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Research Article

Approximation of Fixed Points of Weak Bregman Relatively Nonexpansive Mappings in Banach Spaces

Jiawei Chen, Zhongping Wan, Liuyang Yuan, and Yue Zheng

School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, China

Correspondence should be addressed to Jiawei Chen, j.w.chen713@163.com

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We introduce a concept of weak Bregman relatively nonexpansive mapping which is distinct from Bregman relatively nonexpansive mapping. By using projection techniques, we construct several modification of Mann type iterative algorithms with errors and Halpern-type iterative algorithms with errors to find fixed points of weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings in Banach spaces. The strong convergence theorems for weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings are derived under some suitable assumptions. The main results in this paper develop, extend, and improve the corresponding results of Matsushita and Takahashi (2005) and Qin and Su (2007).

1. Introduction

Throughout this paper, without other specifications, we denote by R the set of real numbers. Let E be a real reflexive Banach space with the dual space E^*. The norm and the dual pair between E^* and E are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $f : E \to R\cup\{+\infty\}$ be proper convex and lower semicontinuous. The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty,+\infty]$ defined by

$$f^*(\xi) = \sup\{\langle \xi, x\rangle - f(x) : x \in E\}. \tag{1.1}$$

We denote by $\operatorname{dom} f$ the domain of f, that is, $\operatorname{dom} f = \{x \in E : f(x) < +\infty\}$. Let C be a nonempty closed and convex subset of E and $T : C \to C$ a nonlinear mapping. Denote by $F(T) = \{x \in C : Tx = x\}$, the set of fixed points of T. T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

In 1967, Brègman [1] discovered an elegant and effective technique for the using of the so-called Bregman distance function D_f (see, Section 2, Definition 2.1) in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique is applied in various ways in order to design and analyze iterative algorithms for solving not only feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria, for computing fixed points of nonlinear mappings, and so on (see, e.g., [1–25], and the references therein).

Nakajo and Takahashi [26] introduced the following modification of the Mann iteration method for a nonexpansive mapping $T : C \to C$ in a Hilbert space H as follows:

$$\begin{gathered}
x_0 \in C, \\
y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\
\overline{C}_n = \{ z \in C : \| z - y_n \| \le \| z - x_n \| \}, \\
\overline{Q}_n = \{ z \in C : \langle x_n - z, x_n - x_0 \rangle \le 0 \}, \\
x_{n+1} = P_{\overline{C}_n \cap \overline{Q}_n} x_0, \quad \forall n \ge 0,
\end{gathered} \tag{1.2}$$

where $\{\alpha_n\} \subset [0,1]$ and $P_{\overline{C}}$ is the metric projection from H onto a closed and convex subset C of H. They proved that $\{x_n\}$ generated by (1.2) converges strongly to a fixed point of T under some suitable assumptions. Motivated by Nakajo and Takahashi [26], Matsushita and Takahashi [27] introduced the following modification of the Mann iteration method for a relatively nonexpansive mapping $T : C \to C$ in a Banach space E as follows:

$$\begin{gathered}
x_0 \in C, \\
y_n = J^{-1}(\alpha_n J(x_n) + (1-\alpha_n) J(T x_n)), \\
\overline{C}_n = \{ z \in C : \phi(z, y_n) \le \phi(z, x_n) \}, \\
\overline{Q}_n = \{ z \in C : \langle J(x_n) - J(x_0), x_n - z \rangle \le 0 \}, \\
x_{n+1} = \Pi_{\overline{C}_n \cap \overline{Q}_n} x_0, \quad \forall n \ge 0,
\end{gathered} \tag{1.3}$$

where $\{\alpha_n\} \subset [0,1]$, $\phi(y,x) = \|y\|^2 - 2\langle y, J(x)\rangle + \|x\|^2$ for all $x, y \in E$, J is the duality mapping of E and Π_C is the generalized projection (see, e.g., [2, 3, 28]) from E onto a closed and convex subset C of E. They also proved that $\{x_n\}$ generated by (1.3) converges strongly to a fixed point of T under some suitable assumptions. Martinez-Yanes and Xu [29] gave a Halpern-type iterative algorithm for a nonexpansive mapping $T : C \to C$ as follows:

$$
\begin{gathered}
x_0 \in C, \\
y_n = \beta_n x_0 + (1-\beta_n) T x_n, \\
\overline{C}_n = \left\{ z \in C : \|z - y_n\|^2 \leq \|z - x_n\|^2 + \beta_n \left(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle \right) \right\}, \\
\overline{Q}_n = \{ z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0 \}, \\
x_{n+1} = P_{\overline{C}_n \cap \overline{Q}_n} x_0, \quad \forall n \geq 0,
\end{gathered}
\tag{1.4}
$$

where $\{\beta_n\} \subset [0,1]$. They derived that $\{x_n\}$ generated by (1.3) converges strongly to a fixed point of T under some suitable assumptions. Qin and Su [30] generalized the results of Martinez-Yanes and Xu [29] to a uniformly convex and uniformly smooth Banach space for a relatively nonexpansive mapping and proposed the following iterative algorithm:

$$
\begin{gathered}
x_0 \in C, \\
y_n = J^{-1}(\beta_n J(x_0) + (1-\beta_n) J(T x_n)), \\
\overline{C}_n = \{ z \in C : \phi(z, y_n) \leq \beta_n \phi(z, x_0) + (1-\beta_n)\phi(z, x_n) \}, \\
\overline{Q}_n = \{ z \in C : \langle J(x_n) - J(x_0), x_n - z \rangle \leq 0 \}, \\
x_{n+1} = \Pi_{\overline{C}_n \cap \overline{Q}_n} x_0, \quad \forall n \geq 0,
\end{gathered}
\tag{1.5}
$$

where $\{\beta_n\} \subset [0,1]$, Π_C is the generalized projection (see, e.g., [2, 3, 28]) from E onto a closed and convex subset C of E. They also obtained that $\{x_n\}$ generated by (1.5) converges strongly to a fixed point of T under some suitable assumptions. In 2003, Butnariu et al. [13] studied several notions of convex analysis: uniformly convexity at a point, total convexity at a point, uniformly convexity on bounded sets, and sequential consistency, which are useful in establishing convergence properties for fixed point and optimization algorithms in infinite dimensional Banach spaces. They established connections between these concepts and used these relations in order to obtain improved convergence results concerning the outer Bregman projection algorithm for solving convex feasibility problems and the generalized proximal point algorithm for optimization in Banach spaces. In 2006, Butnariu and Resmerita [14] presented a Bregman-type iterative algorithms and studied the convergence of the Bregman-type iterative method of solving operator equations. Resmerita [19] investigated the existence of totally convex functions in Banach spaces and, further, established continuity and stability properties of Bregman projections. Very recently, by using Bregman projection, Reich and Sabach [21] presented the following algorithms for finding common zeroes of maximal monotone operators $A_i : E \to 2^{E^*} (i = 1, 2, \dots, N)$ in reflexive Banach space E as follows:

