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Research Article

Cantor Limit Set of a Topological Transformation Group on $\mathbf{S}^1$

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The limit set of a topological transformation group on $\mathbf{S}^1$ generated by two generators is proved to be totally disconnected (or thin) and perfect if the conditions (i–v) are satisfied. A concrete application to a Doubly Periodic Riccati equation is given.

1. Introduction

The conception *limit set* of a *transformation group* plays an important role in both theory and application of modern mathematics.

Assume that Γ is a transformation group (or semigroup) formed with some continuous self-mapping on a Hausdorff space X. For any $x_0 \in X$, the set

$$\Gamma(x_0) = \{\gamma(x_0) \mid \gamma \in \Gamma\} \tag{1.1}$$

is called an *orbit* through x_0 under the action of Γ. For any subset A of X, let

$$\Gamma(A) = \bigcup_{x \in A} \Gamma(x). \tag{1.2}$$

A subset A of X is called a Γ-*invariant set* if

$$\Gamma(A) = A. \tag{1.3}$$

A subset A of X is called the *least invariant set* of Γ if it is a nonempty closed invariant set, in which there is not any nonempty closed proper subset which is Γ-invariant. Based on the continuity of the elements in Γ, it is easy to obtain the following proposition.

Proposition 1.1. *Let A be a least invariant set of Γ. For any $x_0 \in X$, if there is such a point $a \in A$ that*

$$a \in \overline{\Gamma(x_0)}, \tag{1.4}$$

then

$$A \subset \overline{\Gamma(x_0)}. \tag{1.5}$$

For any $x_0 \in X$, a least invariant set A of Γ is called a *limit set* of the orbit $\Gamma(x_0)$ if

$$A \subset \overline{\Gamma(x_0)}. \tag{1.6}$$

It is easy to prove the following proposition.

Proposition 1.2. *If the topology space X is compact, then under the action of the transformation group (or semigroup) Γ, any least invariant set is perfect if it is not finite.*

Therefore, it is quite possible that a limit set A under the action of Γ may be *totally disconnected and perfect* (a *Cantor set*), and that A may be with *fractal structure* when some measure is attached.

It is an important subject in the modern nonlinear science to study the exact structure of the limit set for a given nonlinear system, especially, to determine if the limit set is a totally disconnected and perfect set (for simple, called a Cantor set). However, it is usually not an easy task to do so, because of the strong nonlinearity and nonintegrability of the system. So it is necessary to explore the conditions for the existence of Cantor-type limit set.

As an example, a traditional dynamical system is a continuous (or discrete) group or a semigroup ϕ_t (or ϕ_n), acting on a manifold M. The related group or semigroup is usually generated by a *single generator*. Both the ω-limit set and α-limit set of an orbit through a point x_0 are limit sets of the corresponding group by the present definition [1]. If a least invariant set or a limit set of a dynamical system is Cantor type, then the related complicated motion is described as *deterministic chaos.* This kind of complicated motion has been considered widely. And some methods for determining if the least invariant set of a dynamical system is Cantor type, such as Melnikov function method, have been well developed.

In the study of structures of high-dimensional leaves of a foliation in modern geometry theory, or concretely, for a differential equation in the complex domain, in the study of structures of the foliation formed with the solution manifolds (Riemann surfaces) as its leaves in phase space, it is in need to study the corresponding *monodromy group* or *holonomy group*, which is a kind of representative of the fundamental group of the foliation on its leaf [2–4]. Different to the dynamic system, the monodromy group is usually generated by *several generators*. Clearly, a complicated limit set of a monodromy group should reflect the complicated structure of the corresponding foliation and its leaves.

Besides the interest in geometry, the structure of the least invariant set of the monodromy group usually relates the integrability of the corresponding ordinary differential equation. In different theories on the integrability of ordinary differential equations, such as the analytical theory [3, 5, 6], Lie group theory [7, 8], differential Galois theory [9–11], and so forth, it is commonly implied that, for almost all of the integrable ordinary differential equations, the least invariant sets of foliations in complex domain should have a simple structure, or precisely, the corresponding monodromy groups should be *solvable* [3, 10].

