Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 419341, 11 pages
doi:10.1155/2011/419341

Research Article

A Multiplicity Result for Quasilinear Problems with Nonlinear Boundary Conditions in Bounded Domains

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Received 28 September 2011; Accepted 2 November 2011

Academic Editor: Enrico Obrecht

We study the following quasilinear problem with nonlinear boundary condition $-\Delta_p u - \lambda a(x)u|u|^{p-2} = b(x)u|u|^{\gamma-2}$, in Ω and $(1-\alpha)|\nabla u|^{p-2}(\partial u/\partial n) + \alpha u|u|^{p-2} = 0$, on $\partial\Omega$, where $\Omega \subseteq R^N$ is a connected bounded domain with smooth boundary $\partial\Omega$, the outward unit normal to which is denoted by n. Δ_p is the p-Laplcian operator defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, the functions a and b are sign changing continuous functions in Ω, $1 < p < \gamma < p^*$, where $p^* = Np/(N-p)$ if $N > p$ and ∞ otherwise. The properties of the first eigenvalue $\lambda_1^+(\alpha)$ and the associated eigenvector of the related eigenvalue problem have been studied in (Khademloo, In press). In this paper, it is shown that if $\lambda \leq \lambda_1^+(\alpha)$, the original problem admits at least one positive solution, while if $\lambda_1^+(\alpha) < \lambda < \lambda^*$, for a positive constant λ^*, it admits at least two distinct positive solutions. Our approach is variational in character and our results extend those of Afrouzi and Khademloo (2007) in two aspects: the main part of our differential equation is the p-Laplacian, and the boundary condition in this paper also is nonlinear.

1. Introduction and Results

In this paper, we consider the problem

$$
\begin{aligned}
&-\Delta_p u - \lambda a(x)u|u|^{p-2} = b(x)u|u|^{\gamma-2}, \quad x \in \Omega,\\
&(1-\alpha)|\nabla u|^{p-2}\frac{\partial u}{\partial n} + \alpha u|u|^{p-2} = 0, \quad x \in \partial\Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega \subseteq R^N$ is a connected bounded domain with a smooth boundary $\partial\Omega$, the outward unit normal to which is denoted by n. Δ_p is the p-Laplcianian operator defined by $\Delta_p u =$

$\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The functions a and b are assumed to be sign changing in Ω. Here, we say a function $a(x)$ changes sign if the measure of the sets $\{x \in \Omega; a(x) > 0\}$ and $\{x \in \Omega; a(x) < 0\}$ are both positive. $\lambda \geq 0$ is a real parameter and exponent γ is assumed to satisfy the condition $1 < p < \gamma < p^*$, where $p^* = Np/(N-p)$ if $N > p$ and ∞ otherwise.

A host of literature exists for this type of problem when $p = 2$. For the works concerning with problems similar to (1.1) in the case $p = 2$, we refer to [1–3] and references therein.

The growing attention in the study of the p-Laplace operator is motivated by the fact that it arises in various applications, for example, non-Newtonian fluids, reaction diffusion problems, flow through porus media, glacial sliding, theory of superconductors, biology, and so forth (see [4, 5] and the references therein).

In this paper, we obtain new existence results by using a variational method based on the properties of eigencurves, that is, properties of the map $\lambda \to \mu(\alpha,\lambda)$, where $\mu(\alpha,\lambda)$ denotes the principal eigenvalue of the problem

$$\begin{aligned} -\Delta_p u - \lambda a(x)u|u|^{p-2} &= \mu u|u|^{p-2}, \quad x \in \Omega, \\ (1-\alpha)|\nabla u|^{p-2}\frac{\partial u}{\partial n} + \alpha u|u|^{p-2} &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{1.2}$$

Similar to [6] our method works provided that the eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda a(x)u|u|^{p-2}, \quad x \in \Omega, \\ (1-\alpha)|\nabla u|^{p-2}\frac{\partial u}{\partial n} + \alpha u|u|^{p-2} &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.3}$$

has principal eigenvalues and it can be shown that this occurs on an interval $[\alpha_0, 1]$ where $\alpha_0 \leq 0$. Thus, we are able to obtain existence results for problem (1.2) even in the case of nonlinear Neumann boundary conditions where α is small and negative. Our method depends on using eigencurves to produce an equivalent norm on $W^{1,p}(\Omega)$; such an equivalent norm is also introduced in [2]. The results that we obtain in this paper are generalization of the previous results obtained by Pohozaev and Veron [7].

