Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 352590, 10 pages
doi:10.1155/2011/352590

Research Article

Garding's Inequality for Elliptic Differential Operator with Infinite Number of Variables

Ahmed Zabel[1] and Maryam Alghamdi[2]

[1] *Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, 11884 Cairo, Egypt*
[2] *Department of Mathematics, King Abdulaziz University, P.O. Box 4087, Jeddah 21491, Saudi Arabia*

Correspondence should be addressed to Maryam Alghamdi, m_aljinaidi@hotmail.com

Received 21 July 2010; Accepted 17 November 2010

Academic Editor: Zayid Abdulhadi

We formulate the elliptic differential operator with infinite number of variables and investigate that it is well defined on infinite tensor product of spaces of square integrable functions. Under suitable conditions, we prove Garding's inequality for this operator.

1. Introduction

In order to solve the Dirichlet problem for a differential operator by using Hilbert space methods (sometimes called the direct methods in the calculus of variations), Garding's inequality represents an essential tool [1, 2]. For strongly elliptic differential operators, Garding's inequality was proved by Gärding [3] and its converse by Agmon [4]. One can find a proof for Garding's inequality and its converse in the work of Stummel [5] for strongly semielliptic operators. Two examples for strongly elliptic and semielliptic operators are studied in [6]. More recent results on this subject can be found in [7, 8] for a class of differential operators containing some non-hypoelliptic operators which were first introduced by Dynkin [9] and for differential operators in generalized divergence form (see also [10, 11]).

The aim of this work is to study the existence of the weak solution of the Dirichlet problem for a second-order elliptic differential operator with infinite number of variables.

2. Some Function Spaces

In this paper, we will consider spaces of functions of infinitely many variables, see [12, 13]. For this purpose we introduce the product measure

$$\begin{aligned} d\rho(x) &= \left(p_1(x_1)dx_1\right) \times \left(p_2(x_2)dx_2\right) \times \cdots \\ &= \left(d\rho_1(x_1)\right) \times \left(d\rho_2(x_2)\right) \times \cdots, \\ &\left(p_k(x_k)dx_k = d\rho_k(x_k),\ k = 1,2,\ldots\right) \end{aligned} \tag{2.1}$$

defined on the space $R^\infty = R^1 \times R^1 \times \cdots$ of points $x = (x_k)_{k=1}^\infty, x_k \in R^1$, where $(p_k)_1^\infty$ is a fixed sequence of weights, such that

$$C^2\left(R^1\right) \ni p_k(t) > 0, \quad \int_{R^1} p_k(t)dt = 1. \tag{2.2}$$

For $k = 1,2,\ldots$, we put

$$R^\infty = \underbrace{R^1 \times R^1 \times \cdots \times R^1}_{k-1} \times R^1 \times \cdots \tag{2.3}$$

We can write $x \in R^\infty$, by $x = (x_k, \tilde{x})$, where

$$\tilde{x} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots) \tag{2.4}$$

and $d\rho(x) = d\rho(x_k) \times d\rho(\tilde{x})$.

With respect to $d\rho$ we construct on R^∞ the Hilbert space of functions of infinitely many variables

$$L_2(R^\infty) = L_2\left(R^\infty, d\rho(x)\right), \tag{2.5}$$

which can be understood as the infinite tensor product

$$\bigotimes_{k=1,e}^{\infty} L_2\left(R^1, d\rho_k(x_k)\right) \tag{2.6}$$

with the identity stabilization $e = (e^{(k)})_1^\infty$, $e^{(k)} \in L_2(R^1, d\rho_k(x_k))$, $e^{(k)} = 1$. To say that the function $f \in L_2(R^\infty, d\rho(x))$ is cylindrical, it means that there exist an $m = 1,2,\ldots$, and an $f_c \in L_2(R^m, d\rho_{(m)}(x^{(m)}))$, $(x^{(m)} = (x_1, \ldots, x_m))$, $(d\rho_{(m)}(x^{(m)}) = \bigotimes_1^m d\rho_k(x_k))$, such that $f(x) = f_c(x^{(m)})$, $x \in R^\infty$.