Concretely, for a second-order linear homogeneous ordinary equation with rational function coefficients, it is well known that every second-order linear homogeneous ordinary equation is corresponding to a Riccati equation [12]; if it is integrable in quadratures, then every least invariant set (including the limit set) of the monodromy group for the corresponding Riccati equation should be a finite set, and each finite least invariant set is corresponding to an algebraic curve solution of the Riccati equation. Otherwise, if a limit set of the monodromy group is not finite, especially if it is of Cantor type, then the Riccati equation and its corresponding second-order linear homogeneous ordinary equation is clearly not integrable in quadratures.

In order to investigate a Riccati equation with more complicated coefficients, Guan et al. studied a concrete doubly periodic Riccati equation with a Weierstrass elliptic function coefficient [13–18]; Guan considered its monodromy group and proposed roughly a method to check if its limit set is Cantor type based on numerical result.

In the present paper, this method is improved into a theorem in a more exact and more general form in Section 2. Combining the result in [19], the critical parameter for the existence of Cantor limit set is given exactly in Section 3.

2. The Theorem and Proof

Theorem 2.1. *Let Γ be a transformation group on the 1-dimensional sphere,*

$$\mathbf{S}^1 = \frac{R}{Z}. \tag{2.1}$$

G is generated by two generators $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are both homeomorphic transformations onto $\mathbf{S}^1$. The least invariant set of Γ is totally disconnected and perfect and is also the unique limit set of any orbit $\Gamma(x)$ if all of the following conditions (i–v) are satisfied.

(i) *Both $\mathbf{a}$ and $\mathbf{b}$ have exactly two fixed points in $\mathbf{S}^1, \alpha_1, \alpha_2$, and β_1, β_2, respectively, that is,*

$$\mathbf{a}(\alpha_i) = \alpha_i, \qquad \mathbf{b}(\beta_i) = \beta_i, \quad i = 1, 2, \tag{2.2}$$

where β_1 is located in the inside of one of the two arcs of $\mathbf{S}^1$ separated by α_1 and α_2, while β_2 is located in the inside of the other one.

(ii) *The arcs $\widehat{\alpha_1\beta_1\alpha_2}$ and $\widehat{\alpha_1\beta_2\alpha_2}$ are both invariant sets of $\mathbf{a}$, that is,*

$$\mathbf{a}\colon \widehat{\alpha_1\beta_1\alpha_2} \longmapsto \widehat{\alpha_1\beta_1\alpha_2}, \qquad \mathbf{a}\colon \widehat{\alpha_1\beta_2\alpha_2} \longmapsto \widehat{\alpha_1\beta_2\alpha_2}, \tag{2.3}$$

and the arcs $\widehat{\beta_1\alpha_1\beta_2}$ *and* $\widehat{\beta_1\alpha_2\beta_2}$ *are both invariant sets of* $\mathbf{b}$.

(*iii*) *For both* $i = 1$ *and* $i = 2$,

$$\lim_{n\to\infty} \mathbf{a}^n(\beta_i) = \alpha_2, \qquad \lim_{n\to\infty} \mathbf{a}^{-n}(\beta_i) = \alpha_1,$$
$$\lim_{n\to\infty} \mathbf{b}^n(\alpha_i) = \beta_2, \qquad \lim_{n\to\infty} \mathbf{b}^{-n}(\alpha_i) = \beta_1. \tag{2.4}$$

(*iv*) *The commutator of* $\mathbf{a}$ *and* $\mathbf{b}$,

$$\mathbf{h} = \mathbf{b}^{-1}\mathbf{a}^{-1}\mathbf{b}\mathbf{a}(\in \Gamma), \tag{2.5}$$

has exactly two different fixed points ξ_1 *and* η_1 *located in the inside of the arc* $\widehat{\alpha_1\beta_1}$, *that is,*

$$\xi_1, \eta_1 \in \widehat{(\alpha_1\beta_1)}. \tag{2.6}$$

where ξ_1 *is closer to* α_1 *than* η_1. *In this paper,* $\widehat{(\alpha_1\beta_1)}$ *represents the open arc without two ends* α_1 *and* β_1, *and* $\widehat{[\alpha_1\beta_1]}$ *represent, the closed arc with the two ends* α_1 *and* β_1,

