g}) = (\operatorname{supp}\ f) \cdot (\operatorname{supp}\ g)^*$. □

The following lemma follows from the fact that the product $f \cdot g$ is linear in each variable.

Lemma 2.5 (polarization identity). *For each $f, g \in \ell^2(S)$*

$$\begin{aligned} 4f \cdot \widetilde{g} &= (f+g) \cdot (f+g)^{\sim} - (f-g) \cdot (f-g)^{\sim} \\ &\quad + i(f+ig) \cdot (f+ig)^{\sim} - i(f-ig) \cdot (f-ig)^{\sim}, \end{aligned} \tag{2.12}$$

where $i = \sqrt{-1}$.

Lemma 2.6. *For each $\varphi \in P_{r,f}(S)$, one has $\widetilde{\rho}_r(\varphi) \geq 0$.*

Proof. For each $x, y \in S$

$$\begin{aligned} \langle \widetilde{\rho}_r(\varphi)\delta_x, \delta_y \rangle &= \sum_z \widetilde{\rho}_r(\varphi)\delta_x(z)\overline{\delta_y(z)} = \widetilde{\rho}_r(\varphi)\delta_x(y) = \sum_z \varphi(z)\rho_r(z)\delta_x(y) \\ &= \sum_{zz^* = y^*y} \varphi(z)\delta_x(yz). \end{aligned} \tag{2.13}$$

Now if $xx^* = yy^*$ then for $z = y^*x$ we have $zz^* = y^*xx^*y = y^*y$, and conversely $zz^* = y^*y$ and $x = yz$ imply that $z = zz^*z = y^*yz = y^*x$, and then $x = yy^*x$ and $xz^* = y$, so $y = xz^* = xx^*y$, that is, $yy^* = xx^*yy^* = yy^*xx^* = xx^*$. Hence the last sum is $\varphi(y^*x)$ if $xx^* = yy^*$, and it is zero, otherwise. Summing up,

$$\langle \widetilde{\rho}_r(\varphi)\delta_x, \delta_y \rangle = (\lambda_r(y)\varphi)(x). \tag{2.14}$$

Now for $\xi = \sum_{i=1}^{n} a_i \delta_{x_i} \in \ell_f^2(S)$, we get

$$\langle \widetilde{\rho}_r(\varphi)\xi, \xi\rangle = \sum_{i,j=1}^{n} a_i \overline{a}_j \left\langle \widetilde{\rho}_r(\varphi)\delta_{x_i}, \delta_{x_j}\right\rangle = \sum_{i,j=1}^{n} a_i \overline{a}_j (\lambda_r(x_j)\varphi)(x_i) \geq 0. \tag{2.15}$$

□

Lemma 2.7. *With the above notation,*

$$\widetilde{\lambda}_r(f)\widetilde{\rho}_r(g) = \widetilde{\rho}_r(g)\widetilde{\lambda}_r(f), \tag{2.16}$$

for each $f, g \in \ell^1(S)$.

Proof. Given $f, g \in \ell^1(S)$ and $\xi \in \ell^2(S)$, put $\eta = \widetilde{\rho}_r(g)\xi$ and $\zeta = \widetilde{\lambda}_r(f)\xi$, then $\eta, \zeta \in \ell^2(S)$ and, for each $x \in S$,

$$\begin{aligned}
\widetilde{\lambda}_r(f)\widetilde{\rho}_r(g)\xi(x) &= \sum_{y\in S} f(y)(\lambda_r(y)\eta)(x) \\
&= \sum_{yy^*=xx^*} f(y)\eta(y^*x) = \sum_{yy^*=xx^*} f(y)\sum_{u\in S} g(u)(\rho_r(u)\xi)(y^*x) \\
&= \sum_{yy^*=xx^*} f(y) \sum_{uu^*=x^*yy^*x} g(u)\xi(y^*xu) \\
&= \sum_{yy^*=xx^*} f(y) \sum_{uu^*=x^*x} g(u)\xi(y^*xu), \\
\widetilde{\rho}_r(g)\widetilde{\lambda}_r(f)\xi(x) &= \sum_{u\in S} g(u)(\rho_r(u)\zeta)(x) = \sum_{uu^*=x^*x} g(u)\zeta(xu) \\
&= \sum_{uu^*=x^*x} g(u)\sum_{y\in S} f(y)(\lambda_r(y)\xi)(xu) \\
&= \sum_{uu^*=x^*x} g(u) \sum_{yy^*=xuu^*x^*} f(y)\xi(y^*xu) \\
&= \sum_{uu^*=x^*x} g(u) \sum_{yy^*=xx^*} f(y)\xi(y^*xu),
\end{aligned} \tag{2.17}$$

which are obviously the same. □

Lemma 2.8. *For each $\pi \in \Sigma_r(S)$ and each $\xi \in \mathscr{H}_\pi$, the coefficient function $u = \langle \pi(\cdot)\xi, \xi\rangle$ is in $P_{r,e}(S)$.*

Proof. For each $n \geq 1$, $c_1, \ldots, c_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in S$, noting that π is a restricted representation, we have

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c}_i c_j \lambda_r(x_i)(u)(x_j) &= \sum_{x_i x_i^* = x_j x_j^*} \overline{c}_i c_j u(x_i^* x_j) \\
&= \sum_{x_i x_i^* = x_j x_j^*} \overline{c}_i c_j \langle \pi(x_i^* x_j)\xi, \xi\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{c}_i c_j \langle \pi(x_i)^* \pi(x_j)\xi, \xi\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{c}_i c_j \langle \pi(x_j)\xi, \pi(x_i)\xi\rangle \\
&= \left\| \sum_{i=1}^{n} c_i \pi(x_i)\xi \right\|_2^2 \geq 0,
\end{aligned}
\tag{2.18}
$$

and, regarding π as an element of $\Sigma_0(S_r)$ and using the fact that $u(0) = 0$, we have

$$
\begin{aligned}
\left| \sum_{k=1}^{n} c_k u(x_k) \right|^2 &= \left| \sum_{k=1}^{n} c_k \langle \pi(x_k)\xi, \xi\rangle \right|^2 \\
&\leq \|\xi\|^2 \left\| \sum_{i=1}^{n} c_i \pi(x_i)\xi \right\|_2^2 \\
&= \|\xi\|^2 \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{c}_i c_j \lambda_r(x_i)(u)(x_j) \\
&= \|\xi\|^2 \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{c}_i c_j (u)(x_i^* \cdot x_j),
\end{aligned}
\tag{2.19}
$$

so $u \in P_{0,e}(S_r) = P_{r,e}(S)$. □

The following is proved by R. Godement in the group case [6]. Here we adapt the proof given in [7].

Theorem 2.9. *Let S be a unital inverse semigroup. Given $\varphi \in \ell^\infty(S)$, the following statements are equivalent:*

(i) $\varphi \in P_{r,e}(S)$,

(ii) *there is an $\xi \in \ell^2(S)$ such that $\varphi = \xi \cdot \widetilde{\xi}$.*

Moreover if ξ is of finite support, then so is φ.

Proof. By the above lemma applied to $\pi = \lambda_r$, (ii) implies (i). Also if $\xi \in \ell_f^2(S)$, then by Lemma 2.4, $\xi \cdot \widetilde{\xi}$ is of finite support.