On the collection of functions which are $l = 1,2,\ldots$ times continuously differentiable up to the boundary Γ of R^m for sufficiently large m, we introduce the scalar product

$$(u,v)_l = \sum_{|\alpha| \le l} \left(D^\alpha u, D^\alpha v\right)_{L_2(R^\infty, d\rho(x))}, \tag{2.7}$$

where

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}\cdots}, \quad |\alpha| = \sum_{i=1}^{\infty} \alpha_i. \tag{2.8}$$

The differentiation is taken in the sense of generalized functions, and after the completion we obtain the Sobolev spaces $W_2^l(R^\infty)$, $l = 1, 2, \ldots$.

Sobolev space of order l on R^∞ is defined by

$$W_2^l(R^\infty) = \left\{ u \mid D^{\alpha} u \in L_2\left(R^\infty, d\rho(x)\right) \ \forall \alpha, |\alpha| \le l \right\}, \tag{2.9}$$

$W_2^l(R^\infty)$ endowed with the scalar product (2.7) forming a dense subspace of $L_2(R^\infty, d\rho(x))$, with

$$\|u\|_{L_2(R^\infty, d\rho(x))} \le \|u\|_{W_2^l(R^\infty)} \tag{2.10}$$

for $u \in W_2^l(R^\infty)$.

We use the technique of [13] to construct chains of spaces

$$W_2^l(R^\infty) \subseteq L_2\left(R^\infty, d\rho(x)\right) = W_2^0(R^\infty) \subseteq W_2^{-l}(R^\infty), \quad l = 0, 1, \ldots, \tag{2.11}$$

where $W_2^{-1}(R^\infty)$ are the duals of $W_2^1(R^\infty)$.

3. Elliptic Differential Operator with Infinite Number of Variables

Consider $(a_k)_{k=1}^{\infty}$ to be a sequence of nonnegative locally bounded functions in R^∞ (i.e., they are bounded on each compact subset) with derivatives $(\partial/\partial x_k)a_k \in L_{p,\mathrm{loc}}$ for any $p \ge 1$ and $k = 1, 2, \ldots,$ and for a suitable $x_0 \in R^\infty$ it satisfies the following conditions:

(1) there exists a constant $c_1 > 0$ such that

$$\sum_{k=1}^{\infty} a_k(x_0) \ge c_1, \tag{3.1}$$

(2) let c_1 be the constant in condition (1), and there is n_0 belonging to $\mathbb{N}$ such that

$$\max_{k\in\mathbb{N}} \sup_{x\in R^\infty} |a_k(x) - a_k(x_0)| \le \frac{c_1}{2n_0}. \tag{3.2}$$

Now, we define on $L_2(R^\infty, d\rho(x))$ an elliptic differential operator with infinitely many variables

$$\begin{aligned}(Lu)(x) &= -\sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k(x_k)}}\ \frac{\partial}{\partial x_k}\left(a_k(x)\frac{\partial}{\partial x_k}\left(\sqrt{p_k(x_k)}u(x)\right)\right)\\ &= -\sum_{k=1}^{\infty} D_k(a_k D_k u)(x), \quad u \in W_2^1(R^\infty),\end{aligned} \tag{3.3}$$

where

$$(D_k u)(x) = \frac{1}{\sqrt{p_k(x_k)}}\frac{\partial}{\partial x_k}\left(\sqrt{p_k(x_k)}u(x)\right). \tag{3.4}$$

Theorem 3.1. *Assume that* $(p_k)_{k=1}^{\infty}$ *satisfy the condition that*

$$\sum_{k=1}^{\infty}\left(D_k^2\, 1\right)(x) \tag{3.5}$$

converges in $L_2(R^\infty, d\rho(x))$. *Then the operator* L *in* (3.3) *is well defined and admits a closure in* $L_2(R^\infty, d\rho(x))$.