It can be shown that $\mu(\alpha,\lambda)$ has the variational characterization:

$$\mu(\alpha,\lambda) = \inf\left\{\int_\Omega |\nabla u|^p dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega}|u|^p d\sigma - \lambda\int_\Omega a|u|^p dx;\ u \in W^{1,p}(\Omega),\ \int_\Omega |u|^p dx = 1\right\} \tag{1.4}$$

from whence it follows that

(i) λ is a principal eigenvalue of (1.2) if and only if $\mu(\alpha,\lambda) = 0$,

(ii) $\alpha \to \mu(\alpha,\lambda)$ is an increasing function,

(iii) $\lambda \to \mu(\alpha,\lambda)$ is a concave function with a unique maximum such that $\mu(\alpha,\lambda) \to -\infty$ as $\lambda \to \pm\infty$ [6].

If $\alpha \in (0,1]$, then $\mu(\alpha,\lambda) > 0$, and so $\lambda \to \mu(\alpha,\lambda)$ has exactly one negative zero $\lambda_1^-(\alpha)$ and one positive zero $\lambda_1^+(\alpha)$. Thus, $\lambda_1^-(\alpha)$ and $\lambda_1^+(\alpha)$ are principal eigenvalues for (1.2).

If $\alpha = 0$, then $\mu(0,0) = 0$. If $a(x) > 0$, then $\mu(0,\lambda)$ is decreasing, and if $a(x) < 0$, then $\mu(0,\lambda)$ is increasing. Assume now that $a(x)$ changes sign in Ω: if $\int_\Omega a(x)dx < 0$, there exists a unique $\lambda_1^+(0) > 0$ such that $\mu(0,\lambda^+(0)) = 0$ and $\mu(0,\lambda) > 0$ for $\lambda \in (0,\lambda_1^+(0))$. If $\int_\Omega a(x)dx = 0$, then $\mu(0,0) = 0$ and $\mu(0,\lambda) < 0$ for $\lambda \neq 0$. If $\int_\Omega a(x)dx > 0$, then there exists a unique $\lambda_1^-(0) < 0$ such that $\mu(0,\lambda_1^-(0)) = 0$ and $\mu(0,\lambda) > 0$ for $\lambda \in (\lambda_1^-(0),0)$.

Suppose now that $\int_\Omega a(x)dx < 0$ and that α is small and negative. Then, since $\alpha \to \mu(\alpha,\lambda)$ is increasing, it follows that there still exist principal eigenvalues $\lambda_1^-(\alpha) < \lambda_1^+(\alpha)$ of (2.5), but now both of them are positive.

It can be shown that there exists $\alpha_0 < 0$ such that the above is true for all $\alpha \in (\alpha_0,0)$, but for $\alpha < \alpha_0$, $\mu(\alpha,\lambda) < 0$ for all λ so that principal eigenvalues no longer exist.

Similar considerations show that when $\int_\Omega a(x)dx > 0$, there exists $\alpha_0 < 0$ such that there are principal eigenvalues $\lambda_1^-(\alpha) < \lambda_1^+(\alpha) < 0$ for $\alpha_0 < \alpha < 0$ but when $\int_\Omega a(x)dx = 0$, there are no principal eigenvalues for $\alpha < 0$ (see [6]).

It is easy to see that if $\lambda_1^-(\alpha)$ and $\lambda_1^+(\alpha)$ exist, $\mu(\alpha,\lambda) > 0$ for all $\lambda \in (\lambda_1^-(\alpha),\lambda_1^+(\alpha))$.

Thus, we assume that the following conditions hold:

($\mathcal{A}_1$) $\alpha \in (0,1)$ or that $\int_\Omega a(x)dx \neq 0$ and $\alpha \in (\alpha_0,1]$,

($\mathcal{A}_2$) $a(x) \in L^\infty(\Omega)$,

($\mathcal{B}_1$) $b(x) \in L^\infty(\Omega)$,

($\mathcal{B}_2$) $b^+ \not\equiv 0$,

($\mathcal{B}_3$) $\int_\Omega b(x)(u_1^+)^\gamma dx < 0$,

where u_1^+ is the positive principal eigenfunction corresponding to $\lambda_1^+(\alpha)$.

With these constructions we have the following.

Proposition 1.1. *Assume ($\mathcal{A}_1$), then for every $\lambda \in (0,\lambda_1^+(\alpha))$,*

$$\|u\|_{\alpha,\lambda} := \left(\int_\Omega (|\nabla u|^p - \lambda a(x)|u|^p)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} |u|^p d\sigma\right)^{1/p} \tag{1.5}$$

defines a norm in $W^{1,p}(\Omega)$ which is equivalent to the usual norm of $W^{1,p}(\Omega)$, that is,

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx\right)^{1/p}. \tag{1.6}$$

Proof. See [2]. □

Now we can state our main results.