(*v*) *Under the actions of* $\mathbf{a}$ *and* $\mathbf{b}$, *the points* ξ_1 *and* η_1 *are transformed into other three arcs different to* $\widehat{\alpha_1\beta_1}$:

$$\begin{aligned}
\xi_2 &= \mathbf{a}(\xi_1), & \eta_2 &= \mathbf{a}(\eta_1), & \xi_2, \eta_2 &\in \widehat{(\beta_1\alpha_2)},\\
\xi_3 &= \mathbf{b}(\xi_2), & \eta_3 &= \mathbf{b}(\eta_2), & \xi_3, \eta_3 &\in \widehat{(\beta_2\alpha_2)},\\
\xi_4 &= \mathbf{b}(\xi_1), & \eta_4 &= \mathbf{b}(\eta_1), & \xi_4, \eta_4 &\in \widehat{(\alpha_1\beta_2)}.
\end{aligned} \tag{2.7}$$

Proof. Let

$$I_0 = \widehat{(\xi_1\eta_1)}. \tag{2.8}$$

By the homeomorphic property of transformations and conditions (ii) and (v), we can see

$$\begin{aligned}
\widehat{(\xi_2\eta_2)} &= \mathbf{a}(I_0),\\
\widehat{(\xi_3\eta_3)} &= \mathbf{b}(\mathbf{a}(I_0)),\\
\widehat{(\xi_4\eta_4)} &= \mathbf{b}(I_0)\Big(= \mathbf{a}^{-1}(\mathbf{b}(\mathbf{a}(I_0)))\Big).
\end{aligned} \tag{2.9}$$

Let

$$O(0) = I_0 \bigcup \mathbf{a}(I_0) \bigcup \mathbf{b}(\mathbf{a}(I_0)) \bigcup \mathbf{b}(I_0), \tag{2.10}$$

and let

$$F(0) = \mathbf{S}^1 \setminus O(0); \tag{2.11}$$

then the following facts and conclusions can be derived step by step.

Fact 1. The closed set $F(0)$ is formed with following four separated closed arcs:

$$\begin{aligned} F_{\alpha_1} &= [\widehat{\eta_4\alpha_1\xi_1}], \\ F_{\alpha_2} &= [\widehat{\eta_2\alpha_2\xi_3}], \\ F_{\beta_1} &= [\widehat{\eta_1\beta_1\xi_2}], \\ F_{\beta_2} &= [\widehat{\eta_3\beta_2\xi_4}]. \end{aligned} \tag{2.12}$$

Fact 2. Obviously,

$$\begin{aligned} \mathbf{a}(F_{\alpha_1}) &= F_{\beta_2} \bigcup F_{\beta_1} \bigcup F_{\alpha_1} \bigcup I_0 \bigcup b(I_0), \\ \mathbf{a}^{-1}(F_{\alpha_2}) &= F_{\beta_2} \bigcup F_{\beta_1} \bigcup F_{\alpha_2} \bigcup \mathbf{a}(I_0) \bigcup \mathbf{b}(\mathbf{a}(I_0)), \\ \mathbf{b}(F_{\beta_1}) &= F_{\alpha_2} \bigcup F_{\alpha_1} \bigcup F_{\beta_1} \bigcup I_0 \bigcup \mathbf{a}(I_0), \\ \mathbf{b}^{-1}(F_{\beta_2}) &= F_{\alpha_2} \bigcup F_{\alpha_1} \bigcup F_{\beta_2} \bigcup \mathbf{b}(I_0) \bigcup \mathbf{b}(\mathbf{a}(I_0)). \end{aligned} \tag{2.13}$$