Proof. The mapping

$$L_2\left(R^1, dx_k\right) \ni U(x_k) \longmapsto u(x_k) = p_k^{-1/2}(x_k)U(x_k) \in L_2\left(R^1, d\rho_k(x_k)\right) \tag{3.6}$$

is an isometry between the two spaces of square integrable functions. It carries $(\partial U/\partial x_k)(x_k)$ into the sandwiched (by means of p_k) derivative

$$\begin{aligned}(D_k u)(x_k) &= p_k^{-1/2}(x_k)\frac{\partial}{\partial x_k}\left(p_k^{1/2}(x_k)u(x_k)\right)\\ &= \left(\frac{\partial u}{\partial x_k}\right)(x_k) + (D_k I)(x_k)u(x_k),\end{aligned} \tag{3.7}$$

and it carries

$$\frac{\partial}{\partial x_k}\left(a_k(x)\frac{\partial U}{\partial x_k}(x_k)\right) \tag{3.8}$$

into the corresponding D_k derivative:

$$\begin{aligned}
D_k(a_k(x)D_ku)(x_k) &= p_k^{-1/2}(x_k)\frac{\partial}{\partial x_k}\left(p_k^{1/2}(x_k)a_k(x)(D_ku)(x_k)\right)\\
&= \frac{\partial}{\partial x_k}\left[a_k(x)p_k^{-1/2}(x_k)\frac{\partial}{\partial x_k}\left(p_k^{1/2}(x_k)u(x_k)\right)\right] + (D_k1)(x_k)a_k(x)(D_ku)(x_k)\\
&= a_k(x)\frac{\partial^2u}{\partial x_k^2}(x_k) + \frac{\partial a_k}{\partial x_k}(x)\frac{\partial u}{\partial x_k}(x_k) + \frac{\partial}{\partial x_k}[a_k(x)(D_k1)(x_k)u(x_k)]\\
&\quad + (D_k1)(x_k)a_k(x)(D_ku)(x_k)\\
&= \frac{\partial}{\partial x_k}\left(a_k(x)\frac{\partial u}{\partial x_k}(x_k)\right) + \frac{\partial}{\partial x_k}[a_k(x)(D_k1)(x_k)u(x_k)]\\
&\quad + (D_k1)(x_k)a_k(x)\frac{\partial u}{\partial x_k}(x_k) + a_k(x)(D_k1)^2(x_k)u(x_k)\\
&= \frac{\partial}{\partial x_k}\left(a_k(x)\frac{\partial u}{\partial x_k}(x_k)\right) + a_k(x)u(x_k)\frac{\partial}{\partial x_k}(D_k1)(x_k)\\
&\quad + (D_k1)(x_k)\left[a_k(x)\frac{\partial u}{\partial x_k}(x_k) + \frac{\partial a_k}{\partial x_k}(x)u(x_k)\right]\\
&\quad + (D_k1)(x_k)a_k(x)\frac{\partial u}{\partial x_k}(x_k) + a_k(x)(D_k1)^2(x_k)u(x_k)\\
&= \frac{\partial}{\partial x_k}\left(a_k(x)\frac{\partial u}{\partial x_k}(x_k)\right) + (D_k1)(x_k)\left[2a_k(x)\frac{\partial u}{\partial x_k}(x_k) + u(x_k)\frac{\partial a_k}{\partial x_k}(x)\right]\\
&\quad + \left(D_k^21\right)(x_k)a_k(x)u(x_k).
\end{aligned} \tag{3.9}$$

Denote by $C_{c,0}^{\infty}(R^{\infty})$ the linear span of the set of all cylindrical infinitely differentiable finite functions dense in $W_2^l(R^{\infty})$, that is, all the functions $u \in W_2^l(R^{\infty})$ of the form

$$R^{\infty} \ni x \longmapsto u(x) = u_c(x_1, \ldots, x_n), \tag{3.10}$$

where n depends on u and $u_c \in C_0^{\infty}(R^n)$, $n = 1, 2, \ldots$. Condition (3.5) implies that $D_k1, D_k^21 \in L_2(R^1, d\rho_k(x_k))$, (see [13, Lemma (3.2)]). We note that the action of L on the function $u(x) = u_c(x^{(n)})$ has the form