$$
\begin{aligned}
& x_0 \in E, \\
& y_n^i = \operatorname{Res}_{\lambda_n^i}^f \left(x_n + e_n^i \right), \\
& C_n^i = \left\{ z \in E : D_f\left(z, y_n^i \right) \le D_f\left(z, x_n + e_n^i \right) \right\}, \\
& C_n = \bigcap_{i=1}^N C_n^i, \\
& Q_n = \left\{ z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \le 0 \right\}, \\
& x_{n+1} = \operatorname{proj}_{C_n \cap Q_n}^f x_0, \quad \forall n \ge 0,
\end{aligned}
\tag{1.6}
$$

$$
\begin{aligned}
& x_0 \in E, \\
& \eta_n^i = \xi_n^i + \frac{1}{\lambda_n^i} \left(\nabla f\left(y_n^i \right) - \nabla f(x_n) \right), \quad \xi_n^i \in A_i y_n^i, \\
& \omega_n^i = \nabla f^* \left(\lambda_n^i \eta_n^i + \nabla f(x_n) \right), \\
& C_n^i = \left\{ z \in E : D_f\left(z, y_n^i \right) \le D_f\left(z, \omega_n^i \right) \right\}, \\
& C_n = \bigcap_{i=1}^N C_n^i, \\
& Q_n = \left\{ z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \le 0 \right\}, \\
& x_{n+1} = \operatorname{proj}_{C_n \cap Q_n}^f x_0, \quad \forall n \ge 0.
\end{aligned}
\tag{1.7}
$$

Further, under some suitable conditions, they obtained two strong convergence theorems of maximal monotone operators in reflexive Banach spaces. Reich and Sabach [22] studied the convergence of two iterative algorithms for finitely many Bregman strongly nonexpansive operators in Banach spaces and obtained two strong convergence theorems for finitely many Bregman strongly nonexpansive operators under some assumptions. In [24], Reich and Sabach proposed the following algorithms for finding common fixed points of finitely many Bregman firmly nonexpansive operators $T_i : C \to C$ $(i = 1, 2, \dots, N)$ in reflexive Banach space E as follows: if $\bigcap_{i=1}^N F(T_i) \neq \emptyset$

$$
\begin{aligned}
& x_0 \in E, \\
& Q_0^i = E, \quad i = 1, 2, \dots, N, \\
& y_n^i = T_i\left(x_n + e_n^i \right),
\end{aligned}
$$

$$Q_{n+1}^i = \left\{ z \in Q_n^i : \left\langle \nabla f\left(x_n + e_n^i\right) - \nabla f\left(y_n^i\right), z - y_n^i \right\rangle \leq 0 \right\},$$
$$Q_n = \bigcap_{i=1}^{N} Q_n^i,$$
$$x_{n+1} = \operatorname{proj}_{Q_{n+1}}^f x_0, \quad \forall n \geq 0. \tag{1.8}$$

Under some suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to $\bigcap_{i=1}^{N} F(T_i)$ and applied it to the solution of convex feasibility and equilibrium problems.

Inspired and motivated by the works, we introduce the concept of weak Bregman relatively nonexpansive mappings in reflexive Banach space and give an example to illustrate the existence of weak Bregman relatively nonexpansive mapping and the difference between weak Bregman relatively nonexpansive mapping and Bregman relatively nonexpansive mapping. Secondly, by using the conception of the Bregman projection (see, e.g., [1, 13, 14]), we construct several modification of Mann type iterative algorithms with errors and Halpern-type iterative algorithms with errors to find fixed points of weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings in Banach spaces. The strong convergence theorems for weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings are derived under some suitable assumptions. Moreover, the convergence rate of our algorithms is faster than that of Matsushita and Takahashi [27] and Qin and Su [30]. The main results in this paper develop, extend, and improve the corresponding results in the literature.

2. Preliminaries

Let C be a nonempty closed convex subset of a real reflexive Banach space E, and let $T : C \to C$ be a nonlinear mapping. A point $\omega \in C$ is called an *asymptotic fixed point* of T (see, e.g., [2, 3]) if C contains a sequence $\{x_n\}$ which converges weakly to ω such that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. A point $\omega \in C$ is called an *strong asymptotic fixed point* of T (see, e.g., [2, 3]) if C contains a sequence $\{x_n\}$ which converges strongly to ω such that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. We denote the sets of asymptotic fixed points and strong asymptotic fixed points of T by $\widehat{F}(T)$ and $\widetilde{F}(T)$, respectively. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$. For any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in E$, the *right-hand derivative* of f at x in the direction y defined by

$$f^0(x, y) := \lim_{t \searrow 0} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

f is called *Gâteaux differentiable* at x if, for all $y \in E, \lim_{t\searrow 0}(f(x + ty) - f(x))/t$ exists. In this case, $f^0(x, y)$ coincides with $\nabla f(x)$, the value of the gradient of f at x. f is called *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$. f is called *Fréchet differentiable* at x if this limit is attained uniformly for $\|y\| = 1$. We say f is *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Legendre function $f : E \to (-\infty, +\infty]$ is defined in [7]. From [7], if E is a reflexive Banach space, then f is Legendre if and only if it satisfies the following conditions $(L1)$ and $(L2)$:

$(L1)$ the interior of the domain of f, $\operatorname{int}(\operatorname{dom} f)$, is nonempty, f is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f)$, and $\operatorname{dom} f = \operatorname{int}(\operatorname{dom} f)$,

$(L2)$ the interior of the domain of f^*, $\operatorname{int}(\operatorname{dom} f^*)$, is nonempty, f^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f^*)$, and $\operatorname{dom} f^* = \operatorname{int}(\operatorname{dom} f^*)$.