Theorem 1.2. *Assume ($\mathcal{A}_1$), ($\mathcal{A}_2$), ($\mathcal{B}_1$), and ($\mathcal{B}_2$). Then, for every $\lambda \in (0,\lambda_1^+(\alpha))$, problem* (1.2) *admits at least one positive solution $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Theorem 1.3. *Assume ($\mathcal{A}_1$), ($\mathcal{A}_2$), ($\mathcal{B}_1$), ($\mathcal{B}_2$), and ($\mathcal{B}_3$). Then, problem* (1.2) *has at least one positive solution $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ for $\lambda = \lambda_1^+(\alpha)$.*

Theorem 1.4. *Assume ($\mathcal{A}_1$), ($\mathcal{A}_2$), ($\mathcal{B}_1$), ($\mathcal{B}_2$), and ($\mathcal{B}_3$). Then, there exists $\lambda^* > \lambda_1^+(\alpha)$ such that problem* (1.2) *admits at least two distinct positive weak solutions in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$, whenever $\lambda_1^+(\alpha) < \lambda < \lambda^*$.*

When $\lambda = 0$ and $\alpha = 0$, we have the following.

Corollary 1.5. *Assume* $\int_\Omega b(x)dx < 0$. *Then, the problem*

$$\begin{aligned} -\Delta_p u &= b(x)u|u|^{\gamma-2}, \quad x \in \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.7}$$

has a positive solution.

Throughout this paper, c denotes a positive constant. We will use fibrering method in a similar way to those in [8]. A brief description of the method and the proof of Theorem 1.2 are presented in Section 2. We then study the cases $\lambda = \lambda_1^+(\alpha)$ and $\lambda > \lambda_1^+(\alpha)$ in Sections 3 and 4.

2. The Case When $\lambda < \lambda_1^+(\alpha)$

In this section, motivated by Pohozaev [8], we will introduce the fibrering map as our framework for the study of problem (1.2).

For a (weak) solution of problem (1.2), we mean a function $u \in X = W^{1,p}(\Omega)$ such that for every $v \in W^{1,p}(\Omega)$, there holds

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v - \lambda a(x)|u|^{p-2}uv\right)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega}|u|^{p-2}uv\,d\sigma = \int_\Omega b(x)|u|^{\gamma-2}uv\,dx. \tag{2.1}$$

Now let us define the variational functional corresponding to problem (1.2). We set $I_\lambda : X \to R$ as

$$I_\lambda(u) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p - \lambda a(x)|u|^p\right)dx + \frac{\alpha}{p(1-\alpha)}\int_{\partial\Omega}|u|^p d\sigma - \frac{1}{\gamma}\int_\Omega b(x)|u|^\gamma dx. \tag{2.2}$$

It is easy to see that $I_\lambda \in C^1(X, R)$, and for $v \in X$, there holds

$$\begin{aligned} (I_\lambda'(u), v) &= \int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v - \lambda a(x)|u|^{p-2}uv\right)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega}|u|^{p-2}uv\,d\sigma \\ &= \int_\Omega b(x)|u|^{\gamma-2}uv\,dx. \end{aligned} \tag{2.3}$$

Since $C_0^\infty(\Omega) \subset X$, we know that critical points of I_λ are weak solutions of the problem (1.2). Thus, to prove our main theorems, it suffices to show that I_λ admits critical point. We will do this in this paper. Our main tool is the theory of the fibrering maps. First, we will introduce this map for $I_\lambda : X \to R$.

Let $\phi_u(t) = I_\lambda(tu)$ $(t > 0)$. We refer to such maps as fibrering maps. It is clear that if u is a local minimizer of I_λ, then ϕ_u has a local minimum at $t = 1$.

Lemma 2.1. *Let* $u \in X - \{0\}$ *and* $t > 0$. *Then,* $(I_\lambda'(tu), tu) = 0$ *if and only if* $\phi_u'(t) = 0$.