$$(Lu)(x) = -\sum_{k=1}^{n} D_k(a_kD_ku_c)(x) - u_c(x)\left[\sum_{k=n+1}^{\infty}\left(a_kD_k^21\right)(x) + \sum_{k=n+1}^{\infty}(D_k1)(x)\frac{\partial a_k}{\partial x_k}(x)\right], \tag{3.11}$$

then in view of condition (3.5), the operator $C_{c,0}^{\infty}(R^{\infty}) \ni u(x) \longmapsto (Lu)(x) = -\sum_{k=1}^{\infty} D_k(a_kD_ku)(x) \in L_2(R^{\infty}, d\rho(x))$ is well defined in $L_2(R^{\infty}, d\rho(x))$ and admits a closure which is again denoted by L. □

4. A Garding Inequality

In our consideration, we have an operator of the form

$$(Lu)(x) = -\sum_{k=1}^{\infty} D_k(a_k D_k u)(x) \tag{4.1}$$

with $u \in W_2^1(R^\infty)$.

Lemma 4.1. *The operator L is Hermitian.*

Proof. It is sufficient to verify the Hermitianness on functions of the form $u(x) = u_c(x^{(n)}), v(x) = v_c(x^{(m)})$, where $u_c \in C_0^\infty(R^n), v_c \in C_0^\infty(R^m)$; for example, we take it that $m \le n$.

Using (3.11), we obtain

$$\begin{aligned}
&(Lu,v)_{L_2(R^\infty,d\rho(x))} \\
&= -\int_{R^n}\left[\sum_{k=1}^{n}(D_k a_k D_k u_c)(x)\right]\overline{v_c(x)}p_1(x_1)\cdots p_n(x_n)dx_1\cdots dx_n \\
&\quad -\int_{R^\infty}\left(\sum_{k=n+1}^{\infty}\left[a_k(x)\left(D_k^2 1\right)(x) + (D_k 1)(x)\frac{\partial a_k}{\partial x_k}(x)\right]u_c(x)\right)\overline{v_c(x)}d\rho(x) \\
&= -\sum_{k=1}^{n}\int_{R^n}(D_k a_k D_k u_c)(x)\overline{v_c(x)}p_1(x_1)\cdots p_n(x_n)dx_1\cdots dx_n \\
&\quad -\sum_{k=n+1}^{\infty}\int_{R^\infty}\left(\left[a_k(x)\left(D_k^2 1\right)(x) + (D_k 1)(x)\frac{\partial a_k}{\partial x_k}(x)\right]u_c(x)\right)\overline{v_c(x)}d\rho(x) \\
&= -\sum_{k=1}^{n}A_k - \sum_{k=n+1}^{\infty}B_k,
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
A_k &= \int_{R^n}(D_k a_k D_k u_c)(x)\overline{v_c(x)}p_1(x_1)\cdots p_n(x_n)dx_1\cdots dx_n \\
&= \int_{R^{n-1}}\left(\int_{R^1}(D_k a_k D_k u_c)(x)\overline{v_c(x)}p_k(x_k)dx_k\right)p_1(x_1)\cdots p_{k-1}(x_{k-1}) \\
&\qquad \times p_{k+1}(x_{k+1})\cdots p_n(x_n)dx_1\cdots dx_{k-1}dx_{k+1}\cdots dx_n \\
&= \int_{R^{n-1}}\int_{R^1}\frac{1}{\sqrt{p_k(x_k)}}\,\frac{\partial}{\partial x_k}\left(a_k(x)\frac{\partial}{\partial x_k}\left(\sqrt{p_k(x_k)}u_c(x)\right)\right) \\
&\qquad\qquad \times \overline{v_c(x)}p_k(x_k)dx_k p_1(x_1)\cdots p_{k-1}(x_{k-1}) \\
&\qquad\qquad \times p_{k+1}(x_{k+1})\cdots p_n(x_n)dx_1\cdots dx_{k-1}dx_{k+1}\cdots dx_n
\end{aligned}$$

