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Research Article

Stability of Admissible Functions

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By using the concept of the weak subordination, we examine the stability (a class of analytic functions in the unit disk is said to be stable if it is closed under weak subordination) for a class of admissible functions in complex Banach spaces. The stability of analytic functions in the following classes is discussed: Bloch class, little Bloch class, hyperbolic little Bloch class, extend Bloch class (Q_p), and Hilbert Hardy class (H^2).

1. Introduction

We denote by U the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and by $\mathscr{H}(U)$ the space of all analytic functions in U. A function I, analytic in U, is said to be an inner function if and only if $|I(z)| \leq 1$ such that $|I(e^{i\theta})| = 1$ almost everywhere. We recall that an inner function I can be factored in the form $I = BS$ where B is a Blaschke product and S is a singular inner function takes the form

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\right), \quad z \in U, \tag{1.1}$$

where μ is a finite positive Lebesgue measure.

Let X and Y represent complex Banach spaces. The class of admissible functions $\mathcal{G}(X, Y)$, consists of those functions $f : X \to Y$ that satisfy

$$\|f(x)\| \geq 1, \quad \text{when } \|x\| = 1. \tag{1.2}$$

If f and g are analytic functions with $f, g \in \mathcal{G}(X, Y)$, then f is said to be weakly subordinate to g, written as $f \prec^w g$ if there exist analytic functions $\phi, \omega : U \to X$, with ϕ an inner function ($\|\phi\|_X \leq 1$), so that $f \circ \phi = g \circ \omega$. A class $\mathcal{C}$ of analytic functions in X is said to be stable if it is closed under weak subordination, that is, if $f \in \mathcal{C}$ whenever f and g are analytic functions in X with $g \in \mathcal{C}$ and $f \prec^w g$.

By making use of the above concept of the weak subordination, we examine the stability for a class of admissible functions in complex Banach spaces $\mathcal{G}(X,Y)$. The stability of analytic functions appears in Bloch class, little Bloch class, hyperbolic little Bloch class, extend Bloch class (Q_p), and Hilbert Hardy class (H^2).

2. Stability of Bloch Classes

If f is an analytic function in U, then f is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in U}\left(1-|z|^2\right)|f'(z)| < \infty. \tag{2.1}$$

The space of all Bloch functions is denoted by $\mathcal{B}$. The little Bloch space $\mathcal{B}_0$ consists of those $f \in \mathcal{B}$ such that

$$\lim_{|z|<1}\left(1-|z|^2\right)|f'(z)| = 0. \tag{2.2}$$

The hyperbolic Bloch class $\mathcal{B}^h$ is defined by using the hyperbolic derivative in place of the ordinary derivative in the definition of the Bloch space, where the hyperbolic derivative of an analytic self-map $\varphi : U \to U$ of the unit disk is given by $|\varphi'|/(1-|\varphi|^2)$. That is, $\varphi \in \mathcal{B}^h$ if it is analytic and

$$\sup_{z\in U}\frac{\left(1-|z|^2\right)|\varphi'(z)|}{1-|\varphi(z)|^2} < \infty. \tag{2.3}$$

Similarly, we say $\varphi \in \mathcal{B}_0^h$, the hyperbolic little Bloch class, if $\varphi \in \mathcal{B}^h$ and

$$\lim_{|z|=1}\frac{\left(1-|z|^2\right)|\varphi'(z)|}{1-|\varphi(z)|^2} = 0. \tag{2.4}$$

Note that, for the function $\varphi : U \to X$, we replace $|\cdot|$ by $\|\cdot\|_X$ in the above definitions.

Theorem 2.1. *Let X be a complex Banach space. If X contains all inner functions in U, then $\mathcal{G}(X,X)$ is stable.*

Proof. Suppose that X contains all inner functions $I : U \to X$. Take $g(z) = \phi(z) = z$ and $\omega(z) = I(z)$ $(z \in U)$. Then, ϕ and ω are inner functions and $I = I \circ \phi = g \circ \omega$. Hence, $I \prec^{w} g$, $g \in X$, and $I \in X$. Thus, X is stable. □

Theorem 2.2. *Let $X = \mathcal{H}(U)$ be a space of analytic functions in U which satisfies $\mathcal{B} \subset X$. Then, $\mathcal{G}(U,X)$ is stable.*