Conversely assume that $\varphi \in P_{r,e}(S)$. Choose an approximate identity $\{e_\alpha\}$ for $\ell_r^1(S)$ consisting of positive, symmetric functions of finite support, as constructed in [1, Proposition 3.2]. Let ρ_r be the restricted right regular representation of S, then by the above lemma $\widetilde{\rho}_r(\varphi) \geq 0$. Take $\xi_\alpha = \widetilde{\rho}_r(\varphi)^{1/2}\, e_\alpha \in \ell^2(S)$, then if $1 \in S$ is the identity element, then for each $\alpha \geq \beta$ we have

$$\begin{aligned}\|\xi_\alpha - \xi_\beta\|_2^2 &= \left\langle \widetilde{\rho}_r(\varphi)^{1/2}(e_\alpha - e_\beta), \widetilde{\rho}_r(\varphi)^{1/2}(e_\alpha - e_\beta) \right\rangle \\ &= \langle \widetilde{\rho}_r(\varphi)(e_\alpha - e_\beta), e_\alpha - e_\beta \rangle = \varphi \cdot (e_\alpha - e_\beta) \cdot (e_\alpha - e_\beta)(1) \\ &\leq \|\varphi \cdot (e_\alpha - e_\beta) \cdot (e_\alpha - e_\beta)\|_1 = \|\varphi \cdot (e_\alpha - e_\beta)\|_1 \longrightarrow 0,\end{aligned} \tag{2.20}$$

as $\alpha, \beta \to \infty$, where the last equality follows from [1, Lemma 3.2(ii)]. Hence, there is $\xi \in \ell^2(S)$ such that $\xi_\alpha \to \xi$ in $\ell^2(S)$. Now for each $t \in S$

$$\begin{aligned}\xi \cdot \widetilde{\xi}(t) &= \langle \lambda_r(t^*)\xi, \xi \rangle = \lim_\alpha \left\langle \lambda_r(t^*)\widetilde{\rho}_r(\varphi)^{1/2} e_\alpha, \widetilde{\rho}_r(\varphi)^{1/2} e_\alpha \right\rangle \\ &= \lim_\alpha \left\langle \widetilde{\rho}_r(\varphi)^{1/2}\lambda_r(t^*)\widetilde{\rho}_r(\varphi)^{1/2} e_\alpha, e_\alpha \right\rangle = \lim_\alpha \langle \widetilde{\rho}_r(\varphi)\lambda_r(t^*) e_\alpha, e_\alpha \rangle \\ &= \lim_\alpha \langle \lambda_r(t^*) e_\alpha, \widetilde{\rho}_r(\varphi) e_\alpha \rangle = \lim_\alpha (\overline{\varphi} \cdot (\widetilde{e}_\alpha \cdot \lambda_r(t^*) e_\alpha))(1) \\ &= \lim_\alpha ((\overline{\varphi} \cdot e_\alpha) \cdot \lambda_r(t^*) e_\alpha)(1) = \lim_\alpha \sum_y (\overline{\varphi} \cdot e_\alpha)(y) \lambda_r(t^*) e_\alpha(y^*) \\ &= \lim_\alpha \sum_y (\overline{\varphi} \cdot e_\alpha)(y^*) \lambda_r(t^*) e_\alpha(y) = \lim_\alpha \sum_y (e_\alpha \cdot \check{\varphi})(y) \lambda_r(t^*) e_\alpha(y) \\ &= \lim_\alpha \langle \lambda_r(t^*) e_\alpha, e_\alpha \cdot \check{\varphi} \rangle = \lim_\alpha e_\alpha \cdot (e_\alpha \cdot \check{\varphi})^{\sim}(t) \\ &= \lim_\alpha e_\alpha \cdot \overline{\varphi} \cdot e_\alpha(t) = \overline{\varphi}(t).\end{aligned} \tag{2.21}$$

The last equality follows from the remark after Proposition 3.2 in [1] and the fact that $|e_\alpha \cdot \overline{\varphi} \cdot e_\alpha(t) - \overline{\varphi}(t)| \leq \|e_\alpha \cdot \overline{\varphi} \cdot e_\alpha - \overline{\varphi}\|_1$. Hence $\varphi = \overline{\xi} \cdot (\overline{\xi})^{\sim}$, as required. □

3. Restricted Fourier and Fourier-Stieltjes Algebras

Let S be a unital inverse semigroup and, let $P(S)$ be the set of all bounded positive definite functions on S (see [8] for the group case and [9] for inverse semigroups). We use the notation $P(S)$ with indices r, e, f, and 0 to denote the positive definite functions which are restricted, extendible, of finite support, or vanishing at zero, respectively. Let $B(S)$ be the linear span of $P(S)$. Then $B(S)$ is a commutative Banach algebra with respect to the pointwise multiplication and the following norm [5]:

$$\|u\| = \sup\left\{ \left| \sum_{x \in S} u(x) f(x) \right| : f \in \ell^1(S), \sup_{\pi \in \Sigma(S)} \|\widetilde{\pi}(f)\| \leq 1 \right\} \quad (u \in B(S)). \tag{3.1}$$

Also $B(S)$ coincides with the set of the coefficient functions of elements of $\Sigma(S)$ [5]. If one wants to get a similar result for the set of coefficient functions of elements of $\Sigma_r(S)$, one has to apply the above facts to S_r. But S_r is not unital in general, so one is led to consider a smaller class of bounded positive definite functions on S_r. The results of [3] suggests that these should be the class of extendible positive definite functions on S. Among these, those which vanish at 0 correspond to elements of $P_{r,e}(S)$.

The structure of algebras $B(S)$ and $A(S)$ is far from being well understood, even in special cases. From the results of [4, 10], it is known that for a commutative unital discrete $*$-semigroup S, $B(S) = M(\widehat{S})^{\widehat{}}$ via Bochner theorem [10]. Even in this case, the structure of $A(S)$ seems to be much more complicated than the group case. This is mainly because of the lack of an appropriate analog of the group algebra. If S is a discrete idempotent semigroup with identical involution. Then $\widehat{S}$ is a compact topological semigroup with pointwise multiplication. We believe that in this case $A(S) = L(\widehat{S})^{\widehat{}}$ where $L(\widehat{S})$ is the Baker algebra on $\widehat{S}$ (see e.g., [8]) however we are not able to prove it at this stage. In this section we show that the linear span $B_{r,e}(S)$ of $P_{r,e}(S)$ is a commutative Banach algebra with respect to the pointwise multiplication and an appropriate modification of the above norm. We call this the restricted Fourier-Stieltjes algebra of S and show that it coincides with the set of all coefficient functions of elements of $\Sigma_r(S)$.

As before, the indices e, 0, and f are used to distinguish extendible elements, elements vanishing at 0, and elements of finite support, respectively. We freely use any combination of these indices. Consider the linear span of $P_{r,e,f}(S)$ which is clearly a two-sided ideal of $B_{r,e}(S)$, whose closure $A_{r,e}(S)$ is called the restricted Fourier algebra of S. We show that it is a commutative Banach algebra under pointwise multiplication and norm of $B_r(S)$.

In order to study properties of $B_{r,e}(S)$, we are led by Proposition 2.3 to consider $B_{0,e}(S_r)$. More generally we calculate this algebra for any inverse 0-semigroup T. Let $B_e(T)$ be the linear span of $P_e(T)$ with pointwise multiplication and the norm

$$\|u\| = \sup\left\{ \left|\sum_{x\in T} f(x)u(x)\right| : f \in \ell^1(T),\ \|f\|_{\Sigma(T)} \leq 1 \right\} \quad (u \in B_e(T)), \tag{3.2}$$

and let $B_{0,e}(T)$ be the closed ideal of $B_e(T)$ consisting of elements vanishing at 0. First let us show that $B_e(T)$ is complete in this norm. The next lemma is quite well known and follows directly from the definition of the functional norm.

Lemma 3.1. *If X is a Banach space, $D \subseteq X$ is dense, and $f \in X^*$, then*

$$\|f\| = \sup\{ |f(x)| : x \in D, \|x\| \leq 1 \}. \tag{3.3}$$

Lemma 3.2. *If T is an inverse 0-semigroup (not necessarily unital), then we have the following isometric isomorphism of Banach spaces:*

(i) $B_e(T) \simeq C^*(T)^*$,

(ii) $B_{0,e}(T) \simeq C^*((T)/\mathbb{C}\delta_0)^*$.

In particular $B_e(T)$ and $B_{0,e}(T)$ are Banach spaces.

Proof. (ii) clearly follows from (i). To prove (i), first recall that $P_e(S)$ is affinely isomorphic to $\ell^1(S)^*_+$ [3, 1.1] via

$$\langle u, f\rangle = \sum_{x\in S} f(x)u(x) \quad \left(f \in \ell^1(S), u \in P_e(S)\right). \tag{3.4}$$

This defines an isometric isomorphism τ_0 from $B_e(T)$ into $\ell^1(T)^*$ (with the dual norm). By the brevious lemma, one can lift τ_0 to an isometric isomorphism τ from $B_e(T)$ into $C^*(T)^*$. We only need to check that τ is surjective. Take any $v \in C^*(T)$, and let w be the restriction of v to $\ell^1(T)$. Since $\|f\|_{\Sigma(T)} \leq \|f\|_1$, for each $f \in \ell^1(T)$, The norm of w as a linear functional on $\ell^1(T)$ is not bigger than the norm of v as a functional on $C^*(T)$. In particular, $w \in \ell^1(T)^*$ and so there is a $u \in B_e(T)$ with $\tau_0(u) = w$. Then $\tau(u) = v$, as required.