$$
\begin{aligned}
&= \int_{R^{n-1}} \int_{R^1} \left(\frac{\partial}{\partial x_k} \left(\sqrt{p_k(x_k)} u_c(x) \right) \right) \left(-a_k(x) \frac{\partial}{\partial x_k} \left(\sqrt{p_k(x_k)}\ \overline{v_c(x)} \right) \right) \\
&\qquad \times dx_k p_1(x_1) \cdots p_{k-1}(x_{k-1}) p_{k+1}(x_{k+1}) \cdots p_n(x_n) dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_n \\
&= \int_{R^{n-1}} \int_{R^1} u_c(x) \frac{1}{\sqrt{p_k(x_k)}}\ \frac{\partial}{\partial x_k} \left(a_k(x) \frac{\partial}{\partial x_k} \left(\sqrt{p_k(x_k)}\ \overline{v_c(x)} \right) \right) \\
&\qquad \times p_k(x_k) dx_k p_1(x_1) \cdots p_{k-1}(x_{k-1}) \\
&\qquad \times p_{k+1}(x_{k+1}) \cdots p_n(x_n) dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_n \\
&= \int_{R^n} u_c(x) \overline{(D_k a_k D_k v_c)(x)} p_1(x_1) \cdots p_n(x_n) dx_1 \cdots dx_n, \\
B_k &= \int_{R^\infty} u_c(x) \left[a_k(x) \left(D_k^2 1 \right)(x) + (D_k 1)(x) \frac{\partial a_k}{\partial x_k}(x) \right] \overline{v_c(x)} d\rho(x).
\end{aligned}
\tag{4.3}
$$

Hence, we have

$$
\begin{aligned}
&(Lu, v)_{L_2(R^\infty, d\rho(x))} \\
&= -\sum_{k=1}^{n} \int_{R^n} u_c(x) \overline{(D_k a_k D_k v_c)(x)} p_1(x_1) \cdots p_n(x_n) dx_1 \cdots dx_n \\
&\quad - \sum_{k=n+1}^{\infty} \int_{R^\infty} u_c(x) \left(\left[a_k(x) \left(D_k^2 1 \right)(x) + (D_k 1)(x) \frac{\partial a_k}{\partial x_k}(x) \right] \overline{v_c(x)} \right) d\rho(x) \\
&= (u, Lv)_{L_2(R^\infty, d\rho(x))}.
\end{aligned}
\tag{4.4}
$$

□

Now, we can define on $W_2^1(R^\infty)$ the bilinear form

$$
B(u, v) = (Lu, v)_{L_2(R^\infty, d\rho(x))}, \tag{4.5}
$$

where $L \in \mathcal{L}(W_2^1(R^\infty), W_2^{-1}(R^\infty))$

$$
\begin{aligned}
B(u, v) &= -\sum_{k=1}^{\infty} \int_{R^\infty} \frac{1}{\sqrt{p_k(x_k)}}\ \frac{\partial}{\partial x_k} \left(a_k(x) \frac{\partial}{\partial x_k} \left(\sqrt{p_k(x_k)} u(x) \right) \right) \overline{v(x)} d\rho(x) \\
&= -\sum_{k=1}^{\infty} \int_{R^\infty} \frac{1}{\sqrt{p_k(x_k)}}\ \frac{\partial}{\partial x_k} \left(a_k(x) \frac{\partial}{\partial x_k} \left(\sqrt{p_k(x_k)} u(x) \right) \right) \overline{v(x)} p_k(x_k) dx_k \rho(\tilde{x}) \\
&= -\sum_{k=1}^{\infty} \int_{R^\infty} \frac{\partial}{\partial x_k}\ \left(a_k(x) \frac{\partial}{\partial x_k} \left(\sqrt{p_k(x_k)} u(x) \right) \right) \overline{v(x)} \sqrt{p_k(x_k)} dx_k d\rho(\tilde{x})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \int_{R^\infty} a_k(x) \frac{\partial}{\partial x_k} \left(\sqrt{p_k(x_k)} u(x) \right) \frac{\partial}{\partial x_k} \left(\overline{v(x)} \sqrt{p_k(x_k)} \right) dx_k d\rho(\tilde{x}) \\
&= \sum_{k=1}^{\infty} \int_{R^\infty} a_k(x) p_k(x_k) \frac{1}{\sqrt{p_k(x_k)}} \frac{\partial}{\partial x_k} \left(\sqrt{p_k(x_k)} u(x) \right) \frac{1}{\sqrt{p_k(x_k)}} \\
&\qquad \times \frac{\partial}{\partial x_k} \left(\sqrt{p_k(x_k)}\ \overline{v(x)} \right) dx_k d\rho(\tilde{x}),
\end{aligned}
\tag{4.6}
$$