Proof. The result is an immediate consequence of the fact that

$$\phi_u'(t) = \left(I_\lambda'(tu), u\right) = \frac{1}{t}\left(I_\lambda'(tu), tu\right). \tag{2.4}$$

□

Thus critical points of I_λ correspond to stationary points of the maps ϕ_u that can be given by

$$\phi_u(t) = I_\lambda(tu) = \frac{t^p}{p}A_\lambda(u) - \frac{t^\gamma}{\gamma}B(u), \tag{2.5}$$

where

$$\begin{aligned} A_\lambda(u) &= \int_\Omega \left(|\nabla u|^p - \lambda a(x)|u|^p\right)dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega}|u|^p d\sigma, \\ B(u) &= \int_\Omega b(x)|u|^\gamma dx. \end{aligned} \tag{2.6}$$

Hence,

$$\phi_u'(t) = t^{p-1}A_\lambda(u) - t^{\gamma-1}B(u). \tag{2.7}$$

Thus, if $A_\lambda(u)$ and $B(u)$ have the same sign, ϕ_u has exactly one turning point at

$$t(u) = \left(\frac{A_\lambda(u)}{B(u)}\right)^{1/(\gamma-p)}, \tag{2.8}$$

provided that $B(u) \neq 0$, and if $A_\lambda(u)$ and $B(u)$ have opposite signs, ϕ_u has no turning points. Now substituting (2.8) into (2.5), we get

$$\begin{aligned} I_\lambda(t(u)u) &= \frac{1}{p}\left(\frac{A_\lambda(u)}{B(u)}\right)^{p/(\gamma-p)}A_\lambda(u) - \frac{1}{\gamma}\left(\frac{A_\lambda(u)}{B(u)}\right)^{\gamma/(\gamma-p)}B(u) \\ &= \frac{1}{p}\frac{A_\lambda(u)^{\gamma/(\gamma-p)}}{B(u)^{p/(\gamma-p)}} - \frac{1}{\gamma}\frac{A_\lambda(u)^{\gamma/(\gamma-p)}}{B(u)^{p/(\gamma-p)}} = \left(\frac{1}{p}-\frac{1}{\gamma}\right)\frac{A_\lambda(u)^{\gamma/(\gamma-p)}}{B(u)^{p/(\gamma-p)}} := J_\lambda(u). \end{aligned} \tag{2.9}$$

Lemma 2.2. *Suppose that u_0 is a critical point of J_λ, where $A_\lambda(u_0)$ and $B(u_0)$ have the same sign. Then, $I_\lambda'(t(u_0)u_0) = 0$.*

Proof. Let $u_0 \in X$, $B(u_0) \neq 0$, then

$$
\begin{aligned}
(I'_\lambda(t(u_0)u_0), v) &= \int_\Omega \left(|\nabla(t(u_0)u_0)|^{p-2}\nabla(t(u_0)u_0)\nabla v - \lambda a(x)|t(u_0)u_0|^{p-2}t(u_0)u_0 v \right) dx \\
&\quad - \int_\Omega b(x)|t(u_0)u_0|^{\gamma-2}t(u_0)u_0 v\, dx + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} |t(u_0)u_0|^{p-2}t(u_0)u_0 v\, d\sigma \\
&= t(u_0)^{p-1}(A'_\lambda(u_0), v) - t(u_0)^{\gamma-1}(B'(u_0), v) = \frac{1}{t(u_0)}(J'_\lambda(u_0), v) = 0
\end{aligned}
\tag{2.10}
$$

for all $v \in X$. This completes the proof. □

The following lemma shows that the critical points of the functional I_λ can be found by using the conditional variational problem associated with I_λ.

Lemma 2.3. *Suppose that H is a well-defined functional on X and u_0 is a minimizer of J_λ on*

$$
S := \{u \in X,\ H(u) = c,\ (H'(u), u) \neq 0\}
\tag{2.11}
$$

for some $c \neq 0$. Then, $J'_\lambda(u_0) = 0$.

Proof. If u_0 is a local minimizer of J_λ on S, then u_0 is a solution of the optimization problem:

$$
\text{minimize } J_\lambda(u) \quad \text{subject to } \gamma(u) = 0,
\tag{2.12}
$$

where $\gamma(u) = H(u) - c$. Hence, by the theory of the Lagrange multipliers, there exists $\mu \in R$ such that $J'_\lambda(u_0) = \mu\gamma'(u_0)$. Thus,

$$
(J'_\lambda(u_0), u_0) = \mu(\gamma'(u_0), u_0).
\tag{2.13}
$$

Note that the functional J_λ is 0-homogeneous and the Gateaux derivative of J_λ at the point $v \in X$, $B(v) \neq 0$, in direction v, is zero, that is, $(J'_\lambda(u_0), u_0) = 0$. It then follows that $\mu = 0$ due to $u_0 \in S$. Hence, the proof is complete. □

The following scheme for the investigation of the solvability of (1.2) is based on previous lemmas. First, we will prove the existence of nonzero critical points of J_λ under the constraint given by suitable functional H. This will be an actual critical point of J_λ and it will generate critical point of the Euler functional I_λ which will coincide with the weak solution of problem (1.2).