Since E is reflexive, we know that $(\partial f)^{-1} = \partial f^*$ (see, e.g., [31]). This, by $(L1)$ and $(L2)$, implies

$$\nabla f = (\nabla f^*)^{-1}, \qquad \operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}(\operatorname{dom} f^*),$$
$$\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}(\operatorname{dom} f). \tag{2.2}$$

By Theorem 5.4 [7], conditions $(L1)$ and $(L2)$ also yield that the functions f and f^* are strictly convex on the interior of their respective domains. From now on, we assume that the convex function $f : E \to (-\infty, +\infty]$ is Legendre.

We first recall some definitions and lemmas which are needed in our main results.

Definition 2.1 (see [1, 13]). Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The function $D_f : \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, +\infty)$, defined by

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle \tag{2.3}$$

is called *the Bregman distance* with respect to f.

Remark 2.2 (see [24]). The Bregman distance has the following properties:

(i) the three point identity, for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom} f)$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle, \tag{2.4}$$

(ii) the four point identity, for any $y, \omega \in \operatorname{dom} f$ and $x, z \in \operatorname{int}(\operatorname{dom} f)$,

$$D_f(y, x) - D_f(y, z) - D_f(\omega, x) + D_f(\omega, z) = \langle \nabla f(z) - \nabla f(x), y - \omega \rangle. \tag{2.5}$$

Definition 2.3 (see [1]). Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman projection of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty closed and convex set $C \subset \operatorname{dom} f$ is the necessarily unique vector $\operatorname{proj}_C^f(x) \in C$ satisfying

$$D_f\left(\operatorname{proj}_C^f(x), x\right) = \inf\{D_f(y, x) : y \in C\}. \tag{2.6}$$

Remark 2.4. (i) If E is a Hilbert space and $f(y) = (1/2)\|x\|^2$ for all $x \in E$, then the Bregman projection $\mathrm{proj}_C^f(x)$ is reduced to the metric projection of x onto C.

(ii) If E is a smooth Banach space and $f(y) = (1/2)\|x\|^2$ for all $x \in E$, then the Bregman projection $\mathrm{proj}_C^f(x)$ is reduced to the generalized projection $\Pi_C(x)$ (see, e.g. [3]) which defined by

$$\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y, x), \tag{2.7}$$

where $\phi(y, x) = \|y\|^2 - 2\langle y, J(x)\rangle + \|x\|^2$, J is the normalized duality mapping from E to 2^{E^*}.

Definition 2.5 (see[12, 21]). Let C be a nonempty closed and convex set of $\mathrm{dom}\, f$. The operator $T : C \to \mathrm{int}(\mathrm{dom}\, f)$ with $F(T) \neq \emptyset$ is called:

(i) *quasi-Bregman nonexpansive* if

$$D_f(u, Tx) \leq D_f(u, x), \quad \forall x \in C,\ u \in F(T); \tag{2.8}$$

(ii) *Bregman relatively nonexpansive* if

$$D_f(u, Tx) \leq D_f(u, x), \quad \forall x \in C,\ u \in F(T), \tag{2.9}$$

and $\widehat{F}(T) = F(T)$,

(iii) *Bregman firmly nonexpansive* if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle, \quad \forall x, y \in C, \tag{2.10}$$

or equivalently

$$\begin{aligned} D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \\ \leq D_f(Tx, y) + D_f(Ty, x), \quad \forall x, y \in C. \end{aligned} \tag{2.11}$$

Definition 2.6. Let C be a nonempty closed and convex set of $\mathrm{dom}\, f$. The operator $T : C \to \mathrm{int}(\mathrm{dom}\, f)$ with $F(T) \neq \emptyset$ is called *weak Bregman relatively nonexpansive* if $\widetilde{F}(T) = F(T)$ and

$$D_f(u, Tx) \leq D_f(u, x), \quad \forall x \in C,\ u \in F(T). \tag{2.12}$$

Remark 2.7. It is easy to see that each nonexpansive mapping T is quasi-Bregman nonexpansive mapping with respect to $f(x) = (1/2)\|x\|^2$ for all $x \in E$. Moreover, every relatively nonexpansive mapping T also is Bregman relatively nonexpansive mapping, where

T is called relatively nonexpansive mapping (see, e.g., [32]) if the following conditions are satisfied:

$$\widehat{F}(T) = F(T) \neq \emptyset, \quad \phi(u, Tx) \leq \phi(u, x), \quad \forall x \in C, u \in F(T). \tag{2.13}$$

Now, we give an example which is weak Bregman relatively nonexpansive mapping but not Bregman relatively nonexpansive mapping.

Example 2.8. Let $E = l^2$, $f(x) = (1/2)\|x\|^2$ for all $x \in E$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} |\xi_n|^2 < \infty \right\}, \quad \|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{1/2}, \quad \forall \xi \in l^2, \tag{2.14}$$

and for any $\xi = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_n, \ldots) \in E$, $\langle \xi, \mu \rangle = \sum_{n=1}^{\infty} \xi_n \mu_n$. It is well known that l^2 is a Hilbert space. Let $\{x_n\} \subset E$ be a sequence defined by $x_0 = (1, 0, 0, 0, \ldots)$, $x_1 = (1, 1, 0, 0, \ldots)$, $x_2 = (1, 0, 1, 0, \ldots), \ldots$, $x_n = (\xi_{n,1}, \ldots, \xi_{n,k}, \ldots), \ldots$, where

$$\xi_{n,k} = \begin{cases} 1, & \text{if } k = 1, \; n+1, \\ 0, & \text{otherwise,} \end{cases} \tag{2.15}$$

for all $n \geq 0$.

Define a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \dfrac{n x_n}{n+1}, & \text{if } x = x_n \; (\exists n \geq 1), \\ -x, & \text{if } x \neq x_n \; (\forall n \geq 1), \end{cases} \tag{2.16}$$

for all $n \geq 0$. It is easy to see that $F(T) = \{0\}$, and so, $\{x_n\}$ converges weakly to x_0. Indeed, for any $g = (\zeta_1, \zeta_2 \ldots, \zeta_k, \ldots) \in E$, we have

$$g(x_n - x_0) = \langle g, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \xi_{n,k} = \zeta_{n+1}. \tag{2.17}$$

From $\sum_{n=1}^{\infty} |\zeta_n|^2 < \infty$, it shows that $\lim_{n\to\infty} \zeta_{n+1} = 0$. Moreover,

$$\lim_{n\to\infty} g(x_n - x_0) = \lim_{n\to\infty} \zeta_{n+1} = 0. \tag{2.18}$$

Next, for any $m \neq n$, one has $\|x_n - x_m\| = \sqrt{2} \neq 0$; that is, $\{x_n\}$ is not a Cauchy sequence. Owing to $\|Tx_n - x_n\| = \|x_n\|/(n+1)$, we obtain

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{2.19}$$

Then, x_0 is an asymptotic fixed point of T, but $x_0 \notin F(T) = \{0\}$. So, T is not Bregman relatively nonexpansive mapping.