then

$$
B(u,v) = \sum_{k=1}^{\infty} \int_{R^\infty} a_k(x) D_k u(x) \overline{D_k v(x)} d\rho(x). \tag{4.7}
$$

Lemma 4.2. *The bilinear form* (4.7) *is continuous on* $W_2^1(R^\infty)$.

Proof. For $u, v \in C_{c,0}^\infty(R^\infty)$,

$$
\begin{aligned}
|B(u,v)| &\le \sum_{k=1}^{\infty} \int_{R^\infty} a_k(x) |D_k u(x)| |D_k v(x)| d\rho(x) \\
&\le \max_{k\in\mathbb{N}} \sup_{x\in R^\infty} a_k(x) \sum_{k=1}^{\infty} \int_{R^\infty} |D_k u(x)| |D_k v(x)| d\rho(x) \\
&\le \max_{k\in\mathbb{N}} \sup_{x\in R^\infty} a_k(x) \sum_{k=1}^{\infty} \left(\int_{R^\infty} |D_k u(x)|^2 d\rho(x) \right)^{1/2} \left(\int_{R^\infty} |D_k v(x)|^2 d\rho(x) \right)^{1/2} \\
&\le c \sum_{k=1}^{\infty} \|D_k u\|_{L_2(R^\infty, d\rho(x))} \|D_k v\|_{L_2(R^\infty, d\rho(x))} \\
&\le c \|u\|_{W_2^1(R^\infty)} \|v\|_{W_2^1(R^\infty)}.
\end{aligned}
\tag{4.8}
$$

Thus B has a continuous extension onto $W_2^1(R^\infty)$ which is again denoted by B. □

Theorem 4.3. *Suppose that* L *is given as in* (4.1). *In particular assume that* (3.5) *holds. Then there exist positive constants* $c_0 > 0$ *and* $c_1 \ge 0$ *such that*

$$
B(u,u) \ge c_0 \|u\|^2_{W_2^1(R^\infty)} - c_1 \|u\|^2_{L_2\left(R^\infty, d\rho(x)\right)} \tag{4.9}
$$

holds for all $u \in W_2^1(R^\infty)$.

Proof. For $u \in C^{\infty}_{c,0}(R^{\infty})$,

$$\begin{aligned} B(u,u) &= \sum_{k=1}^{\infty} \int_{R^{\infty}} a_k(x) |D_k u(x)|^2 d\rho(x) \\ &= \sum_{k=1}^{\infty} a_k(x_0) \int_{R^{\infty}} |D_k u(x)|^2 d\rho(x) - \sum_{k=1}^{\infty} \int_{R^{\infty}} (a_k(x_0) - a_k(x)) |D_k u(x)|^2 d\rho(x) \\ &\geq \sum_{k=1}^{\infty} a_k(x_0) \int_{R^{\infty}} |D_k u(x)|^2 d\rho(x) - \sum_{k=1}^{\infty} \int_{R^{\infty}} |a_k(x_0) - a_k(x)| |D_k u(x)|^2 d\rho(x), \end{aligned} \tag{4.10}$$