We know that the restriction map $\tau : B_{0,e}(S_r) \to B_{r,e}(S)$ is a surjective linear isomorphism. Also τ is clearly an algebra homomorphism ($B_{0,e}(S_r)$ is an algebra under pointwise multiplication [3, 3.4], and the surjectivity of τ implies that the same fact holds for $B_{r,e}(S)$). Now we put the following norm on $B_r(S)$

$$\|u\|_r = \sup\left\{ \left|\sum_{x\in S} f(x)u(x)\right| : f \in \ell^1_r(S), \|f\|_{\Sigma_r(S)} \right\} \quad (u \in B_r(S)), \tag{3.5}$$

then using the fact that $B_{0,e}(S_r)$ is a Banach algebra (it is a closed subalgebra of $B(S_r)$ which is a Banach algebra [5, Theorem 3.4]) we have the following. □

Lemma 3.3. *The restriction map $\tau : B_{0,e}(S_r) \to B_{r,e}(S)$ is an isometric isomorphism of normed algebras. In particular, $B_{r,e}(S)$ is a commutative Banach algebra under pointwise multiplication and above norm.*

Proof. The second assertion follows from the first and Lemma 2.4 applied to $T = S_r$. For the first assertion, we only need to check that τ is an isometry. But this follows directly from [1, Theorem 3.2] and the fact that $\Sigma_r(S) = \Sigma_0(S_r)$. □

Corollary 3.4. *$B_{r,e}(S)$ is the set of coefficient functions of elements of $\Sigma_r(S)$.*

Proof. Given $u \in P_{r,e}(S)$, let v be the extension by zero of u to a function on S_r, then $v \in P_{0,e}(S_r)$, so there is a cyclic representation $\pi \in \Sigma(S_r)$, say with cyclic vector $\xi \in \mathscr{H}_\pi$, such that $v = \langle \pi(\cdot)\xi, \xi\rangle$ (see the proof of [3, 3.2]). But

$$0 = v(0) = \langle \pi(0)\xi, \xi\rangle = \langle \pi(0^*0)\xi, \xi\rangle = \|\pi(0)\xi\|, \tag{3.6}$$

that is, $\pi(0)\xi = 0$. But ξ is the cyclic vector of π, which means that for each $\eta \in \mathscr{H}_\pi$, there is a net of elements of the form $\sum_{i=1}^n c_i\pi(x_i)\xi$, converging to η in the norm topology of $\mathscr{H}_\pi$, and

$$\pi(0)\sum_{i=1}^n c_i\pi(x_i)\xi = \sum_{i=1}^n c_i\pi(0)\xi = 0, \tag{3.7}$$

so $\pi(0)\eta = 0$, and so $\pi(0) = 0$. This means that $\pi \in \Sigma_0(S_r) = \Sigma_r(S)$. Now a standard argument, based on the fact that $\Sigma_r(S) = \Sigma_0(S_r)$ is closed under direct sum, shows that each $u \in B_{r,e}(S)$ is a coefficient function of some element of $\Sigma_r(S)$. The converse follows from Lemma 2.8. □

Corollary 3.5. *One has the isometric isomorphism of Banach spaces $B_{r,e}(S) \simeq C_r(S)^*$.*

Proof. We have the following of isometric linear isomorphisms: first $B_{r,e}(S) \simeq B_{0,e}(S_r)$ (Lemma 3.3), then $B_{0,e}(S_r) \simeq (C^*(S_r)/\mathbb{C}\delta_0)^*$ (Lemma 3.2, applied to $T = S_r$), and finally $C^*_r(S) \simeq C^*(S_r)/\mathbb{C}\delta_0$ [1, Theorem 4.1]. □

Next, as in [5], we give an alternative description of the norm of the Banach algebra $B_{r,e}(S)$. For this we need to know more about the universal representation of S_r. The universal representation ω of S_r is the direct sum of all cyclic representations corresponding to elements of $P_e(S_r)$. To be more precise, this means that given any $u \in P_e(S_r)$ we consider u as a positive linear functional on $\ell^1(S_r)$, then by [7, 21.24], there is a cyclic representation $\{\widetilde{\pi}_u, \mathcal{H}_u, \xi_u\}$ of $\ell^1(S_r)$, with $\pi_u \in \Sigma(S_r)$, such that

$$\langle u, f\rangle = \langle \widetilde{\pi}_u(f)\xi_u, \xi_u\rangle \quad \Big(f \in \ell^1(S_r)\Big). \tag{3.8}$$

Therefore π_u is a cyclic representation of S_r and $u = \langle \pi_u(\cdot)\xi_u, \xi_u\rangle$ on S_r. Now ω is the direct sum of all π_u's, where u ranges over $P_e(S_r)$. There is an alternative construction in which one can take the direct sum of π_u's with u ranging over $P_{0,e}(S_r)$ to get a subrepresentation ω_0 of ω. Clearly $\omega \in \Sigma(S_r)$ and $\omega_0 \in \Sigma_0(S_r)$. It follows from [3, 3.2] that the set of coefficient functions of ω and ω_0 are $B_e(S_r)$ and $B_{0,e}(S_r) = B_{r,e}(S)$, respectively. As far as the original semigroup S is concerned, we prefer to work with ω_0, since it could be considered as an element of $\Sigma_r(S)$. Now $\widetilde{\omega}_0$ is a nondegenerate $*$- representation of $\ell_r^1(S)$ which uniquely extends to a nondegenerate representation of the restricted full C^*-algebra $C_r^*(S)$, which we still denote by $\widetilde{\omega}_0$. We gather some of the elementary facts about $\widetilde{\omega}_0$ in the next lemma.

Lemma 3.6. *With the above notation, we have the following.*

(i) *$\widetilde{\omega}_0$ is the direct sum of all nondegenerate representations π_u of $C_r^*(S)$ associated with elements $u \in C_r^*(S)_+^*$ via the GNS, construction, namely, $\widetilde{\omega}_0$ is the universal representation of $C_r^*(S)$. In particular, $C_r^*(S)$ is faithfully represented in $\mathcal{H}_{\omega_0}$.*

(ii) *The von Numann algebras $C_r^*(S)^{**}$ and the double commutant of $C_r^*(S)$ in $\mathcal{B}(\mathcal{H}_{\omega_0})$ are isomorphic. They are generated by elements $\widetilde{\omega}_0(f)$, with $f \in \ell_r^1(S)$, as well as by elements $\omega_0(x)$, with $x \in S$.*

(iii) *Each representation π of $C_r^*(S)$ uniquely decomposes as $\pi = \pi^{**} \circ \omega_0$.*

(iv) *For each $\pi \in \Sigma_r(S)$ and $\xi, \eta \in \mathcal{H}_\pi$, let $u = \langle \pi(\cdot)\xi, \eta\rangle$, then $u \in C_r(S)^*$ and*

$$\langle T, u\rangle = \langle \widetilde{\pi}^{**}(T)\xi, \eta\rangle \quad \left(T \in C_r^*(S)^{**}\right). \tag{3.9}$$

Proof. Statement (i) follows by a standard argument. Statement (iii) and the first part of (ii) follow from (i), and the second part of (ii) follows from the fact that both sets of elements described in (ii) have clearly the same commutant in $\mathcal{B}(\mathcal{H}_{\omega_0})$ as the set of elements $\widetilde{\omega}_0(u)$, with $u \in C_r^*(S)$ which generate $C_r^*(S)''$. The first statement of (iv) follows from Lemma 2.8 and Corollary 3.5. As for the second statement, first note that for each $f \in \ell_r^1(S)$, $\widetilde{\omega}_0(f)$ is the image of f under the canonical embedding of $C_r^*(S)$ in $C_r^*(S)^{**}$. Therefore, by (iii),

$$\begin{aligned}\langle \widetilde{\omega}_0(f), u\rangle &= \langle u, f\rangle = \sum_{x\in S} f(x)u(x)\\ &= \langle \widetilde{\pi}(f)\xi, \eta\rangle = \langle \widetilde{\pi}^{**} \circ \widetilde{\omega}_0(f)\xi, \eta\rangle.\end{aligned} \tag{3.10}$$