Proof of Theorem 1.2. Suppose that $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{B}_1)$, $(\mathcal{B}_2)$ satisfy. It follows from variational characterization of $\mu(\alpha, \lambda)$ that $A_\lambda(u) \geq 0$ for all $u \in X$ and $0 \leq \lambda < \lambda_1^+(\alpha)$. Moreover, we have

$$
(A'_\lambda(u), u) = pA_\lambda(u), \quad u \in X.
\tag{2.14}
$$

Hence, for using Lemma 2.3, it is sufficient to consider the case $A_\lambda(u) = c \neq 0$, for example $c = 1$. In this case, we have

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{\gamma}\right) \frac{1}{B(u)^{p/(\gamma-p)}}. \tag{2.15}$$

From the necessary condition for the existence of $t(u)$, we have $B(u) > 0$. It follows that we must consider a critical point with $B(u) > 0$.

The functional $J_\lambda(u)$ is nonnegative and so bounded below, hence we can look for positive local minimizer for $J_\lambda(u)$ on X.

Let us consider variational problem:

$$\mathcal{M}_\lambda = \sup\{B(u); A_\lambda(u) = 1,\ B(u) > 0\}. \tag{2.16}$$

Note that for $0 \leq \lambda < \lambda_1^+(\alpha)$, this set is not empty, and from Lemma 2.3, the solution of this problem is a minimizer of $J_\lambda(u)$ on X.

Suppose $\{u_n\}$ is the maximizing sequence of this problem. It follows from the equivalent property in Proposition 1.1, $\{u_n\}$ is bounded and so we may assume that $u_n \rightharpoonup u_0$ in X. Since X may be compactly embedded in $L^p(\Omega)$, $L^\gamma(\Omega)$, and $L^p(\partial\Omega)$, we have $u_n \to u_0$ in $L^p(\Omega)$, $L^\gamma(\Omega)$ and $L^p(\partial\Omega)$. Hence,

$$B(u_n) \longrightarrow B(u_0) = \mathcal{M}_\lambda > 0. \tag{2.17}$$

Moreover, we have

$$A_\lambda(u_0) \leq \liminf A_\lambda(u_n) = 1. \tag{2.18}$$

Here, the weak lower semicontinuity of the equivalent norm was used.

Assume that $A_\lambda(u_0) < 1$. By using the map:

$$L(t) = A_\lambda(tu_0), \tag{2.19}$$

we have $L(1) < 1$ and $\lim_{t\to\infty} L(t) = \infty$ and so $L(t_0) = 1$ for some $t_0 > 1$, that is, $A_\lambda(t_0 u_0) = 1$. So we derive

$$B(t_0 u_0) = t_0^\gamma B(u_0) > \mathcal{M}_\lambda, \tag{2.20}$$

which is a contradiction. Hence, $A_\lambda(u_0) = 1$ and so u_0 is a maximizer. As Lemma 2.2, the result would follow by considering $u_1 = t(u_0)u_0 \in X$. Then, u_1 is a weak solution of (1.2) and $u_1 \geq 0$ in Ω. Now, following the bootstrap argument (used, e.g., in [9]), we prove $u_1 \in L^\infty(\Omega)$. Then, we can apply the Harnack inequality due to Trudinger [5] in order to get $u_1 > 0$ in Ω (cf. [9]). □

3. The Case When $\lambda = \lambda_1^+(\alpha)$

Proof of Theorem 1.3. If $\lambda = \lambda_1^+(\alpha)$, It is easy to see that the set $\{u \in X; A_\lambda(u) = 1\}$ is unbounded and we are forced to require an additional assumption $(\mathcal{B}_3)$. In this case, again we are looking for a maximizing of the problem:

$$\mathcal{M}_{\lambda_1^+(\alpha)} = \sup\left\{B(u); A_{\lambda_1^+(\alpha)}(u) = 1,\ B(u) > 0\right\}. \tag{3.1}$$

Suppose $\{u_n\}$ is a maximizing sequence of this problem. First, we investigate the case when $\{u_n\}$ is unbounded. Then, we may assume without loss of generality that $\|u_n\| \to \infty$. So, we obtain

$$1 = A_{\lambda_1^+(\alpha)}(u_n) = \|u_n\|^p A_{\lambda_1^+(\alpha)}(v_n), \tag{3.2}$$

where $v_n = u_n/\|u_n\|$. Thus, we have

$$A_{\lambda_1^+(\alpha)}(v_n) = \frac{A_{\lambda_1^+(\alpha)}(u_n)}{\|u_n\|^p} = \frac{1}{\|u_n\|^p} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.3}$$