Proof. Suppose that $X = \mathcal{H}(U)$ and $\mathcal{B} \subset X$. Let $f \in X$, $\mathcal{E} = \{m + ni : m,n \in \mathbb{Z}\}$ and $F = \{z \in U : f(z) \in \mathcal{E}\}$. Since F is a countable subset of U, it has capacity zero and therefore the universal covering map I from U onto $U \setminus F$ is an inner function (see, e.g., Chapter 2 of [1]). Set $g = f \circ I$, then the image of g is contained in $\mathbb{C}/\mathcal{E}$. Consequently, see [2], g is a Bloch function. Since $\mathcal{B} \subset X$, we have that $g = f \circ I \in X$ even though $f \in X$. Thus, X is stable. □

Theorem 2.3. *Let $X = \mathscr{H}(U)$ be a space of analytic functions in U and $f \in \mathcal{G}(X, X)$. If $I : U \to X$ satisfying $I(0) = \Theta$ (the zero element in X) and $\|f(I(z))\|_X < 1$, then $\mathcal{G}(X, X)$ is stable.*

Proof. Assume that $X = \mathscr{H}(U)$, $f \in \mathcal{G}(X, X)$, and $I : U \to X$ with $I(0) = \Theta$. Then (see [3])

$$\|f(I(z))\|_X < 1 \Longrightarrow \|I(z)\|_X < 1, \tag{2.5}$$

hence I is an inner function in X. By putting $g := f \circ I$, we obtain that $g \in X$ even though $f \in X$. Thus, X is stable and consequently yields the stability of $\mathcal{G}(X, X)$. □

Next we discuss the stability of the spaces $\mathcal{B}_0$ and $\mathcal{B}_0^h$. An analytic self-map φ of U induces a linear operator $C_\varphi : \mathscr{H}(U) \to \mathscr{H}(U)$, defined by $C_\varphi f = f \circ \varphi$. This operator is called the composition operator induced by φ.

Recall that a linear operator $T : X \to Y$ is said to be bounded if the image of a bounded set in X is a bounded subset of Y, while T is compact if it takes bounded sets to sets with compact closure. Furthermore, if T is a bounded linear operator, then it is called weakly compact, if T takes bounded sets in X to relatively weakly compact sets in Y. By using the operator $C_\varphi f$, we have the following result.

Theorem 2.4. *If $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is compact, then it is stable.*

Proof. Assume the analytic self-map φ of U and $f \in \mathcal{B}_0$; thus, we have the linear operator $C_\varphi : \mathcal{B}_0 \to \mathcal{B}_0$, defined by $C_\varphi f = f \circ \varphi := g$. By the assumption, we obtain that g is compact function in $\mathcal{B}_0$. Hence, φ is an inner function [4, Corollary 1.3] which implies the stability of $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$. □

Theorem 2.5. *Let φ be holomorphic self-map of U such that*

$$\lim_{|z|=1} \frac{\left(1 - |z|^2\right)|\varphi'(z)|}{1 - |\varphi(z)|^2} = 0. \tag{2.6}$$

Then, $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is stable.

Proof. Assume the analytic self-map φ of U and $f \in \mathcal{B}_0$; hence, in virtue of [5, Theorem 4.7], it is implied that the composition operator $C_\varphi f$ on $\mathcal{B}_0$ is compact. Thus we pose that $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is stable. □

Theorem 2.6. *Consider φ is a holomorphic self-map of U, satisfying the following condition: for every $\epsilon > 0$, there exists $0 < r < 1$ such that*

$$\frac{\left(1 - |z|^2\right)|\varphi'(z)|}{1 - |\varphi(z)|^2} < \epsilon \tag{2.7}$$

when $|\varphi| > r$. Then, $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is stable.

Proof. Assume the analytic self-map φ of U and $f \in \mathcal{B}_0$; hence, in virtue of [5, Theorem 4.8], it is yielded that the composition operator $C_\varphi f$ on $\mathcal{B}_0$ is compact. Hence, we obtain that $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is stable. □

Theorem 2.7. *If $C_\varphi f$ is weakly compact in $\mathcal{B}_0$, then $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is stable.*

Proof. According to [5, Theorem 4.10], we have that $C_\varphi f$ is compact in $\mathcal{B}_0$ and, consequently, $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is stable. □

Theorem 2.8. *Let φ be holomorphic self-map of U. If the function*

$$\tau_\varphi(z) := \frac{\left(1-|z|^2\right)|\varphi'(z)|}{1-|\varphi(z)|^2} \tag{2.8}$$

is bounded, then $\mathcal{G}(\mathcal{B}_0^h, \mathcal{B}_0^h)$ is stable.