Taking limit in $\|\cdot\|_{\Sigma_r}$ we get the same relation for any $f \in C_r^*(S)$, and then, using (ii), by taking limit in the ultraweak topology of $C_r^*(S)^{**}$, we get the desired relation. □

Lemma 3.7. *Let 1 be the identity of S, then for each $u \in P_{r,e}(S)$ one has $\|u\|_r = u(1)$.*

Proof. As $\|\delta_e\|_{\Sigma_r} = 1$ and $u(1) = \lambda_r(1)u(1) \geq 0$, we have $\|u\|_r \geq |u(1)| = u(1)$. Conversely, by the proof of Corollary 3.4, there is $\pi \in \Sigma_r(S)$ and $\xi \in \mathscr{H}_\pi$ such that $u = \langle \pi(\cdot)\xi, \xi\rangle$. Hence $u(1) = \langle \pi(1)\xi, \xi\rangle = \|\xi\|^2 \geq \|u\|_r$. □

Lemma 3.8. *For each $\pi \in \Sigma_r(S)$ and $\xi, \eta \in \mathscr{H}_\pi$, consider $u = \langle \pi(\cdot)\xi, \eta\rangle \in B_{r,e}(S)$, then $\|u\|_r \leq \|\xi\| \cdot \|\eta\|$. Conversely each $u \in B_{r,e}(S)$ is of this form and one may always choose ξ, η so that $\|u\|_r = \|\xi\| \cdot \|\eta\|$.*

Proof. The first assertion follows directly from the definition of $\|u\|_r$ (see the paragraph after Lemma 3.6). The first part of the second assertion is the content of Corollary 3.4. As for the second part, basically the proof goes as in [9]. Consider u as an element of $C_r^*(S)^*$ and let $u = v \cdot |u|$ be the polar decomposition of u, with $v \in C_r^*(S)^{**}$ and $|u| \in C_r^*(S)_+^* = P_{r,e}(S)$, and the dot product is the module action of $C_r^*(S)^{**}$ on $C_r^*(S)^*$. Again, by the proof of Corollary 3.4, there is a cyclic representation $\pi \in \Sigma_r(S)$, say with cyclic vector η, such that $|u| = \langle \pi(\cdot)\eta, \eta\rangle$. Put $\xi = \tilde{\pi}^{**}(v)\eta$, then $\|\xi\| \leq \|\eta\|$ and, by Lemma 3.6(iv) applied to $|u|$,

$$\begin{aligned} u(x) &= \langle \omega_0(x), u\rangle = \langle \omega_0(x)v, |u|\rangle \\ &= \left\langle \tilde{\pi}^{**} \circ \omega_0(x)(v)\eta, \eta\right\rangle = \left\langle \pi(x)\xi, \eta\right\rangle, \end{aligned} \tag{3.11}$$

and, by Corollary 3.5 and Lemma 3.7,

$$\|u\|_r = \||u|\| = |u|(1) = \left\|\eta\right\|^2 \geq \|\xi\| \cdot \left\|\eta\right\|. \tag{3.12}$$

□

Note that the above lemma provides an alternative (direct) way of proving the second statement of Lemma 3.3 (just take any two elements u, v in $B_{r,e}(S)$ and represent them as coefficient functions of two representations such that the equality holds for the norms of both u and v, then use the tensor product of those representations to represent uv and apply the first part of the lemma to uv). Also it gives the alternative description of the norm on $B_{r,e}(S)$ as follows.

Corollary 3.9. *For each $u \in B_{r,e}(S)$,*

$$\|u\|_r = \inf\left\{\|\xi\| \cdot \left\|\eta\right\| : \xi, \eta \in \mathscr{H}_\pi,\ \pi \in \Sigma_r(S),\ u = \left\langle \pi(\cdot)\xi, \eta\right\rangle\right\}. \tag{3.13}$$

Corollary 3.10. *For each $u \in B_{r,e}(S)$,*

$$\|u\|_r = \sup\left\{\left|\sum_{n=1}^{\infty} c_n u(x_n)\right| : c_n \in \mathbb{C}, x_n \in S\ (n \geq 1), \left\|\sum_n c_n \delta_{x_n}\right\|_{\Sigma_r} \leq 1\right\}. \tag{3.14}$$

Proof. Just apply Kaplansky's density theorem to the unit ball of $C_r^*(S)^{**}$. □

Corollary 3.11. *The unit ball of $B_{r,e}(S)$ is closed in the topology of pointwise convergence.*

Proof. If $u \in B_{r,e}(S)$ with $\|u\|_r \leq 1$, then for each $n \geq 1$, each $c_1, \ldots, c_n \in \mathbb{C}$ and each $x_1, \ldots, x_n \in S$,

$$\left| \sum_{k=1}^{n} c_k u(x_k) \right| \leq \left\| \sum_{k=1}^{n} c_k \delta_{x_k} \right\|_{\Sigma_r}. \tag{3.15}$$

If $u_\alpha \to u$, pointwise on S with $u_\alpha \in B_{r,e}(S)$, $\|u_\alpha\|_r \leq 1$, for each α, then all u_α's satisfy above inequality, and so does u. Hence, by above corollary, $u \in B_{r,e}(S)$ and $\|u\|_r \leq 1$. □

Lemma 3.12. *For each $f, g \in \ell^2(S)$, $f \cdot \tilde{g} \in B_{r,e}(S)$ and if $\|\cdot\|_r$ is the norm of $B_{r,e}(S)$, $\|f \cdot \tilde{g}\|_r \leq \|f\|_2 \cdot \|g\|_2$.*

Proof. The first assertion follows from polarization identity of Lemma 2.5 and the fact that for each $h \in \ell^2(S)$, $h \cdot \tilde{h}$ is a restricted extendible positive definite function (Theorem 2.9). Now if $u = f \cdot \tilde{g}$, then

$$\begin{aligned}
\|u\|_r &= \sup\left\{ \left| \sum_{y \in S} u(y)\varphi(y) \right| : \varphi \in \ell_r^1(S), \|\varphi\|_{\Sigma_r} \leq 1 \right\} \\
&= \sup\left\{ \left| \sum_{y \in S} \langle \lambda_r(y^*) f, g \rangle \varphi(y) \right| : \varphi \in \ell_r^1(S), \|\varphi\|_{\Sigma_r} \leq 1 \right\} \\
&= \sup\left\{ \left| \sum_{y \in S} \langle f, \lambda_r(y) g \rangle \varphi(y) \right| : \varphi \in \ell_r^1(S), \|\varphi\|_{\Sigma_r} \leq 1 \right\} \\
&= \sup\left\{ \left| \left\langle f, \tilde{\lambda}_r(\varphi) g \right\rangle \right| : \varphi \in \ell_r^1(S), \|\varphi\|_{\Sigma_r} \leq 1 \right\} \\
&= \sup_{\|\varphi\|_{\Sigma_r} \leq 1} \left\| \tilde{\lambda}_r(\varphi) \right\| \|f\|_2 \|g\|_2 \leq \|f\|_2 \cdot \|g\|_2.
\end{aligned} \tag{3.16}$$

□

The next theorem extends Eymard's theorem [9, 3.4] to inverse semigroups.

Theorem 3.13. *Consider the following sets:*

$$\begin{aligned}
E_1 &= \left\langle f \cdot \tilde{g} : f, g \in \ell_f^2(S) \right\rangle, \\
E_2 &= \left\langle h \cdot \tilde{h} : h \in \ell_f^2(S) \right\rangle, \\
E_3 &= \left\langle P_{r,e,f}(S) \right\rangle, \\
E_4 &= \left\langle P(S) \cap \ell^2(S) \right\rangle,
\end{aligned}$$

$$E_5 = \left\langle h \cdot \tilde{h} : h \in \ell^2(S) \right\rangle,$$
$$E_6 = \left\langle f \cdot \tilde{g} : f, g \in \ell^2(S) \right\rangle. \tag{3.17}$$

Then $E_1 \subseteq E_2 \subseteq E_3 \subseteq E_4 \subseteq E_5 \subseteq E_6 \subseteq B_{r,e}(S)$, and the closures of all of these sets in $B_{r,e}(S)$ are equal to $A_{r,e}(S)$.