Since $\|v_n\| = 1$, we may assume that $v_n \rightharpoonup v_0$ in X. Again, since X may be compactly embedded in $L^p(\Omega)$, $L^\gamma(\Omega)$ and $L^p(\partial\Omega)$, we have

$$\begin{gathered}
\lim_{n\to\infty} \int_\Omega a(x)|v_n|^p dx = \int_\Omega a(x)|v_0|^p dx, \\
\lim_{n\to\infty} \int_\Omega b(x)|v_n|^\gamma dx = \int_\Omega a(x)|v_0|^\gamma dx, \\
\lim_{n\to\infty} \int_{\partial\Omega} |v_n|^p d\sigma = \int_{\partial\Omega} |v_0|^p d\sigma.
\end{gathered} \tag{3.4}$$

So,

$$\int_\Omega a(x)|v_0|^p dx \geq \frac{1}{\lambda_1^+(\alpha)}, \tag{3.5}$$

And, therefore, $v_0 \not\equiv 0$. Returning to $A_{\lambda_1^+(\alpha)}(v_n)$, we have also

$$0 \leq A_{\lambda_1^+(\alpha)}(v_0) \leq \liminf_{n\to\infty} A_{\lambda_1^+(\alpha)}(v_n) = 0. \tag{3.6}$$

Due to simplicity of $\lambda_1^+(\alpha)$, there exists $c \neq 0$ such that $v_0(x) = cu_1^+(x)$. Therefore, we have

$$0 < \mathcal{M}_{\lambda_1^+(\alpha)} = \lim_{n\to\infty} B(u_n) = \lim_{n\to\infty} \|u_n\|^\gamma B(v_n), \tag{3.7}$$

that implies $|c|^\gamma B(u_1^+) \geq 0$, which contradicts $(\mathcal{B}_3)$. Therefore, $\{u_n\}$ is bounded and so $u_n \rightharpoonup u_0$ in X and $u_n \to u_0$ in $L^p(\Omega)$, $L^\gamma(\Omega)$ and $L^p(\partial\Omega)$. Hence, $B(u_n) \to B(u_0) = \mathcal{M}_{\lambda_1^+(\alpha)} > 0$, and so $u_0 \neq 0$.

Moreover, we obtain $0 \leq A_{\lambda_1^+(\alpha)}(u_0) \leq 1$. Here, the variational characteristic of $\lambda_1^+(\alpha)$ and the weak lower semicontinuity of the norm were used. Let us now consider the case $A_{\lambda_1^+(\alpha)}(u_0) = 0$. Again, we obtain $u_0 = cu_1^+$ for some $c \neq 0$, and so $B(u_0) = |c|^\gamma B(u_1^+) = \mathcal{M}_{\lambda_1^+(\alpha)} > 0$ which contradicts $(\mathcal{B}_3)$.

Now we prove that $A_{\lambda_1^+(\alpha)}(u_0) = 1$. Suppose otherwise, then by using $L(t)$ in (2.20), we obtain some $t_0 > 1$ such that $A_{\lambda_1^+(\alpha)}(t_0 u_0) = 1$. By direct calculation, we get a contradiction like (4.1). This yields that u_0 is a nonnegative solution of problem (1.2). Using the same ideas as the proof of Theorem 1.2, we have proved Theorem 1.3. □

4. The Case When $\lambda > \lambda_1^+(\alpha)$

As it is proved in [10], we will show that for $\lambda > \lambda_1^+(\alpha)$ but close enough to $\lambda_1^+(\alpha)$, we have two distinct positive solutions for problem (1.2). The existence of one of them is obtained by using the following lemma.

Lemma 4.1. *Under the assumption of Theorem 1.4, there exists a maximizer u_1 of the problem:*

$$\sup\{B(u); A_\lambda(u) = 1,\ B(u) > 0\}, \tag{4.1}$$

whenever $\lambda \in (\lambda_1^+, \lambda_1^+(\alpha) + \delta)$, for some $\delta > 0$. Moreover, $u_1 \in X \cap L^\infty(\Omega)$ is a positive weak solution of problem (1.2).

Proof. First note that using $L(t)$ in (2.20), it is easy to see that u_1 is a maximizer of problem (4.2) if and only if u_1 is a maximizer of the problem:

$$\mathcal{M}_\lambda = \sup\{B(u); A_\lambda(u) \leq 1,\ B(u) > 0\}. \tag{4.2}$$

Now suppose that the result is false. Then, there exists a sequence $\{\delta_k\}$ such that $\delta_k \to 0$ and problem (4.1) has no solution for $\lambda_1^+(\alpha) + \delta_k$. For simplicity, we use $\lambda_k = \lambda_1^+(\alpha) + \delta_k$. Let $\{u_n^k\}$ be a maximizing sequence of this problem, that is,

$$A_{\lambda_k}\left(u_n^k\right) \leq 1, \quad B\left(u_n^k\right) \longrightarrow \mathcal{M}_{\lambda k} > 0. \tag{4.3}$$

We prove that if $\{u_n^k\}$ be bounded or unbounded, we arrive at a contradiction and so the lemma is proved.