Fact 3. From (i), (ii), and (iii), it follows that

$$\lim_{n\to\infty} \mathbf{a}^n(x) = \alpha_2, \qquad \lim_{n\to\infty} \mathbf{a}^{-n}(x) = \alpha_1, \quad \forall x \in \mathbf{S}^1 \setminus \{\alpha_1, \alpha_2\}, \tag{2.14}$$

and that

$$\lim_{n\to\infty} \mathbf{b}^n(x) = \beta_2, \qquad \lim_{n\to\infty} \mathbf{b}^{-n}(x) = \beta_1, \quad \forall x \in \mathbf{S}^1 \setminus \{\beta_1, \beta_2\}. \tag{2.15}$$

Fact 4. Clearly, the commutator $\mathbf{h}$ of $\mathbf{a}$ and $\mathbf{b}$ is also a homeomorphic transformation. From the condition (iv), it follows that, either

$$\lim_{n\to-\infty} \mathbf{h}^n(x) = \xi_1, \qquad \lim_{n\to+\infty} \mathbf{h}^n(x) = \eta_1 \tag{2.16}$$

or

$$\lim_{n \to -\infty} \mathbf{h}^n(x) = \eta_1, \qquad \lim_{n \to +\infty} \mathbf{h}^n(x) = \xi_1 \tag{2.17}$$

if

$$x \in \mathbf{S}^1 \setminus \{\xi_1, \eta_1\}. \tag{2.18}$$

Fact 5. Let Z_0 represent the set of all nonzero integers, and let

$$F_\alpha = F_{\alpha_1} \bigcup F_{\alpha_2}, \qquad F_\beta = F_{\beta_1} \bigcup F_{\beta_2}; \tag{2.19}$$

then the family of transformations $\{\mathbf{a}^m\}_{m \in Z_0}$ changes F_β into a family of separated closed arcs in F_α, which are condensed at α_1 and α_2, and the family of transformation $\{\mathbf{b}^m\}_{m \in Z_0}$ changes F_α into a family of separated closed arcs in F_β with β_1 and β_2 as their condensation points.

Fact 6. Let

$$\begin{aligned} F_\alpha(1) &= \bigcup_{m \in Z_0} \mathbf{a}^m(F_\beta) \subset F_\alpha, \\ F_\beta(1) &= \bigcup_{m \in Z_0} \mathbf{b}^m(F_\alpha) \subset F_\beta, \end{aligned} \tag{2.20}$$

and let

$$\begin{gathered} O(1) = \bigcup_{m \in Z} \left[\mathbf{a}^m(O(0)) \bigcup \mathbf{b}^m(O(0))\right], \\ F(1) = \overline{F_\alpha(1)} \bigcup \overline{F_\beta(1)}; \end{gathered} \tag{2.21}$$

then it follows that

$$F(1) = \mathbf{S}^1 \setminus O(1). \tag{2.22}$$

For an integer n greater than 1, we may inductively let

$$\begin{gathered} F_\alpha(n) = \bigcup_{m \in Z_0} \mathbf{a}^m\big(F_\beta(n-1)\big) \subset F_\alpha(n-1), \\ F_\beta(n) = \bigcup_{m \in Z_0} \mathbf{a}^m(F_\alpha(n-1)) \subset F_\beta(n-1), \\ O(n) = \bigcup_{m \in Z} \left[\mathbf{a}^m(O(n-1)) \bigcup \mathbf{b}^m(O(n-1))\right] \supset O(n-1), \\ F(n) = \overline{F_\alpha(n)} \bigcup \overline{F_\beta(n)} \Big(= \mathbf{S}^1 \setminus O(n)\Big). \end{gathered} \tag{2.23}$$

Clearly, the closed set $F(n)$ is formed with a family of separated closed arcs and their condensation points. The ends of these closed arcs belong to the set

$$\Gamma(\xi_1)\bigcup\Gamma(\eta_1), \tag{2.24}$$

and these condensation points belong to the set

$$V = \Gamma(\alpha_1)\bigcup\Gamma(\alpha_2)\bigcup\Gamma(\beta_1)\bigcup\Gamma(\beta_2). \tag{2.25}$$

Obviously,

$$F(n) \supset F(n+1). \tag{2.26}$$

Fact 7. Let $P(n)$ be the set of the ends of all separated closed arcs in $F(n)$, and let

$$B = \Gamma(\xi_1)\bigcup\Gamma(\eta_1). \tag{2.27}$$

From the Fact 3 obtained, it follows that

$$\begin{gathered} P(n) \subset P(n+1) \subset B, \\ B = \bigcup_n P(n). \end{gathered} \tag{2.28}$$

Both B and $\overline{B}$ are obvious invariants under the action of Γ.