Proof. The inclusion $E_1 \subseteq E_2$ follows from Lemma 2.5, and the inclusions $E_2 \subseteq E_3$ and $E_4 \subseteq E_5$ follow from Theorem 2.9. The inclusions $E_3 \subseteq E_4$ and $E_5 \subseteq E_6$ are trivial. Now E_1 is dense in E_6 by Lemma 3.12, and the fact that $\ell^2_f(S)$ is dense in $\ell^2(S)$. Finally $\overline{E}_3 = A_{r,e}(S)$, by definition, and $E_3 \subseteq E_2 \subseteq E_1$ by Theorem 2.9; hence $\overline{E}_i = A_{r,e}(S)$, for each $1 \leq i \leq 6$. □

Lemma 3.14. *$P_{r,e}(S)$ separates the points of S.*

Proof. We know that S_r has a faithful representation (namely the left regular representation Λ), so $P_e(S_r)$ separates the points of S_r [3, 3.3]. Hence $P_{0,e}(S_r) = P_{r,e}(S)$ separates the points of $S_r \setminus \{0\} = S$. □

Proposition 3.15. *For each $x \in S$ there is $u \in A_{r,e}(S)$ with $u(x) = 1$. Also $A_{r,e}(S)$ separates the points of S.*

Proof. Given $x \in S$, let $u = \delta_{(x^*x)} \cdot \tilde{\delta}_{x^*} \in E_1 \subseteq A_{r,e}(S)$, then $u(x) = 1$. Also given $y \neq x$ and u as above, if $u(y) \neq 1$, then u separates x and y. If $u(y) = 1$, then use above lemma to get some $v \in B_{r,e}(S)$ which separates x and y. Then $u(x) = u(y) = 1$, so $(uv)(x) = v(x) \neq v(y) = (uv)(y)$; that is, $uv \in A_{r,e}(S)$ separates x and y. □

Proposition 3.16. *For each finite subset $K \subseteq S$, there is $u \in P_{r,e,f}(S)$ such that $u|_K \equiv 1$.*

Proof. For $F \subseteq S$, let $F_e = \{x^*x : x \in F\}$ and note that $F \subseteq F \cdot F_e$ (since $x = x(x^*x)$, for each $x \in F$). Now given a finite set $K \subseteq S$, put $F = K \cup K^* \cup K_e$; then since $K_e = K_e^*$ we have $F = F^*$, and since $K_e = (K^*)_e$ and $(K_e)_e = K_e$ we have $F_e \subseteq F$. Hence $K \subseteq F \subseteq F \cdot F$. Now $F \cdot F$ is a finite set, and if $f = \chi_F$, then $u = f \cdot \tilde{f} = \chi_F \cdot \tilde{\chi}_F = \chi_{F \cdot F^*} = \chi_{F \cdot F} \in P_{r,e,f}(S)$ and $u|_K \equiv 1$. □

Corollary 3.17. *$B_{r,e,f}(S) = \langle P_{r,e,f}(S) \rangle$ and $\overline{B_{r,f}(S)} = A_{r,e}(S)$.*

Proof. Clearly $\langle P_{r,e,f}(S) \rangle \subseteq B_{r,e,f}(S)$. Now if $v \in B_{r,e,f}(S)$, then $v = \sum_{i=1}^{4} \alpha_i v_i$, for some $\alpha_i \in \mathbb{C}$ and $v_i \in P_{r,e,f}(S) (1 \leq i \leq 4)$. Let $K = \operatorname{supp}(v) \subseteq S$ and $u \in P_{r,e,f}(S)$ be as in the above proposition, then $u|_K \equiv 1$ so $v = uv = \sum_{i=1}^{4} \alpha_i (uv_i)$ is in the linear span of $P_{r,e,f}(S)$. □

4. Fourier and Fourier-Stieltjes Algebras of Associated Groupoids

We observed in Section 1 that one can naturally associate a (discrete) groupoid S_a to any inverse semigroup S. The Fourier and Fourier-Stieltjes algebras of (topological and measured) groupoids are studied in [11–14]. It is natural to ask if the results of these papers, applied to the associated groupoid S_a of S, could give us some information about the associated algebras on S. In this section we explore the relation between S and its associated groupoid S_a and resolve some technical difficulties which could arise when one tries to relate the corresponding function algebras. We also investigate the possibility of assigning graph groupoids to S and find relations between the corresponding C^*-algebras.

Let us recall some general terminology and facts about groupoids. There are two parallel approaches to the theory of groupoids, theory of measured groupoids, and theory of locally compact groupoids (compare [13] with [14]). Here we deal with discrete groupoids (like S_a), and so basically it doesn't matter which approach we take, but the topological approach is more suitable here. Even if one wants to look at the topological approach, there are two different interpretations about what we mean by a "representation" (compare [12] with [13]). The basic difference is that whether we want representations to preserve multiplications *everywhere* or just *almost everywhere* (with respect to a Borel measure on the unit space of our groupoid which changes with each representation). Again the "everywhere approach" is more suitable for our setting. This approach, mainly taken by [11, 12], is the best fit for the representation theory of inverse semigroups (when one wants to compare representation theories of S and S_a). Even then, there are some basic differences which one needs to deal with them carefully.

We mainly follow the approach and terminology of [12]. As we only deal with discrete groupoids we drop the topological considerations of [12]. This would simplify our short introduction and facilitate our comparison. A (discrete) groupoid is a set G with a distinguished subset $G^2 \subseteq G \times G$ of pairs of multiplicable elements, a multiplication map: $G^2 \to G$; $(x,y) \mapsto xy$, and an inverse map : $G \to G$; $x \mapsto x^{-1}$, such that for each $x,y,z \in G$

(i) $(x^{-1})^{-1} = x$,

(ii) if $(x,y),(y,z)$ are in G^2, then so are $(xy,z),(x,yz)$, and $(xy)z = x(yz)$,

(iii) (x^{-1},x) is in G^2 and if (x,y) is in G^2 then $x^{-1}(xy) = y$,

(iv) if (y,x) is in G^2, then $(yx)x^{-1} = y$.

For $x \in G$, $s(x) = x^{-1}x$ and $r(x) = xx^{-1}$ are called the source and range of x, respectively. $G^0 = s(G) = r(G)$ is called the unit space of G. For each $u,v \in G^0$ we put $G^u = r^{-1}(u)$, $G_v = s^{-1}(v)$, and $G^u_v = G_v \cap G^u$. Note that for each $u \in G^0$, G^u_u is a (discrete) group, called the isotropy group at u. Any (discrete) groupoid G is endowed with left and right Haar systems $\{\lambda_u\}$ and $\{\lambda^u\}$, where λ_u and λ^u are simply counting measures on G_u and G^u, respectively. Consider the algebra $c_{00}(G)$ of finitely supported functions on G. We usually make this into a normed algebra using the so-called I-norm

$$\|f\|_I = \max\left\{\sup_{u\in G^0}\sum_{x\in G_u}|f(x)|,\ \sup_{u\in G^0}\sum_{x\in G^u}|f(x)|\right\} \quad (f \in c_{00}(G)), \tag{4.1}$$

where the above supremums are denoted, respectively, by $\|f\|_{I,s}$ and $\|f\|_{I,r}$. Note that in general $c_{00}(G)$ is not complete in this norm. We show the completion of $c_{00}(G)$ in $\|\cdot\|_I$ by $\ell^1(G)$. There are also natural C^*-norms in which one can complete $c_{00}(G)$ and get a C^*-algebra. Two-well known groupoid C^*-algebras obtained in this way are the full and reduced groupoid C^*-algebras $C^*(G)$ and $C^*_L(G)$. Here we briefly discuss their construction and refer the reader to [14] for more details.

A Hilbert bundle $\mathscr{H} = \{\mathscr{H}_u\}$ over G^0 is just a field of Hilbert spaces indexed by G^0. A representation of G is a pair $\{\pi,\mathscr{H}^\pi\}$ consisting of a map π and a Hilbert bundle $\mathscr{H}^\pi = \{\mathscr{H}^\pi_u\}$ over G^0 such that, for each $x,y \in G$,

(i) $\pi(x) : \mathscr{H}^{\pi}_{s(x)} \longrightarrow \mathscr{H}^{\pi}_{r(x)}$ is a surjective linear isometry,

(ii) $\pi(x^{-1}) = \pi(x)^*$,

(iii) if (x, y) is in G^2, then $\pi(xy) = \pi(x)\pi(y)$.