For any strong convergent sequence $\{y_n\} \subset l^2$ such that $y_n \to y_0$ and $\|Ty_n - y_n\| \to 0$ as $n \to \infty$. Then, there exists a sufficiently large nature number M such that $y_n \neq x_m$ for any $n, m > M$. Thus, $Ty_n = -y_n$ for $n > M$, which implies that $2y_n \to 0$ and $y_n \to y_0 = 0$ as $n \to \infty$. That is, $y_0 = 0$ is a strong asymptotic fixed point of T, and so, $\widetilde{F}(T) = F(T) = \{0\}$. Since

$$\begin{aligned} D_f(0, Tx) &= f(0) - f(Tx) - \langle \nabla f(Tx), 0 - Tx \rangle = -\frac{1}{2}\|Tx\|^2 + \langle Tx, Tx \rangle = \frac{1}{2}\|Tx\|^2 \\ &\leq \frac{1}{2}\|x\|^2 = f(0) - f(x) - \langle \nabla f(x), 0 - x \rangle = D_f(0, x), \quad x \in E. \end{aligned} \tag{2.20}$$

Therefore, T is a weak Bregman relatively nonexpansive mapping.

Definition 2.9 (see [12]). Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. f is called:

(i) *totally convex at* $x \in \operatorname{int}(\operatorname{dom} f)$ if its modulus of total convexity at x; that is, the function $\nu_f : \operatorname{int}(\operatorname{dom} f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\} \tag{2.21}$$

is positive whenever $t > 0$,

(ii) *totally convex* if, it is totally convex at every point $x \in \operatorname{int}(\operatorname{dom} f)$,

(iii) *totally convex on bounded sets* if $\nu_f(B, t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \operatorname{int}(\operatorname{dom} f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(B, t) := \inf\{\nu_f(x, t) : x \in B \cap \operatorname{dom} f\}. \tag{2.22}$$

Definition 2.10 (see [12, 21]). The function $f : E \to (-\infty, +\infty]$ is called:

(i) *cofinite* if $\operatorname{dom} f^* = E^*$,

(ii) *sequentially consistent* if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first is bounded, and

$$\lim_{n \to \infty} D_f(y_n, x_n) = 0 \Longrightarrow \lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{2.23}$$

Lemma 2.11 (see [21, Proposition 2.3]). *If $f : E \to (-\infty, +\infty]$ is Fréchet differentiable and totally convex, then f is cofinite.*

Lemma 2.12 (see [14, Theorem 2.10]). *Let $f : E \to (-\infty, +\infty]$ be a convex function whose domain contains at least two points. Then, the following statements hold:*

(i) *f is sequentially consistent if and only if it is totally convex on bounded sets,*

(ii) *if f is lower semicontinuous, then f is sequentially consistent if and only if it is uniformly convex on bounded sets,*

(iii) *if f is uniformly strictly convex on bounded sets, then it is sequentially consistent and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain, and the Fréchet derivative f' is uniformly continuous on bounded sets.*

Lemma 2.13 (see [20, Proposition 2.1]). *Let $f : E \to R$ be a uniformly Fréchet differentiable and bounded on bounded subsets of E. Then, ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.14 (see [21, Lemma 3.1]). *Let $f : E \to R$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ is also bounded.*

Lemma 2.15 (see [21, Proposition 2.2]). *Let $f : E \to R$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$, and let C be a nonempty closed convex subset of E. Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded and any weak subsequential limit of $\{x_n\}_{n=1}^{\infty}$ belongs to C. If $D_f(x_n, x_0) \leq D_f(\mathrm{proj}_C^f(x_0), x_0)$ for any $n \in N$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\mathrm{proj}_C^f(x_0)$.*

In [23], Reich and Sabach proved the following result.

Lemma 2.16 (see [23, Lemma 15.5]). *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty closed convex subset of $\mathrm{int}(\mathrm{dom}\ f)$ and $T : C \to C$ a Bregman firmly nonexpansive mapping with respect to f. Then, $F(T)$ is closed and convex.*

Motivated by Lemma 2.16, we get the similar result for quasi-Bregman nonexpansive mapping.

Proposition 2.17. *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty closed convex subset of $\mathrm{int}(\mathrm{dom}\ f)$ and $T : C \to C$ a quasi-Bregman nonexpansive mapping with respect to f. Then, $F(T)$ is closed and convex.*

Proof. Without loss of generality, set $F(T)$ is nonempty. Firstly, we show that $F(T)$ is closed. Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in $F(T)$ such that $x_n \to \overline{x}$. By the definition of quasi-Bregman nonexpansive mapping, we have

$$D_f(x_n, T\overline{x}) \leq D_f(x_n, \overline{x}), \quad n \geq 0. \tag{2.24}$$

Since $f : E \to (-\infty, +\infty]$ is a Legendre function, f is continuous at $\overline{x} \in C \subset \mathrm{int}(\mathrm{dom}\ f)$. Then, from the definition of Bregman distance,

$$\begin{aligned} &\lim_{n \to \infty} D_f(x_n, T\overline{x}) = D_f(\overline{x}, T\overline{x}), \\ &\lim_{n \to \infty} D_f(x_n, \overline{x}) = D_f(\overline{x}, \overline{x}) = 0. \end{aligned} \tag{2.25}$$

From (2.24) and (2.25), it follows that $D_f(\overline{x}, T\overline{x}) = 0$, and so, from [7, Lemma 7.3(vi), page 642], $T\overline{x} = \overline{x}$. Therefore, $\overline{x} \in F(T)$, and so, $F(T)$ is closed.

We now show that $F(T)$ is convex. For any $x, y \in F(T)$ and $t \in (0,1)$, it yields that $z = tx + (1-t)y \in C$. From the definition of quasi-Bregman nonexpansive mapping, it follows that

$$\begin{aligned} D_f(z, Tz) &= f(z) - f(Tz) - \langle \nabla f(Tz), tx + (1-t)y - T(tx + (1-t)y) \rangle \\ &= f(z) + tD_f(x, Tz) + (1-t)D_f(y, Tz) - tf(x) - (1-t)f(y) \\ &\leq f(z) + tD_f(x, z) + (1-t)D_f(y, z) - tf(x) - (1-t)f(y) \\ &= \langle \nabla f(z), z - tx - (1-t)y \rangle = 0. \end{aligned} \tag{2.26}$$

Again, from [7, Lemma 7.3(vi), page 642], we get $Tz = z$. Therefore, $F(T)$ is convex. This completes the proof. □

From the definitions of Bregman distance and the Fenchel conjugate of f, we have the following result.