The first case: $\{u_n^k\}$ is bounded. Thus, $u_n^k \rightharpoonup u_0^k$ in X for some $u_0^k \in X$ and $u_n^k \to u_0^k$ in $L^p(\Omega)$, $L^\gamma(\Omega)$, and $L^p(\partial\Omega)$. Hence, $B(u_0^k) = \mathcal{M}_{\lambda k} > 0$ and the weak lower semicontinuity of the norm gives $A_{\lambda_k}(u_0^k) \leq 1$. Therefore, u_0^k is a solution of problem (4.5), which is a contradiction.

The case $\{u_n^k\}$ is unbounded. Then we may assume that $\|u_n^k\| \to \infty$. Let $v_n^k = u_n^k / \|u_n^k\|$. Then, $v_n^k \rightharpoonup v_0^k$ in X for some $v_0^k \in X$ and $v_n^k \to v_0^k$ in $L^p(\Omega)$, $L^\gamma(\Omega)$ and $L^p(\partial\Omega)$. This implies that

$$B\left(u_n^k\right) = \left\|u_n^k\right\|^\gamma B\left(v_n^k\right) \longrightarrow \mathcal{M}_{\lambda k} > 0. \tag{4.4}$$

Therefore, $B(v_0^k) \geq 0$. Furthermore, $\|u_n^k\|^p A_{\lambda_k}(v_n^k) \leq 1$ and so $A_{\lambda_k}(v_n^k) \leq 1/\|v_n^k\|^p$. Thus, we arrive at

$$A_{\lambda_k}\left(v_0^k\right) \leq \liminf_{n\to\infty} A_{\lambda_k}\left(v_n^k\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{4.5}$$

which implies

$$\int_\Omega \left|\nabla v_0^k\right|^p dx \leq 1, \tag{4.6}$$

$$\lambda_k \int_\Omega a(x)\left|v_0^k\right|^p dx \geq 1. \tag{4.7}$$

It is a direct consequent of the comactly embedding of X in $L^p(\Omega)$, $L^\gamma(\Omega)$ and $L^p(\partial\Omega)$. Now, we pass to the limit for $k \to \infty$. Then, $\lambda_k \to \lambda_1^+(\alpha)$ and since $\{v_0^k\}$ is a bounded sequence, we may assume that $v_0^k \rightharpoonup v_0$ in X for some $v_0 \in X$ and $v_0^k \to v_0$ in $L^p(\Omega)$, $L^\gamma(\Omega)$, and $L^p(\partial\Omega)$.

It follows from (4.10) that

$$\lambda_1^+(\alpha)\int_\Omega a(x)|v_0|^p dx \geq 1, \tag{4.8}$$

and from the variational characteristic of $\lambda_1^+(\alpha)$ and (4.6) that

$$0 \leq A_{\lambda_1^+(\alpha)}(v_0) \leq 0, \tag{4.9}$$

which contradicts $(\mathcal{B}_3)$.

Hence, for some $\delta > 0$, problem (4.5) has at least one nonnegative solution u_1 for any $\lambda \in (\lambda_1^+(\alpha), \lambda_1^+(\alpha) + \delta)$. □

In order to find the second positive solution, we consider the minimizing problem:

$$\mathcal{N}_\lambda = \inf\{A_\lambda(u); B(u) = -1\}. \tag{4.10}$$

Note that this set is empty for $\lambda < \lambda_1^+(\alpha)$ (because of the variational characterization of $\lambda_1^+(\alpha)$). Hence, this second problem does not have a solution for $\lambda < \lambda_1^+(\alpha)$.

Lemma 4.2. *Under the assumption of Theorem 1.4, there exists $\varepsilon > 0$ such that for $\lambda \in (\lambda_1^+, \lambda_1^+(\alpha)+\varepsilon)$, problem* (1.1) *has a nonnegative solution u_2 satisfying $A_\lambda(u_2) < 0$.*

Proof. First note that using the auxiliary function $L(t) = |t|^\gamma B(u)$ and the assumption $(\mathcal{B}_3)$, it is easy to see that $B(t_1 u_1^+) = -1$ for some $t_1 < 1$. Thus, the set $\{u \in X; B(u) = -1\}$ is nonempty. Also, for this t_1, we have

$$A_\lambda\left(t_1 u_1^+\right) = |t_1|^p\left(\lambda_1^+(\alpha) - \lambda\right)\int_\Omega a(x)\left|u_1^+\right|^p dx < 0, \tag{4.11}$$

for $\lambda > \lambda_1^+(\alpha)$.