We usually just refer to π as the representation, and it is always understood that there is a Hilbert bundle involved. We denote the set of all representations of G by $\Sigma(G)$. Note that here a representation corresponds to a (continuous) Hilbert bundle, where as in the usual approach to (locally compact or measured) categories representations are given by measurable Hilbert bundles (see [12] for more details).

A natural example of such a representation is the left regular representation L of G. The Hilbert bundle of this representation is $L^2(G)$ whose fiber at $u \in G^0$ is $L^2(G^u, \lambda^u)$. In our case that G is discrete, this is simply $\ell^2(G^u)$. Each $f \in c_{00}(G)$ could be regarded as a section of this bundle (which sends $u \in G^0$ to the restriction of f to G^u). Also G acts on bounded sections ξ of $L^2(G)$ via

$$L_x\xi(y) = \xi\left(x^{-1}y\right) \quad \left(x \in G, y \in G^{r(x)}\right). \tag{4.2}$$

Let $E^2(G)$ be the set of sections of $L^2(G)$ vanishing at infinity. This is a Banach space under the supnorm and contains $c_{00}(G)$. Furthermore, it is a canonical $c_0(G^0)$-module via

$$b\xi(x) = \xi(x)b(r(x)) \quad \left(x \in G, \xi \in E^2(G), b \in c_0\left(G^0\right)\right). \tag{4.3}$$

Now $E^2(G)$, with the $c_0(G)$-valued inner product

$$\langle \xi, \eta\rangle(u) = \langle L(\cdot)\xi^u \circ s(\cdot), \eta^u \circ r(\cdot)\rangle, \tag{4.4}$$

is a Hilbert C^*-module. The action of $c_{00}(G)$ on itself by left convolution extends to a $*$-anti representation of $c_{00}(G)$ in $E^2(G)$, which is called the left regular representation of $c_{00}(G)$ [12, Proposition 10]. The map $f \mapsto L_f$ is a norm decreasing homomorphism from $(c_{00}(G), \|\cdot\|_{I,r})$ into $\mathcal{B}(E^2(G))$. Also the former has a left bounded approximate identity $\{e_\alpha\}$ consisting of positive functions such that $\{L_{e_\alpha}\}$ tends to the identity operator in the strong operator topology of the later [12, Proposition 11]. The closure of the image of $c_{00}(G)$ under L is a C^*-subalgebra $C^*_L(G)$ of $\mathcal{B}(E^2(G))$ which is called the reduced C^*-algebra of G. We should warn the reader that $\mathcal{B}(E^2(G))$ is merely a C^*-algebra and, in contrast with the Hilbert space case, it is not a von Neumann algebra in general. The above construction simply means that we have used the representation L to introduce an auxiliary C^*-norm on $c_{00}(G)$ and took the completion of $c_{00}(G)$ with respect to this norm. A similar construction using all nondegenerate $*$-representations of $c_{00}(G)$ in Hilbert C^*-modules yields a C^*-completion $C^*(G)$ of $c_{00}(G)$, called the full C^*-algebra of G.

Next one can define positive definiteness in this context. Let $\pi \in \Sigma(G)$, for bounded sections ξ, η of $\mathscr{H}^{\pi}$, the function

$$x \longmapsto \langle \pi(x)\xi(s(x)), \eta(r(x))\rangle, \tag{4.5}$$

on G (where the inner product is taken in the Hilbert space $\mathscr{H}^{\pi}_{r(x)}$) is called a coefficient function of π. A function $\varphi \in \ell^{\infty}(G)$ is called positive definite if for all $u \in G^0$ and all $f \in c_{00}(G)$

$$\sum_{x,y\in G^u} \varphi\left(y^{-1}x\right) f(y)\overline{f}(x) \geq 0, \tag{4.6}$$

or, equivalently, for each $n \geq 1$, $u \in G^0$, $x_1, \ldots x_n \in G^u$, and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$

$$\sum_{i,j=1}^{n} \overline{\alpha}_i \alpha_j \varphi\left(x_i^{-1}x_j\right) \geq 0. \tag{4.7}$$

We denote the set of all positive definite functions on G by $P(G)$. The linear span $B(G)$ of $P(G)$ is called the Fourier-Stieltjes algebra of G. It is equal to the set of all coefficient functions of elements of $\Sigma(G)$ [12, Theorem 1]. It is a unital commutative Banach algebra [12, Theorem 2] under pointwise operations and the norm $\|\varphi\| = \inf \|\xi\|\|\eta\|$, where the infimum is taken over all representations $\varphi = \langle \pi(\cdot)\xi \circ s(\cdot), \eta \circ r(\cdot)\rangle$. On the other hand each $\varphi \in B(G)$ could be considered as a completely bounded linear operator on $C^*(G)$ via

$$\langle \varphi, f\rangle = \varphi \cdot f \quad (\varphi \in B(G), f \in c_{00}(G)), \tag{4.8}$$

such that $\|\varphi\|_{\infty} \leq \|\varphi\|_{\text{cb}} \leq \|\varphi\|$ [12, Theorem 3]. The last two norms are equivalent on $B(G)$ (they are equal in the group case, but it is not known if this is the case for groupoids). Following [12] we denote $B(G)$ endowed with cb-norm with $\mathcal{B}(G)$. This is known to be a Banach algebra (This is basically [13, Theorem 6.1] adapted to this framework [12, Theorem 3]).

There are four candidates for the Fourier algebra $A(G)$. The first is the closure of the linear span of the coefficients of $E^2(G)$ in $B(G)$ [14], the second is the closure of $\mathcal{B}(G) \cap c_{00}(G)$ in $\mathcal{B}(G)$ [12], the third is the closure of the of the subalgebra generated by the coefficients of $E^2(G)$ in $B(G)$, and the last one is the completion of the normed space of the quotient of $E^2(G)\widehat{\otimes}E^2(G)$ by the kernel of θ from $E^2(G)\widehat{\otimes}E^2(G)$ into $c_0(G)$ induced by the bilinear map $\theta : c_{00}(G) \times c_{00}(G) \to c_0(G)$ defined by

$$\theta(f, g) = g * \check{f} \quad (f, g \in c_{00}(G)). \tag{4.9}$$

These four give rise to the same algebra in the group case. We refer the interested reader to [12] for a comparison of these approaches. Here we adapt the third definition. Then $A(G)$ is a Banach subalgebra of $B(G)$ and $A(G) \subseteq c_0(G)$.

Now we are ready to compare the function algebras on inverse semigroup S and its associated groupoid S_a. We would apply the above results to $G = S_a$. First let us look at the representation theory of these objects. As a set, S_r compared to S_a has an extra zero element. Moreover, the product of two nonzero elements of S_r is 0, exactly when it is undefined in S_a. Hence it is natural to expect that $\Sigma(S_a)$ is related to $\Sigma_0(S_r) = \Sigma_r(S)$. The major difficulty to make sense of this relation is the fact that representations of S_a are defined through Hilbert bundles, where as restricted representations of S are defined in Hilbert spaces. But a careful interpretation shows that these are two sides of one coin.

Lemma 4.1. *One has* $\Sigma_r(S) = \Sigma(S_a)$.