Lemma 2.18. *Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and proper convex lower semi-continuous. Then, for all $z \in E$,*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.27}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.19 (see [14, Corollary 4.4]). *Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex on* int(dom f). *Let $x \in$ int(dom f) and $C \subset$ int(dom f) a nonempty closed convex set. If $\widehat{x} \in C$, then the following statements are equivalent:*

(i) *the vector $\widehat{x}$ is the Bregman projection of x onto C with respect to f,*

(ii) *the vector $\widehat{x}$ is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0, \quad \forall y \in C, \tag{2.28}$$

(iii) *the vector $\widehat{x}$ is the unique solution of the inequality*

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x), \quad \forall y \in C. \tag{2.29}$$

3. Main Results

In this section, we introduce several modification of Mann-type iterative algorithms with errors and Halpern-type iterative algorithms with errors to find fixed points of weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings in Banach spaces. The strong convergence theorems for weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings are proved under some suitable conditions.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real reflexive Banach space E and $f : E \to R$ a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subset of E, and let $T : C \to C$ be a weak Bregman relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{gathered}
x_0 \in C, \quad Q_0 = C, \\
z_n = \nabla f^*\left(\beta_n \nabla f(T(x_n + e_n)) + \left(1 - \beta_n\right)\nabla f(x_n + e_n)\right), \\
y_n = \nabla f^*\left(\alpha_n \nabla f(x_n + e_n) + (1 - \alpha_n)\nabla f(z_n)\right), \\
C_n = \left\{z \in C_{n-1} \cap Q_{n-1} : D_f\left(z, y_n\right) \leq D_f(z, x_n + e_n)\right\}, \\
C_0 = \left\{z \in C : D_f\left(z, y_0\right) \leq D_f(z, x_0)\right\}, \\
Q_n = \left\{z \in C_{n-1} \cap Q_{n-1} : \left\langle \nabla f(x_0) - \nabla f(x_n), z - x_n \right\rangle \leq 0\right\}, \\
x_{n+1} = \operatorname{proj}^f_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ such that $\liminf_{n\to\infty}(1 - \alpha_n)\beta_n > 0$, and $\{e_n\}$ is an error sequence in E with $e_n \to 0$ as $n \to \infty$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the point $\operatorname{proj}^f_{F(T)}(x_0)$, where $\operatorname{proj}^f_{F(T)}(x_0)$ is the Bregman projection of C onto $F(T)$.

Proof. By Proposition 2.17, it follows that $F(T)$ is a nonempty closed and convex subset of E. It is easy to verify that C_0, C_1, Q_0, and Q_1 are closed and convex. Suppose that C_k and Q_k $(k \geq 1)$ are closed and convex. Then, $C_k \cap Q_k$ is closed and convex. For any $z \in C_{k+1}, y \in Q_{k+1}$,

$$\begin{aligned}
&D_f\left(z, y_{k+1}\right) \leq D_f(z, x_{k+1} + e_{k+1}) \\
&\Longleftrightarrow f(z) - f\left(y_{k+1}\right) - \left\langle \nabla f\left(y_{k+1}\right), z - y_{k+1}\right\rangle \\
&\qquad \leq f(z) - f(x_{k+1} + e_{k+1}) - \left\langle \nabla f(x_{k+1} + e_{k+1}), z - (x_{k+1} + e_{k+1})\right\rangle \\
&\Longleftrightarrow \left\langle \nabla f(x_{k+1} + e_{k+1}), z - (x_{k+1} + e_{k+1})\right\rangle - \left\langle \nabla f\left(y_{k+1}\right), z - y_{k+1}\right\rangle \\
&\qquad \leq f\left(y_{k+1}\right) - f(x_{k+1} + e_{k+1}) \\
&\Longleftrightarrow \left\langle \nabla f(x_{k+1} + e_{k+1}) - \nabla f\left(y_{k+1}\right), z - y_{k+1}\right\rangle \\
&\qquad \leq f\left(y_{k+1}\right) - f(x_{k+1} + e_{k+1}) - \left\langle \nabla f(x_{k+1} + e_{k+1}), y_{k+1} - (x_{k+1} + e_{k+1})\right\rangle \\
&\Longleftrightarrow \left\langle \nabla f(x_{k+1} + e_{k+1}) - \nabla f\left(y_{k+1}\right), z - y_{k+1}\right\rangle \leq D_f\left(y_{k+1}, x_{k+1} + e_{k+1}\right), \\
&\left\langle \nabla f(x_0) - \nabla f(x_k), y - x_k\right\rangle \leq 0,
\end{aligned} \tag{3.2}$$

which implies that C_{k+1} and Q_{k+1} are closed and convex. As a consequence, C_n and Q_n are closed and convex for all $n \geq 0$. Taking $p \in F(T)$ arbitrarily,

$$\begin{aligned}
&D_f(p, y_n) \\
&\quad = D_f(p, \nabla f^*(\alpha_n \nabla f(x_n + e_n) + (1-\alpha_n)\nabla f(z_n))) \\
&\quad \leq \alpha_n D_f(p, x_n + e_n) + (1-\alpha_n) D_f(p, z_n) \\
&\quad = \alpha_n D_f(p, x_n + e_n) + (1-\alpha_n) D_f(p, \nabla f^*(\beta_n \nabla f(T(x_n + e_n)) + (1-\beta_n)\nabla f(x_n + e_n))) \\
&\quad \leq \alpha_n D_f(p, x_n + e_n) + (1-\alpha_n)\left[(1-\beta_n) D_f(p, x_n + e_n) + \beta_n D_f(p, T(x_n + e_n))\right] \\
&\quad \leq \alpha_n D_f(p, x_n + e_n) + (1-\alpha_n)\left[(1-\beta_n) D_f(p, x_n + e_n) + \beta_n D_f(p, x_n + e_n)\right] \\
&\quad = D_f(p, x_n + e_n),
\end{aligned} \tag{3.3}$$

that is, $p \in C_n$, and so, $F(T) \subset C_n$ for all $n \geq 0$. We now show that $F(T) \subset Q_n$ for all $n \geq 0$. Clearly, $F(T) \subset Q_0 = C$. Assume that $F(T) \subset Q_k$ for all $k \geq 0$. Note that $x_{k+1} = \mathrm{proj}^f_{C_k \cap Q_k}(x_0)$, and we have

$$\langle \nabla f(x_0) - \nabla f(x_{k+1}), x_{k+1} - z \rangle \geq 0, \quad z \in C_k \cap Q_k. \tag{3.4}$$