Proof. Assume the analytic self-map φ of U and $f \in \mathcal{B}_0^h$. Since $\tau_\varphi(z)$ is bounded, then in virtue of [4, Theorem 1.2], it is yielded that φ is an inner function. By putting $g := f \circ \varphi$, where $g \in \mathcal{B}_0^h$, we have the desired result. □

Theorem 2.9. *If $\varphi \in \mathcal{B}_0^h$, then $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is stable.*

Proof. Following [4], it will be shown that there are inner functions $\varphi \in \mathcal{B}_0^h$; then, $C_\varphi : \mathcal{B}_0 \to \mathcal{B}_0$ is compact (see [4, 5]). Thus, $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is stable. □

Theorem 2.10. *Let φ be self-map in U and $w : (0,1] \to (0,\infty)$ be continuous with $\lim_{t=0} w(t) = 0$ satisfying*

$$\lim_{|z|\uparrow 1} \frac{\left(1-|z|^2\right)|\varphi'(z)|}{w\left(1-|\varphi(z)|^2\right)} = 0, \tag{2.9}$$

then $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is stable.

Proof. According to [5, Theorem 5.15], we pose that φ is inner. Thus, in view of [4, Corollary 1.3], $C_\varphi : \mathcal{B}_0 \to \mathcal{B}_0$ is compact; hence, $\mathcal{G}(\mathcal{B}_0, \mathcal{B}_0)$ is stable. □

Remark 2.11. The Schwarz-Pick Lemma implies

(i) C_φ maps $\mathcal{B}$ to $\mathcal{B}$;

(ii) $0 \le |\tau_\varphi(z)| \le 1$;

(iii) $\varphi \in \mathcal{B}_0$ if C_φ maps $\mathcal{B}_0 \to \mathcal{B}_0$ and conversely, $f, \varphi \in \mathcal{B}_0 \Rightarrow f \circ \varphi \in \mathcal{B}_0$.

3. Stability of the Hilbert Hardy Space

In this section, we assume that $f \in H^2$, where H^2 is the Hilbert Hardy space on U, that is, the set of all analytic functions on U with square summable Taylor coefficients. It is well known that each such φ (self-map in U) induces a bounded composition operator $C_\varphi f = f \circ \varphi$ on

H^2. Moreover, Joel Shapiro obtained the following characterization of inner functions [6]: the function $\varphi : U \to U$ is inner if and only if

$$\|C_\varphi f\|_e = \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}, \tag{3.1}$$

where $|C_\varphi f|_e$ denotes the essential norm of $C_\varphi f$.

Theorem 3.1. *Let φ be self-map of U and $f \in H^2$. If* (1.1) *holds, then $\mathcal{G}(H^2, H^2)$ is stable.*

Proof. Assume the analytic self-map φ of U and $f \in H^2$. Condition (1.1) implies that φ is an inner function. By setting $g := C_\varphi f = f \circ \varphi$ and, consequently, $g \in H^2$, it is yielded that $\mathcal{G}(H^2, H^2)$ is stable. □

Next we will show that the compactness of $C_\varphi f$ introduces the stability of $\mathcal{G}(H^2, H^2)$. Two positive (or complex) measures μ and ν defined on a measurable space (Ω, Σ) are called singular if there exist two disjoint sets A and B in Σ whose union is Ω such that μ is zero on all measurable subsets of B while ν is zero on all measurable subsets of A.

Theorem 3.2. *If the composition operator $C_\varphi f : H^2 \to H^2$ is compact, then $\mathcal{G}(H^2, H^2)$ is stable.*

Proof. Since $C_\varphi : H^2 \to H^2$ is compact, all the Aleksandrov measures of φ are singular absolutely continuous with respect to the arc-length measure (see [7, 8]). Thus, in view of [4, Remark 1], φ is inner. By letting $g := C_\varphi f = f \circ \varphi$, and, consequently, $g \in H^2$, it is yielded that $\mathcal{G}(H^2, H^2)$ is stable. □

Theorem 3.3. *If φ has values never approach the boundary of U, then $\mathcal{G}(H^2, H^2)$ is stable.*

Proof. Assume the composition operator $C_\varphi : H^2 \to H^2$. Since φ has values never approach the boundary of U:

$$\|\varphi\|_\infty = \sup\{|\varphi|,\ z \in U\} < 1, \tag{3.2}$$

C_φ is compact on H^2 (see [9, 10]). Hence, φ is an inner function and $\mathcal{G}(H^2, H^2)$ is stable. □

Remark 3.4. (i) It is well known that if C_φ is compact on H^2, then it is compact on H^p for all $0 < p < \infty$ (see [9, Theorem 6.1]).