Proof. Let E be the set of idempotents of S. First let us show that each $\pi \in \Sigma_r(S)$ could be regarded as an element of $\Sigma(S_a)$. Indeed, for each $x \in S$, $\pi(x) : \mathcal{H}_\pi \to \mathcal{H}_\pi$ is a partial isometry, so if we put $\mathcal{H}_u = \pi(u)\mathcal{H}_\pi$ $(u \in E)$, then we could regard $\pi(x)$ as an isomorphism from $\mathcal{H}_{x^*x} \to \mathcal{H}_{xx^*}$. Using the fact that the unit space of S_a is $S_a^0 = E$, it is easy now to check that $\pi \in \Sigma(S_a)$. Conversely suppose that $\pi \in \Sigma(S_a)$, then for each $x \in S_a$, $\pi(x) : \mathcal{H}_{s(x)} \to \mathcal{H}_{r(x)}$ is an isomorphism of Hilbert spaces. Let $\mathcal{H}_\pi$ be the direct sum of all Hilbert spaces $\mathcal{H}_u$, $u \in E$, and define $\pi(x)(\xi_u) = (\eta_v)$, where

$$\eta_v = \begin{cases} \pi(x)\xi_{x^*x}, & \text{if } v = xx^* \\ 0, & \text{otherwise} \end{cases} \quad (x \in S, v \in E), \tag{4.10}$$

then we claim that

$$\pi(x)\pi(y) = \begin{cases} \pi(xy), & \text{if } x^*x = yy^* \\ 0, & \text{otherwise} \end{cases} \quad (x, y \in S). \tag{4.11}$$

First let us assume that $x^*x = yy^*$, then $\pi(xy)(\xi_u) = (\theta_v)$, where $\theta_v = 0$, except for $v = xyy^*x^* = xx^*$, for which $\theta_v = \pi(xy)\xi_{y^*x^*xy} = \pi(xy)\xi_{y^*y}$. On the other hand, $\pi(y)(\xi_u) = (\eta - v)$, where $\eta_v = 0$, except for $v = yy^*$, for which $\eta_v = \pi(y)\xi_{y^*y}$, and $\pi(x)(\eta_v) = (\zeta_w)$, with $\zeta_w = 0$, except for $w = xx^*$, for which $\zeta_w = \pi(x)\eta_{x^*x} = \pi(x)\eta_{yy^*} = \pi(x)\pi(y)\xi_{y^*y}$. Hence $\pi(xy)(\xi_u) = \pi(x)\pi(y)(\xi_u)$, for each $(\xi_u) \in \mathcal{H}_\pi$. Next assume that $x^*x \neq yy^*$, then the second part of the above calculation clearly shows that $\pi(x)\pi(y)(\xi_u) = 0$. This shows that π could be considered as an element of $\Sigma_r(S)$. Finally it is clear that these two embeddings are inverse of each other. □

Next, $S_r = S_a \cup \{0\}$ as sets, and for each bounded map $\varphi : S_r \to \mathbb{C}$ with $\varphi(0) = 0$, it immediately follows from the definition that $\varphi \in P(S_a)$ if and only if $\varphi \in P_0(S_r)$. Hence by above lemma we have the following.

Theorem 4.2. *The Banach spaces* $B_r(S) = B_0(S_r)$ *and* $B(S_a)$ *are isometrically isomorphic.*

This combined with [12, Theorem 2] (applied to $G = S_a$) shows that $B_r(S)$ is indeed a Banach algebra under pointwise multiplication and the above linear isomorphism is also an isomorphism of Banach algebras. By [12, Theorem 1] now we conclude by the following.

Corollary 4.3. $B_r(S)$ *is the set of coefficient functions of* $\Sigma_r(S)$.

There are several other canonical ways to associate a groupoid (besides S_a) to S. Two natural candidates are the universal groupoid [15] and the graph groupoid [16]. The latter is indirectly related to S_r as it used the idea of adding a zero element to S. There is a vast literature on graph C^*-algebras for which we refer the interested reader to [17] and references therein.

To associate a graph groupoid to S it is more natural to start with a (countable) discrete semigroup S without involution and turn it into an inverse semigroup using the idea of [16, Section 3]. Let S be such a semigroup, and let $S^* = \{x^* : x \in S\}$ be a copy of S. Let $s(x) = xx^*$ and $r(x) = x^*x$ be defined formally. Put $E = \{r(x) : x \in S\} \cup \{s(x) : x \in S\}$. Add a zero element

0 which multiplies everything to 0. Let $\mathcal{E}$ be a directed graph with set of vertices being E, the set of direct and inverse edges are S and S^*, respectively. Let $S_{\mathcal{E}}$ be the graph semigroup of $\mathcal{E}$. The inverse 0-semigroup generated by $S \cup S^*$ is defined as the inverse semigroup generated by $S \cup S^*$ subject to $0^* = 0$, $(x^*)^* = x$, $(xy)^* = y^*x^*$, and $x^*y = 0$ unless $x = y$, for $x, y \in S$.

Lemma 4.4. *$S_{\mathcal{E}}$ is an inverse semigroup, and if $\widetilde{S}$ is the inverse 0-semigroup generated by $S \cup S^*$, then $S_{\mathcal{E}} = \widetilde{S}_r$.*

Proof. The graph semigroup $S_{\mathcal{E}}$ is the semigroup generated by $E \cup S \cup S^* \cup \{0\}$ subject to the following relations [16]:

(i) 0 is a zero for $S_{\mathcal{E}}$,

(ii) $s(x)x = x = xr(x)$ and $r(x)x^* = x^* = x^*s(x)$, for all $x \in S$,

(iii) $ab = 0$ if $a, b \in E \cup S \cup S^*$ and $r(a) \neq s(b)$,

(iv) $x^*y = 0$ if $x, y \in S$ and $x \neq y$,

where, in (iii), the source and range of elements in E and S^* are defined naturally. Then $S_{\mathcal{E}}$ is an inverse semigroup [16, Propositions 3.1]. Clearly the $(\widetilde{S}_r, \cdot)$ satisfies all the above relations, and the identity map is a semigroup isomorphism from $S_{\mathcal{E}}$ onto $\widetilde{S}_r$. □

Let T be the set of all pairs (α, β) of finite paths in $\mathcal{E}$ with $r(\alpha) = r(\beta)$ together with a zero element z; then T is naturally an inverse semigroup and $T = S_{\mathcal{E}}$ [16, Propositions 3.2]. Consider those paths of the form (x, x) where $x \in S$ and let E_f be the set of all those idempotents e for which there are finitely many $x \in S$ with $s(x) = e$. Let I be the closed ideal of $\ell^1(T)$ generated by δ_z and elements of the form $\delta_{(e,e)} - \sum_{s(x)=e} \delta_{(x,x)}$ for $e \in E_f$. Then $C_0^*(S_{\mathcal{E}})$ is the universal C^*-algebra of $\ell^1(T)/I$ [16].

Theorem 4.5. *Let $C^*(\mathcal{E})$ be the graph C^*-algebra of $\mathcal{E}$. Then $C^*(\mathcal{E})$ is a quotient of $C_r^*(\widetilde{S})$.*

Proof. By the above lemma and the fact that $T = S_{\mathcal{E}}$, there is a isometric epimorphism $\phi : \ell^1(\widetilde{S}_r)/\mathbb{C}\delta_0 \to \ell^1(T)/I$. let J be the closure of $\ker(\phi)$ in the C^*-norm of $C_r^*(\widetilde{S})$. Then $C_0^*(S_{\mathcal{E}}) \cong C_r^*(\widetilde{S})/J$. Now the result follows from the fact that $C_0^*(S_{\mathcal{E}}) \cong C^*(\mathcal{E})$ [16, Corollary 3.9]. □

A (locally finite) directed graph $\mathcal{E}$ is cofinal if given vertex v and infinite path α, there is a finite path β with $s(\beta) = v$ and $r(\beta) = r(\alpha)$. It has no sinks if there are no edges emanating from any vertex.

Theorem 4.6. *When $\mathcal{E}$ has no loops, then $C_r^*(\widetilde{S})$ is approximately finite dimensional. If moreover $\mathcal{E}$ is cofinal, then $C_r^*(\widetilde{S})$ is simple.*

Proof. If $\mathcal{E}$ has no loops, we have $I = \mathbb{C}z$, hence $J = 0$ and epimorphism ϕ in the proof of the above theorem is an algebra isomorphism. Hence $C_r^*(\widetilde{S}) \cong C^*(\mathcal{E})$. But since $\mathcal{E}$ has no loops, $C^*(\mathcal{E})$ is an AF-algebra [18, Theorem 2.4]. Now assume that $\mathcal{E}$ is also cofinal, then $\mathcal{E}$ has no sinks hence $C_r^*(\widetilde{S}) \cong C^*(\mathcal{E})$ is simple by [18, Corollary 3.11]. □

It also follows from [18, Corollary 3.11] that if $\mathcal{E}$ has no sinks and is cofinal, but it has a loop, then $C_r^*(\widetilde{S})$ is purely infinite. However this case never happens for the directed graph $\mathcal{E}$ constructed above, as it has no loops when $xx^* = e$ or $x^*x = e$ implies that $x = e$, and has no sink when $xx^* = e$ implies that $x = e$, for each $x \in S$ and $e \in E$, but these two conditions are clearly equivalent, and both are equivalent to $S = E$. A concrete example is $S = (\mathbb{N}, \max)$ with $n = n^*$. Also a sufficient condition for $\mathcal{E}$ to be cofinal is that S is finitely transitive; namely, for each $e, f \in E$ there are finitely many $x_i \in S$, $1 \le i \le n$ with $s(x_1) = e, r(x_n) = f$ and $r(x_i) = s(x_{i+1})$, for $1 \le i \le n-1$. Let us say that S is N-transitive if we could always find such a finite path with $n \le N$. A concrete example of a 1-transitive semigroup is the Brandt semigroup B_2 consisting of all pairs (i, j), $i, j \in \{0, 1\}$, plus zero element, with $(i, j)(k, l) = (i, l)$ if $j = k$, and zero otherwise.