and using conditions (1) and (2),

$$\begin{aligned} B(u,u) &\geq c_1 \sum_{k=1}^{\infty} \int_{R^{\infty}} |D_k u(x)|^2 d\rho(x) - \max_{k \in \mathbb{N}} \sup_{x \in R^{\infty}} |a_k(x_0) - a_k(x)| \sum_{k=1}^{\infty} \int_{R^{\infty}} |D_k u(x)|^2 d\rho(x) \\ &\geq c_1 \left[\|u\|^2_{W_2^1(R^{\infty})} - \|u\|^2_{L_2(R^{\infty}, d\rho(x))} \right] - \frac{c_1}{2n_0} \sum_{K=1}^{\infty} \int_{R^{\infty}} |D_k u(x)|^2 d\rho(x) \\ &\geq c_1 \left[\|u\|^2_{W_2^1(R^{\infty})} - \|u\|^2_{L_2(R^{\infty}, d\rho(x))} \right] - \frac{c_1}{2n_0} \|u\|^2_{W_2^1(R^{\infty})} \\ &= c_1 \left(1 - \frac{1}{2n_0} \right) \|u\|^2_{W_2^1(R^{\infty})} - c_1 \|u\|^2_{L_2(R^{\infty}, d\rho(x))}, \end{aligned} \tag{4.11}$$

and with $c_0 = c_1(1 - 1/2n_0)$, we finally obtain (4.9). □

5. Conclusions

In view of our recent achievement, we recommend to extend this approach to include the linear partial differential operators in generalized divergence form $\sum_{\alpha,\beta \in \Gamma} D^{\alpha}(a_{\alpha\beta}(\cdot) D^{\beta})$, where Γ is finite, and nonempty collection of $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i = 1, 2, \ldots$, and $a_{\alpha\beta}$ $(\alpha, \beta \in \Gamma \times \Gamma)$ are real locally bounded functions on R^{∞}.

References

[1] N. Jacob, *Pseudo Differential Operators and Markov Processes*, vol. 1, Imperial College Press, London, UK, 2001, Fourier analysis and semigroup.

[2] K. Yosida, *Functional Analysis*, Grundlehren der mathematischen Wissenschaften Bd 123, Springer, New York, NY, USA, 4th edition, 1974.

[3] L. Gårding, "Dirichlet's problem for linear elliptic partial differential equations," *Mathematica Scandinavica*, vol. 1, pp. 55–72, 1953.

[4] S. Agmon, "The coerciveness problem for integro-differential forms," *Journal d'Analyse Mathématique*, vol. 6, pp. 183–223, 1958.

[5] F. Stummel, *Rand-und Eigenwertaufgaben in Sobolewschen Räumen*, vol. 102 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1969.

[6] N. Jacob, "On Dirichlet's boundary value problem for some formally hypoelliptic differential operators," *Journal of Australian Mathematical Society Series A*, vol. 44, no. 3, pp. 324–349, 1988.

[7] N. Jacob, "On generalized Dirichlet problems," *Mathematica Scandinavica*, vol. 55, no. 2, pp. 245–252, 1984.
[8] N. Jacob and B. Schomburg, "On Garrding's inequality," *Aequationes Mathematicae*, vol. 31, no. 1, pp. 7–17, 1986.
[9] E. B. Dynkin, "Harmonic functions associated with several Markov processes," *Advances in Applied Mathematics*, vol. 2, no. 3, pp. 260–283, 1981.
[10] K. Doppel and N. Jacob, "A nonhypoelliptic Dirichlet problem from stochastics," *Annales Academiae Scientiarum Fennicae. Series A I. Mathematica*, vol. 8, no. 2, pp. 375–389, 1983.
[11] K. Doppel and N. Jacob, "On an evolution equation for a nonhypoelliptic linear partial differential operator from stochastics," *Zeitschrift für Analysis und ihre Anwendungen*, vol. 3, no. 5, pp. 425–433, 1984.
[12] J. M. Berezanskiĭ, *Expansions in Eigenfunctions of Selfadjoint Operators*, vol. 17, American Mathematical Society, Providence, RI, USA, 1968.
[13] Y. M. Berezanskiĭ, *Selfadjoint Operators in Spaces of Functions of Infinitely Many Variables*, vol. 63 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1986.