Fact 8. From the Facts 3 and 4, it follows that

$$\overline{B} = \overline{V}. \tag{2.29}$$

Fact 9. Clearly, by the construction of the sets $O(n)$,

$$\Gamma(I_0) = \Gamma(O(0)) = \bigcup_{n\in N} O(n), \tag{2.30}$$

and for any elements $\mathbf{a}_1$ and $\mathbf{a}_2$ in Γ, if

$$\mathbf{a}_1(I_0)\bigcap\mathbf{a}_2(I_0) \neq \emptyset \text{ (empty set)}, \tag{2.31}$$

then

$$\mathbf{a}_1(I_0) = \mathbf{a}_2(I_0). \tag{2.32}$$

$\Gamma(I_0)$ is an Γ invariant open set.

Fact 10. By the definition of B,

$$\overline{B}\bigcap\Gamma(I_0) = \emptyset, \tag{2.33}$$

and from Facts 3, 4, and 8,

$$\overline{V} = \overline{B} \subset \overline{\Gamma(I_0)}. \tag{2.34}$$

Fact 11. Let

$$L = \overline{V}\left(= \overline{B}\right). \tag{2.35}$$

Then from Fact 10, it is easy to see that there is no inner point in the closed set L. And from the Fact 3 and 4, it follows that

$$L \subseteq \overline{\Gamma(x)}, \quad \forall x \in S^1, \tag{2.36}$$

and that there is no isolated point in L. Therefore, L, as a limit set of Γ, is a totally disconnected and perfect set.

Fact 12. From Facts 3 and 4, it is easy to see that

$$L = \overline{V} = \overline{B} = \overline{\Gamma(\alpha_1)} = \overline{\Gamma(\alpha_2)} = \overline{\Gamma(\beta_1)} = \overline{\Gamma(\beta_2)} = \overline{\Gamma(\xi_1)} = \overline{\Gamma(\eta_1)}. \tag{2.37}$$

It follows that L is a least invariant closed set of Γ.

The theorem has been proved through the above facts. □

3. Application of the Theorem

In [13] we considered a doubly periodic Riccati equation in the complex domain:

$$\frac{dz}{dt} = z^2 - \lambda\varrho(t), \quad z, t \in C, \tag{3.1}$$

where the coefficient $\varrho(t)$ is a Weierstrass elliptic function satisfying

$$\begin{aligned} \left[\varrho'(t)\right]^2 &= 4\left[\varrho^2(t) - 1\right]\varrho(t), \quad \varrho(0) = 0, \\ \varrho(t+T) &= \varrho(t+iT) = \varrho(t), \quad \forall t \in C, \end{aligned} \tag{3.2}$$

where

$$T = \frac{\Gamma(1/2)\Gamma(1/4)}{2\Gamma(3/4)} \tag{3.3}$$

is the real period of $\varrho(t)$.

Because of the doubly periodicity, (3.1) can be treated as a differential equation defined on the torus $\mathbf{T}^2$ [18].

By the numerical results, we guessed that the solution space may have a fractal limit set when $\lambda < 0$.

In [18], Guan had qualitatively proved the existence of the fractal structure for $\lambda \in (-\infty, \lambda_0)$, where the critical value λ_0 was evaluated approximately as -0.227.