(ii) C_φ is compact on H^∞ if and only if $\|\varphi\|_\infty < 1$ (see [10, Theorem 2.8]).

Theorem 3.5. *If $\varphi \in \mathcal{B}_0^h$ is univalent then, $\mathcal{G}(L_a^p, H^q)$, $0 < p < q < \infty$ is stable, where L_a^p, H^q are the classical Bergman and Hardy spaces.*

Proof. Since φ is univalent, $C_\varphi : L_a^p \to H^q$ is compact for all $0 < p < q < \infty$ (see [11, Theorem 6.4]). In view of Remark 3.4, we obtain that φ is an inner function; hence, $\mathcal{G}(L_a^p, H^q)$ is stable. □

Next, we use the angular derivative criteria to discuss the stability of admissible functions. Recall that φ has angular derivative at $\zeta \in \partial U$ if the nontangential $\lim w = f(\zeta) \in \partial U$ exists and if $(f(z) - f(\zeta))/(z - \zeta)$ converges to some $\mu \in \mathbb{C}$ as $z \to \zeta$ nontangentially.

Theorem 3.6. *If φ satisfies both the angular derivative criteria and*

$$\sup\{n_{\varphi}(w) : r < |w| < 1, 0 < r < 1\} < \infty, \tag{3.3}$$

where $n_{\varphi}(w)$ is the number of points in $\varphi^{-1}(w)$ with multiplicity counted, then $\mathcal{G}(H^2, H^2)$ is stable.

Proof. According to [12, Corollary 3.6], we have that C_{φ} is compact on H^2. Again in view of Remark 3.4, we obtain that φ is inner and, consequently, $\mathcal{G}(H^2, H^2)$ is stable. □

4. Stability of Q_p Class

For $0 < p < 1$, an analytic function f in U is said to belong to the space Q_p if

$$\sup_{a\in U} \int_U \left|f'(z)\right|^2 g(z,a)^p dA(z) < \infty, \tag{4.1}$$

where $dA(z) = dxdy = rdrd\theta$ is the Lebesgue area measure and g denotes the Green function for the disk given by

$$g(z;a) = \log \frac{1-\overline{a}z}{a-z}, \qquad a, z \in U, \; a \neq z. \tag{4.2}$$

The spaces Q_p are conformally invariant. In [13], It was shown that $Q_p = \mathcal{B}$ for all p, while $Q_1 = BMOA$, the space of those $f \in H^1$ whose boundary values have bounded mean oscillation on ∂U (see [14]). For $0 < \alpha < 1$, Λ^{α} is the Lipschitz space, consisting of those $f \in \mathcal{H}(U)$, which are continuous in U and satisfy

$$\left|f(z_1) - f(z_2)\right| \leq C|z_1 - z_2|^{\alpha}, \qquad z_1, z_2 \in U, \tag{4.3}$$

for some $C = C(f) > 0$. In this section, we will show the stability of functions belong to the spaces Q_1 and Λ^{α}.

Theorem 4.1. *If $f \in Q_1$, then $\mathcal{G}(Q_1, Q_1)$ is stable.*

Proof. In the similar manner of Theorem 2.2, we pose an inner function φ on U. Now, in view of [15, Theorem H], yields $g := f \circ \varphi \in Q_1$, even though $f \in Q_1$. Thus, Q_1 is stable. □

Theorem 4.2. *If $f \in \Lambda^{\alpha}$, $0 < \alpha < 1$ such that*

$$\left|f(z)\right| = O\left((1-|z|)^{\alpha}\right) \tag{4.4}$$

for some z, then $\mathcal{G}(\Lambda^{\alpha}, \Lambda^{\alpha})$ is stable.

Proof. Again as in Theorem 2.2, we obtain an inner function φ on U. Now in view of [15, Theorem K], yields $g := f \circ \varphi \in \Lambda^{\alpha}$, even though $f \in \Lambda^{\alpha}$. Thus, Λ^{α} is stable. □

5. Conclusion

From above, we conclude that the composition operator C_φ, of admissible functions in different complex Banach spaces, plays an important role in stability of these spaces. It was shown that the compactness of this operator implied the stability, when φ is an inner function on the unit disk U. Furthermore, weakly compactness imposed the stability of Bloch spaces. In addition, noncompactness leaded to the stability for some spaces such as Q_p-spaces and Lipschitz spaces.

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