Therefore,

$$\langle \nabla f(x_0) - \nabla f(x_{k+1}), x_{k+1} - p \rangle \geq 0, \quad p \in F \subset C_k \cap Q_k, \tag{3.5}$$

which yields that $p \in Q_{k+1}$. Then, $F(T) \subset Q_n$ for all $n \geq 0$. Consequently, $F(T) \subset C_n \cap Q_n$ and $C_n \cap Q_n$ is nonempty closed and convex for all $n \geq 0$. Moreover, $\{x_n\}$ is well defined.

Secondly, we show that $\{x_n\}$ is a Cauchy sequence and bounded. Since

$$\langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0, \quad \forall z \in Q_n, \tag{3.6}$$

it follows that $x_n = \mathrm{proj}^f_{Q_n}(x_0)$. Therefore, by $x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}(x_0) \in Q_n$,

$$D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0). \tag{3.7}$$

Taking $p \in F(T)$ arbitrarily. From Lemma 2.19, it yields that

$$D_f\left(p, \mathrm{proj}^f_{Q_n}(x_0)\right) + D_f\left(\mathrm{proj}^f_{Q_n}(x_0), x_0\right) \leq D_f(p, x_0). \tag{3.8}$$

Moreover, one has

$$D_f(x_n, x_0) \leq D_f(p, x_0) - D_f(p, x_n) \leq D_f(p, x_0). \tag{3.9}$$

Hence, $\{D_f(x_n, x_0)\}$ is bounded and so $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are also bounded. From (3.7), it shows that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists. In the light of $x_m \in Q_{m-1} \subset Q_n$ for any $m > n$, by Lemma 2.19,

$$D_f\left(x_m, \operatorname{proj}^f_{Q_n}(x_0)\right) + D_f\left(\operatorname{proj}^f_{Q_n}(x_0), x_0\right) \le D_f(x_m, x_0), \tag{3.10}$$

that is,

$$D_f(x_m, x_n) \le D_f(x_m, x_0) - D_f(x_n, x_0). \tag{3.11}$$

Consequently, one has

$$\lim_{n\to\infty} D_f(x_m, x_n) = 0. \tag{3.12}$$

Since f is totally convex on bounded subsets of E, by Lemma 2.12 and (3.12), we have

$$\lim_{n\to\infty} \|x_m - x_n\| = 0. \tag{3.13}$$

Thus, $\{x_n\}$ is a Cauchy sequence, and so,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.14}$$

Since $e_n \to 0$ as $n \to \infty$, one has

$$\lim_{n\to\infty} \|(x_{n+1} + e_{n+1}) - (x_n + e_n)\| = 0, \qquad \lim_{n\to\infty} \|x_{n+1} - (x_n + e_n)\| = 0. \tag{3.15}$$

Let $x_n \to \overline{\omega} \in C$. Then, $x_n + e_n \to \overline{\omega}$.

Thirdly, we show that $\{x_n\}$ converges strongly to a point of $F(T)$. Since f is uniformly Fréchet differentiable on bounded subsets of E, from Lemma 2.12, ∇f is norm-to-norm uniformly continuous on bounded subsets of E. So, by (3.15),

$$\lim_{n\to\infty} \left\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\right\| = 0. \tag{3.16}$$

It follows from $x_{n+1} \in C_n$ that

$$D_f(x_{n+1}, y_n) \le D_f(x_{n+1}, x_n + e_n). \tag{3.17}$$

By the uniformly Fréchet differentiable of f on bounded subsets of E, f is also uniformly continuous on bounded subsets of E. Hence, from (3.12) and $\lim_{n\to\infty} e_n = 0$,

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n + e_n) = \lim_{n\to\infty} \left(f(x_{n+1}) - f(x_n + e_n) - \left\langle \nabla f(x_n + e_n), x_{n+1} - (x_n + e_n)\right\rangle\right) = 0. \tag{3.18}$$

As a consequence, $\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0$ and so, $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$. Moreover, one has

$$\lim_{n\to\infty}\left\|\nabla f(x_{n+1}) - \nabla f(y_n)\right\| = 0. \tag{3.19}$$

Since $\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$, $\|x_n - y_n\| \to 0$ and $y_n \to \overline{\omega}$ as $n \to \infty$. Noticing that

$$\begin{aligned}
&\left\|\nabla f(x_{n+1}) - \nabla f(y_n)\right\| \\
&\quad = \left\|\nabla f(x_{n+1}) - \left(\alpha_n \nabla f(x_n + e_n) + (1-\alpha_n)\nabla f(z_n)\right)\right\| \\
&\quad \ge (1-\alpha_n)\left\|\nabla f(x_{n+1}) - \nabla f(z_n)\right\| - \alpha_n\left\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\right\| \\
&\quad = (1-\alpha_n)\left\|\nabla f(x_{n+1}) - \left(\beta_n \nabla f(T(x_n + e_n)) + \left(1-\beta_n\right)\nabla f(x_n + e_n)\right)\right\| \\
&\qquad - \alpha_n\left\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\right\| \\
&\quad \ge -\alpha_n\left\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\right\| + (1-\alpha_n)\beta_n\left\|\nabla f(x_{n+1}) - \nabla f(T(x_n + e_n))\right\| \\
&\qquad - (1-\alpha_n)\left(1-\beta_n\right)\left\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\right\|.
\end{aligned} \tag{3.20}$$

Therefore,

$$\begin{aligned}
&(1-\alpha_n)\beta_n\|\nabla f(x_{n+1}) - \nabla f(T(x_n + e_n))\| \\
&\quad \le \|\nabla f(x_{n+1}) - \nabla f(y_n)\| + \alpha_n\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\| \\
&\qquad + (1-\alpha_n)\left(1-\beta_n\right)\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\|.
\end{aligned} \tag{3.21}$$

In view of $\liminf_{n\to\infty}(1-\alpha_n)\beta_n > 0$ and from both (3.16) and (3.19), one has

$$\lim_{n\to\infty}\left\|\nabla f(x_{n+1}) - \nabla f(T(x_n + e_n))\right\| = 0. \tag{3.22}$$

Furthermore, we have

$$\lim_{n\to\infty}\|x_{n+1} - T(x_n + e_n)\| = 0, \tag{3.23}$$

and so, by (3.14),

$$\lim_{n\to\infty}\|(x_n + e_n) - T(x_n + e_n)\| = 0. \tag{3.24}$$

Since $x_n \to \overline{\omega}$ and $e_n \to 0$, we get $\overline{\omega} \in \widetilde{F}(T) = F(T)$.