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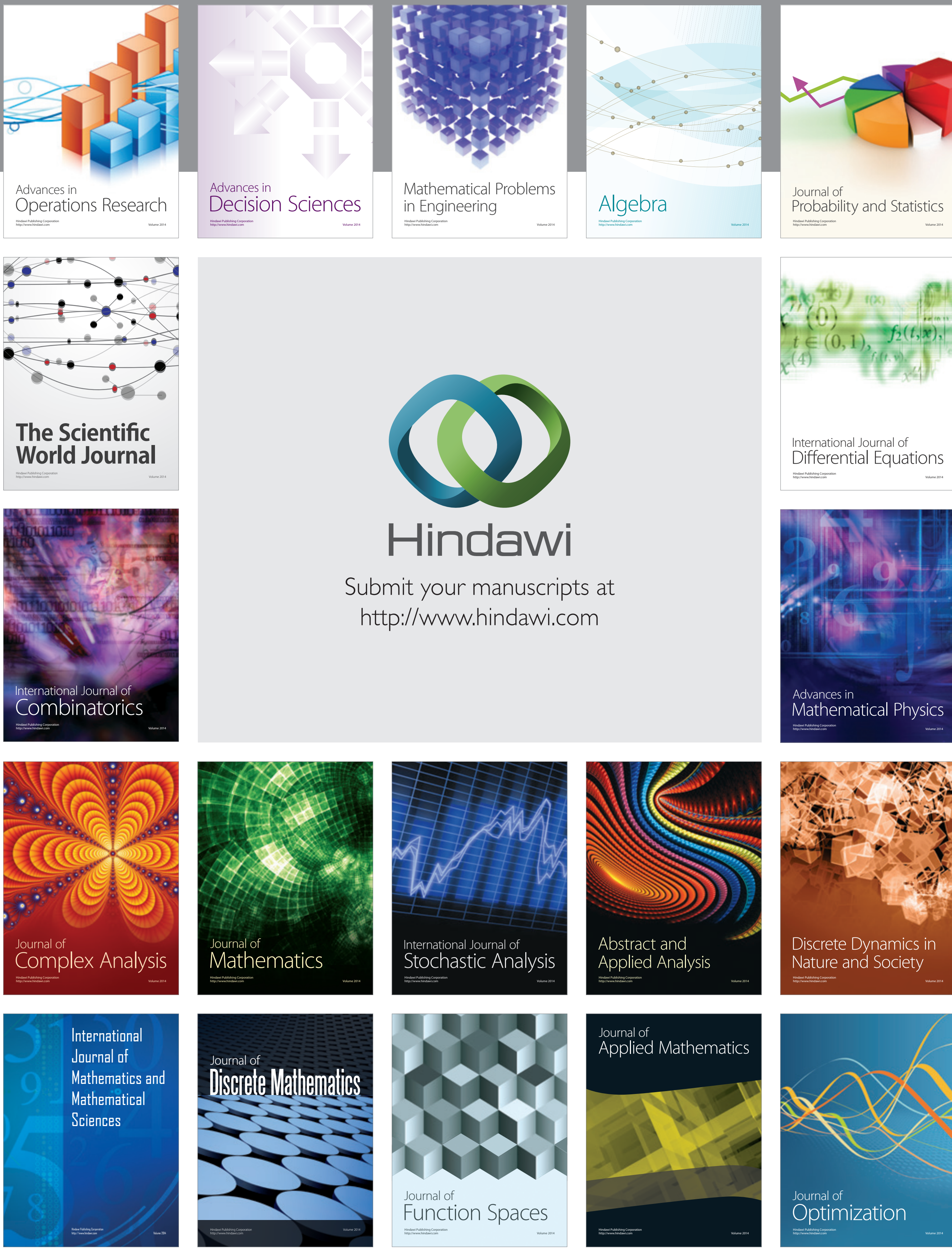
References

[1] M. Amini and A. Medghalchi, "Restricted algebras on inverse semigroups. I. Representation theory," *Mathematische Nachrichten*, vol. 279, no. 16, pp. 1739–1748, 2006.
[2] M. V. Lawson, *Inverse Semigroups: The Theory of Partial Symmetries*, World Scientific, River Edge, NJ, USA, 1998.
[3] R. J. Lindahl and P. H. Maserick, "Positive-definite functions on involution semigroups," *Duke Mathematical Journal*, vol. 38, pp. 771–782, 1971.
[4] C. F. Dunkl and D. E. Ramirez, "L^∞-representations of commutative semitopological semigroups," *Semigroup Forum*, vol. 7, no. 1–4, pp. 180–199, 1974.
[5] M. Amini and A. Medghalchi, "Fourier algebras on topological foundation $*$-semigroups," *Semigroup Forum*, vol. 68, no. 3, pp. 322–334, 2004.
[6] R. Godement, "Les fonctions de type positif et la théorie des groupes," *Transactions of the American Mathematical Society*, vol. 63, pp. 1–84, 1948.
[7] G. K. Pedersen, *C^*-Algebras and Their Automorphism Groups*, vol. 14 of *London Mathematical Society Monographs*, Academic Press, London, UK, 1979.
[8] M. L. Bami, "Bochner's theorem and the Hausdorff moment theorem on foundation topological semigroups," *Canadian Journal of Mathematics*, vol. 37, no. 5, pp. 785–809, 1985.
[9] P. Eymard, "L'algèbre de Fourier d'un groupe localement compact," *Bulletin de la Société Mathématique de France*, vol. 92, pp. 181–236, 1964.
[10] M. L. Bami, "Representations of foundation semigroups and their algebras," *Canadian Journal of Mathematics*, vol. 37, no. 1, pp. 29–47, 1985.
[11] K. J. Oty, "Fourier-Stieltjes algebras of r-discrete groupoids," *Journal of Operator Theory*, vol. 41, no. 1, pp. 175–197, 1999.
[12] A. L. T. Paterson, "The Fourier algebra for locally compact groupoids," *Canadian Journal of Mathematics*, vol. 56, no. 6, pp. 1259–1289, 2004.
[13] J. Renault, "The Fourier algebra of a measured groupoid and its multipliers," *Journal of Functional Analysis*, vol. 145, no. 2, pp. 455–490, 1997.
[14] A. Ramsay and M. E. Walter, "Fourier-Stieltjes algebras of locally compact groupoids," *Journal of Functional Analysis*, vol. 148, no. 2, pp. 314–367, 1997.
[15] A. N. T. Paterson, *Groupoids, Inverse Semigroups, and Their Operator Algebras*, Birkhauser, Boston, Mass, USA, 1998.
[16] A. L. T. Paterson, "Graph inverse semigroups, groupoids and their C^*-algebras," *Journal of Operator Theory*, vol. 48, supplement 3, pp. 645–662, 2002.
[17] I. Raeburn, *Graph Algebras*, vol. 103 of *CBMS Regional Conference Series in Mathematics*, Published for the Conference Board of the Mathematical Sciences, Washington, DC, USA; American Mathematical Society, Providence, RI, USA, 2005.
[18] A. Kumjian, D. Pask, and I. Raeburn, "Cuntz-Krieger algebras of directed graphs," *Pacific Journal of Mathematics*, vol. 184, no. 1, pp. 161–174, 1998.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