Again, we assume that the result is not true. Then, there exists $\varepsilon_k \to 0$ such that for $\lambda_k = \lambda_1^+(\alpha) + \varepsilon_k$, problem (1.1) has no solution.

Let $\{u_n^k\}$ be a minimizing sequence of this problem, that is, $B(u_n^k)=-1, \quad A_{\lambda_k}(u_n^k) \longrightarrow \mathcal{N}_{\lambda k}<0.$

Assume that $\{u_n^k\}$ is bounded. Using similar argument as in the proof of Lemma 4.1, we find a solution of problem (1.1) which is a contradiction. Let us assume $\{u_n^k\}$ is bounded. Again, we consider

$$v_n^k = \frac{u_n^k}{\left\|u_n^k\right\|}, \quad \text{with } \left\|u_n^k\right\| \longrightarrow \infty, \tag{4.12}$$

$\{v_n^k\}$ is bounded, and so $v_n^k \rightharpoonup v_0^k$ in X for some $v_0^k \in X$. Thus, $B(v_n^k) = \dfrac{u_n^k}{\|u_n^k\|^\gamma} = \dfrac{-1}{\|u_n^k\|^\gamma} \longrightarrow 0$.

Letting $n \to \infty$, we arrive at $B(v_0^k) = 0$. It follows easily that $A_{\lambda_k}(u_n^k) = \|u_n^k\|^p A_{\lambda_k}(v_n^k) \leq 0$, and so $A_{\lambda_k}(v_n^k) \leq 0$. As might be expected, we arrive at a contradiction with assumption $(\mathcal{B}_3)$, by using similar steps as in the proof of Theorem 1.3. This completes the proof. □

Proof of Theorem 1.4. Let $\eta = \min\{\varepsilon, \delta\}$, $\lambda^* = \lambda_1^+(\alpha) + \eta$ and $w_1 = t(u_1)u_1$ and $w_2 = t(u_2)u_2$, where $t(u_i)$ is defined by (2.16).

It easy to see that $A_\lambda(w_1) > 0$ and $A_\lambda(w_2) \leq 0$, and so $w_1 \not\equiv w_2$. Thus, w_1 and w_2 are two distinct non negative weak solutions to (1.2). Other properties of w_1 and w_2 follow in the same way as in Section 2. □

References

[1] G. A. Afrouzi and S. Khademloo, "The Nehari manifold for a class of indefinite weight semilinear elliptic equations," *Bulletin of the Iranian Mathematical Society*, vol. 33, no. 2, pp. 49–95, 2007.

[2] K. J. Brown and B. Ko, "The existence of positive solutions for a class of indefinite weight semilinear elliptic boundary value problems," *Nonlinear Analysis*, vol. 39, no. 5, pp. 587–597, 2000.

[3] S. I. Pohozhaev and A. Tesei, "Existence of positive solutions of some nonlinear Neumann problems," *Doklady Mathematics of the Russian Academy of Sciences*, vol. 363, no. 4, pp. 450–453, 1998.

[4] F. Cîrstea, D. Motreanu, and V. Rădulescu, "Weak solutions of quasilinear problems with nonlinear boundary condition," *Nonlinear Analysis*, vol. 43, pp. 623–636, 2001.

[5] N. S. Trudinger, "On Harnack type inequalities and their application to quasilinear elliptic equations," *Communications on Pure and Applied Mathematics*, vol. 20, pp. 721–747, 1967.

[6] S. Khademloo, "Eigenvalue problems for the *p*-Laplace operator with nonlinear boundary condition," *Australian Journal of Basic and Applied Sciences*. In press.

[7] S. I. Pohozaev and L. Veron, "Multiple positive solutions of some quasilinear Neumann problems," *Applicable Analysis*, vol. 74, no. 3-4, pp. 363–391, 2000.

[8] S. I. Pohozhaev, "On the global fibration method in nonlinear variational problems," in *Proceedings of the Steklov Institute of Mathematics*, vol. 219, pp. 286–334, 1997.

[9] P. Drabek, "Strongly nonlinear degenerated and singular elliptic problems," *Pitman Research Notes in Mathematics Series*, vol. 343, pp. 112–146, 1996.

[10] K. J. Brown and Y. Zhang, "The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function," *Journal of Differential Equations*, vol. 193, no. 2, pp. 481–499, 2003.