Finally, we show $\overline{\omega} = \operatorname{proj}^f_{F(T)}(x_0)$. Since $\operatorname{proj}^f_{F(T)}(x_0) \in F(T) \subset C_n \cap Q_n$, it follows from $x_{n+1} = \operatorname{proj}^f_{(C_n\cap Q_n)}(x_0)$ that $D_f(x_{n+1}, x_0) \le D_f(\operatorname{proj}^f_{F(T)}(x_0), x_0)$. By Lemma 2.15, $x_n \to \operatorname{proj}^f_{F(T)}(x_0)$ as $n \to \infty$. Therefore, $\{x_n\}$ and $\{y_n\}$ converge strongly to $\operatorname{proj}^f_{F(T)}(x_0)$. This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a real reflexive Banach space E and $f : E \to R$ a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subset of E, and let $T : C \to C$ be a Bregman relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ such that $\liminf_{n\to\infty}(1-\alpha_n)\beta_n > 0$, and $\{e_n\}$ is an error sequence in E with $e_n \to 0$ as $n \to \infty$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.1) converge strongly to the point $\mathrm{proj}^f_{F(T)}(x_0)$, where $\mathrm{proj}^f_{F(T)}(x_0)$ is the Bregman projection of C onto $F(T)$.*

Proof. As in the proof of Theorem 3.1, we know that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $\overline{\omega} \in C$, and so,

$$\lim_{n\to\infty} \|(x_n + e_n) - T(x_n + e_n)\| = 0. \tag{3.25}$$

Then, for any subsequence $\{x_{nk}\}$ of $\{x_n\}$ converges weakly to $\overline{\omega}$,

$$\lim_{k\to\infty} \|(x_{nk} + e_{nk}) - T(x_{nk} + e_{nk})\| = 0. \tag{3.26}$$

Therefore, $\overline{\omega} \in \widehat{F}(T) = F(T)$. By the similar proof of Theorem 3.2, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $\mathrm{proj}^f_{F(T)}(x_0)$. This completes the proof. □

If $\alpha_n \equiv 0$, $e_n \equiv 0$, and $f(x) = (1/2)\|x\|^2$ for all $x \in E$, $n \geq 0$, then from Remark 2.4 and Theorem 3.1, we have the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real reflexive, smooth, and strictly convex Banach space E, and let $T : C \to C$ be a relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{gathered}
x_0 \in C, \qquad Q_0 = C, \\
y_n = J^{-1}\left(\beta_n J(T(x_n)) + (1-\beta_n) J(x_n)\right), \\
C_n = \left\{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\right\}, \\
C_0 = \left\{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\right\}, \\
Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle J(x_0) - J(x_n), z - x_n\rangle \leq 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{3.27}$$

where J is the duality mapping on E, $\{\beta_n\} \subset [0,1]$ such that $\liminf_{n\to\infty}\beta_n > 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the point $\Pi_{F(T)}(x_0)$, where $\Pi_{F(T)}(x_0)$ is the generalized projection (see, e.g., [2, 3, 28]) of C onto $F(T)$.

In [27], Matsushita and Takahashi proved the following result.

Theorem MT (see [27, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real uniformly convex and uniformly smooth Banach space E, and let $T : C \to C$ be a relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence of real numbers such that $0 \leq \alpha_n < 1$*

and $\limsup_{n\to\infty}\alpha_n < 1$. *Then, the sequence* $\{x_n\}$ *generated by* (1.3) *converges strongly to the point* $\Pi_{F(T)}(x_0)$, *where* $\Pi_{F(T)}(x_0)$ *is the generalized projection (see, e.g., [2, 3, 28]) of* C *onto* $F(T)$.

Remark 3.4. Corollary 3.3 extends Theorem MT [27] from uniformly convex and uniformly smooth Banach spaces to reflexive, smooth, and strictly convex Banach space.

Now, we investigate convergence theorems for Halpern-type iterative algorithms with errors.

Theorem 3.5. *Let* C *be a nonempty closed convex subset of a real reflexive Banach space* E *and* $f: E \to R$ *a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subset of* E, *and let* $T: C \to C$ *be a weak Bregman relatively nonexpansive mapping such that* $F(T) \neq \emptyset$. *Define a sequence* $\{x_n\}$ *in* C *by the following algorithm:*

$$\begin{gathered}
x_0 \in C, \quad Q_0 = C, \\
z_n = \nabla f^*\left(\beta_n \nabla f(x_0) + (1-\beta_n)\nabla f(T(x_n + e_n))\right), \\
y_n = \nabla f^*\left(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(x_n + e_n)\right), \\
C_n = \left\{z \in C_{n-1} \cap Q_{n-1} : D_f(z, y_n) \leq (1-\alpha_n\beta_n) D_f(z, x_n + e_n) + \alpha_n\beta_n D_f(z, x_0)\right\}, \\
C_0 = \left\{z \in C : D_f(z, y_0) \leq D_f(z, x_0)\right\}, \\
Q_n = \left\{z \in C_{n-1} \cap Q_{n-1} : \left\langle \nabla f(x_0) - \nabla f(x_n), z - x_n \right\rangle \leq 0\right\}, \\
x_{n+1} = \operatorname{proj}^f_{(C_n \cap Q_n)} x_0, \quad \forall n \geq 0,
\end{gathered} \tag{3.28}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *such that* $\liminf_{n\to\infty}\alpha_n > 0$ *and* $\lim_{n\to\infty}\beta_n = 0$, *and* $\{e_n\}$ *is an error sequence in* E *with* $e_n \to 0$ *as* $n \to \infty$. *Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the point* $\operatorname{proj}^f_{F(T)}(x_0)$, *where* $\operatorname{proj}^f_{F(T)}(x_0)$ *is the Bregman projection of* C *onto* $F(T)$.