In [19], using the symmetries of (3.1), Guan et al. obtained the exact formulae of two generators E and U of the monodromy group G of (3.1), which are represented by the *Möbius* transformations on the extended complex plane $\widehat{C}$ as follows:

$$\begin{aligned} E(c) &= e^{2\tau}c, \\ U(c) &= \frac{i \sinh \tau + c \cosh \tau}{\cosh \tau - ic \sinh \tau}, \end{aligned} \qquad \forall c \in \widehat{C}, \tag{3.4}$$

where the parameter τ depends on the parameter λ in (3.1) through

$$\sinh^2(\tau) = \cos\left(\frac{\sqrt{1+4\lambda}}{2}\pi\right). \tag{3.5}$$

In addition, it is proved further that

$$\tau = \mu T, \tag{3.6}$$

where μ is the *Floquet exponent* of the related second-order linear ordinary differential equation

$$\ddot{u} - \lambda \varrho(t) u = 0. \tag{3.7}$$

Therefore, (3.4), (3.5), and (3.6) give the exact relation between the generators of the monodromy group, the Floquet exponent and the parameter λ in the equation. In the theory of the differential equations, this is a very rare case that these exact relations could be obtained.

Now by the exact relation (3.4) and (3.5), we may see that, if

$$\lambda \le 0, \text{ or } \lambda \in [2n(2n-1), 2n(2n+1)], \quad n \in N, \tag{3.8}$$

then the extended imaginary axis, the left half complex plane corresponding to the negative real part, and the right half complex plane corresponding to the positive real part are all

G-invariant sets. And the extended imaginary axis is just homeomorphic to $\mathbf{S}^1$, in which the points $c_1 = 0, c_2 = \infty, c_3 = i$, and $c_4 = -i$ are, respectively, the fixed points of E and U. Notice that, if and only if the real parameter λ satisfies

$$\lambda < -\frac{1}{4}, \tag{3.9}$$

then

$$\cos\left(\frac{\sqrt{1+4\lambda}}{2}\pi\right) > 1, \tag{3.10}$$

so that, either

$$e^{2\tau} > 3 + 2\sqrt{2}, \tag{3.11}$$

or

$$0 < e^{2\tau} < 3 - 2\sqrt{2}, \tag{3.12}$$

and that the commutator of E and U, $H = U^{-1}E^{-1}UE$, has exactly two different fixed points on the extended imaginary axis,

$$\begin{aligned}\xi_1 &= \frac{\left[1 - e^{2\tau} + \sqrt{(1-e^{2\tau})^2 - 4e^{2\tau}}\right]i}{2e^{2\tau}},\\ \eta_1 &= \frac{\left[1 - e^{2\tau} - \sqrt{(1-e^{2\tau})^2 - 4e^{2\tau}}\right]i}{2e^{2\tau}}.\end{aligned} \tag{3.13}$$

In this case, either the two fixed points are both located in the interval between the points $c_4 = -i$ and $c_1 = 0$, if $e^{2\tau} > 3 + 2\sqrt{2}$ is hold, or they are both located in the interval between $c_3 = i$ and $c_2 = \infty$, if $0 < e^{2\tau} < 3 - 2\sqrt{2}$ is hold. It is easy to prove further that the conditions (i)–(v) are all satisfied if $\lambda < -1/4$. Therefore, the least invariant set of G on the extended imaginary axis is totally disconnected and perfect. So, by the theorem obtained, the critical value of λ for the existence of the Cantor limit set can be exactly determined as $\lambda_0 = -1/4$.

Now, it can be seen that, if $\lambda < -1/4$, the doubly periodic Riccati equation (3.1) is not integrable by quadratures, and the limit set of foliation of the equation is like a *Cantor book*, since each point in the limit set of the monodromy group is corresponding to a piece of leaf of the foliation's limit set [18].

The related Hausdorff dimension of the least invariant set of the monodromy group is evaluated in [18] through some measure consideration.

4. Conclusion

The theorem obtained provides the condition for determining if a transformation group on $\mathbf{S}^1$ with two generators has a Cantor-like limit set. The example in Section 3 shows this theorem can be applied to the study of the complexity of the limit sets of foliations. This theorem can also be applied to the theory of discrete groups to determine if a discrete group is a Fuchsian group of the second kind [20].

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