Proof. By Proposition 2.17, it follows that $F(T)$ is a nonempty closed and convex subset of E. It is easy to see that C_n is closed and Q_n is closed and convex for all $n \geq 0$. For any $z \in C_n$, $n \geq 1$,

$$\begin{aligned}
& D_f(z, y_n) \leq (1-\alpha_n\beta_n) D_f(z, x_n + e_n) + \alpha_n\beta_n D_f(z, x_0) \\
& \Longleftrightarrow f(z) - f(y_n) - \left\langle \nabla f(y_n), z - y_n \right\rangle \\
& \qquad \leq (1-\alpha_n\beta_n)\left(f(z) - f(x_n + e_n) - \left\langle \nabla f(x_n + e_n), z - x_n - e_n \right\rangle\right) \\
& \qquad\quad + \alpha_n\beta_n\left(f(z) - f(x_0) - \left\langle \nabla f(x_0), z - x_0 \right\rangle\right) \\
& \Longleftrightarrow (1-\alpha_n\beta_n) f(x_n + e_n) + \alpha_n\beta_n f(x_0) - f(y_n) \\
& \qquad \leq \left\langle \nabla f(y_n), z - y_n \right\rangle - (1-\alpha_n\beta_n)\left\langle \nabla f(x_n + e_n), z - x_n - e_n \right\rangle - \alpha_n\beta_n\left\langle \nabla f(x_0), z - x_0 \right\rangle,
\end{aligned} \tag{3.29}$$

which implies that C_n is closed and convex for all $n \geq 1$. Since, for any $z \in C_0$,

$$\begin{aligned}
&D_f(z, y_0) \leq D_f(z, x_0) \\
&\Longleftrightarrow f(z) - f(y_0) - \langle \nabla f(y_0), z - y_0 \rangle \leq f(z) - f(x_0) - \langle \nabla f(x_0), z - x_0 \rangle \\
&\Longleftrightarrow \langle \nabla f(x_0), z - x_0 \rangle - \langle \nabla f(y_0), z - y_0 \rangle \leq f(y_0) - f(x_0) \\
&\Longleftrightarrow \langle \nabla f(x_0), z - x_0 \rangle - \langle \nabla f(y_0), z - x_0 + x_0 - y_0 \rangle \leq f(y_0) - f(x_0) \\
&\Longleftrightarrow \langle \nabla f(x_0) - \nabla f(y_0), z - x_0 \rangle + D_f(x_0, y_0) \leq 0,
\end{aligned} \tag{3.30}$$

which shows that C_0 is closed and convex. As a consequence, C_n is closed and convex for all $n \geq 0$. Taking $p \in F(T)$ arbitrarily, by Lemma 2.18,

$$\begin{aligned}
D_f(p, y_n) &= D_f\left(p, \nabla f^*(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(x_n + e_n))\right) \\
&\leq \alpha_n D_f(p, z_n) + (1-\alpha_n) D_f(p, x_n + e_n) \\
&= (1-\alpha_n) D_f(p, x_n + e_n) + \alpha_n D_f\left(p, \nabla f^*\left(\beta_n \nabla f(x_0) + (1-\beta_n)\nabla f(T(x_n + e_n))\right)\right) \\
&\leq (1-\alpha_n) D_f(p, x_n + e_n) + \alpha_n \beta_n D_f(p, x_0) + \alpha_n(1-\beta_n) D_f(p, T(x_n + e_n)) \\
&\leq (1-\alpha_n) D_f(p, x_n + e_n) + \alpha_n \beta_n D_f(p, x_0) + \alpha_n(1-\beta_n) D_f(p, x_n + e_n) \\
&= (1-\alpha_n\beta_n) D_f(p, x_n + e_n) + \alpha_n \beta_n D_f(p, x_0),
\end{aligned} \tag{3.31}$$

that is, $p \in C_n$, and so, $F(T) \subset C_n$ for all $n \geq 0$. As in the proof of Theorem 3.1, we get $F(T) \subset Q_n$ for all $n \geq 0$, $\{x_n\}$ is a Cauchy sequence, $\{x_n\}, \{y_n\}$, and $\{z_n\}$ are also bounded, and thus,

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n + e_n) = 0, \qquad \lim_{n\to\infty} \|(x_{n+1} + e_{n+1}) - (x_n + e_n)\| = 0, \tag{3.32}$$

$$\lim_{n\to\infty} \|x_{n+1} - (x_n + e_n)\| = 0. \tag{3.33}$$

Consequently, $F(T) \subset C_n \cap Q_n$ and $C_n \cap Q_n$ is nonempty closed and convex for all $n \geq 0$. Moreover, $\{x_n\}$ is well defined. Set $x_n \to \overline{\omega} \in C$.

Secondly, we show that $\{x_n\}$ converges strongly to a point of $F(T)$. Since f is uniformly Fréchet differentiable on bounded subsets of E, from Lemma 2.12, ∇f is norm-to-norm uniformly continuous on bounded subsets of E. So, by (3.33),

$$\lim_{n\to\infty} \left\| \nabla f(x_{n+1}) - \nabla f(x_n + e_n) \right\| = 0. \tag{3.34}$$

In view of $x_{n+1} = \mathrm{proj}^f_{(C_n \cap Q_n)}(x_0) \in C_n$, we have

$$D_f(x_{n+1}, y_n) \leq (1-\alpha_n\beta_n) D_f(x_{n+1}, x_n + e_n) + \alpha_n\beta_n D_f(x_{n+1}, x_0). \tag{3.35}$$

Due to $\lim_{n\to\infty}\beta_n = 0$, from (3.32), one has

$$\lim_{n\to\infty} D_f\left(x_{n+1}, y_n\right) = 0. \tag{3.36}$$

Therefore, $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$. Moreover, one has

$$\lim_{n\to\infty}\left\|\nabla f(x_{n+1}) - \nabla f\left(y_n\right)\right\| = 0. \tag{3.37}$$

Since $\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$, by (3.32) and (3.33),

$$\lim_{n\to\infty}\left\|x_n - y_n\right\| = 0, \tag{3.38}$$

and thus, $y_n \to \overline{\omega}$ as $n \to \infty$. Noticing that

$$\begin{aligned}
&\left\|\nabla f(x_{n+1}) - \nabla f\left(y_n\right)\right\| \\
&\quad = \left\|\nabla f(x_{n+1}) - \left(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(x_n + e_n)\right)\right\| \\
&\quad \ge \alpha_n\left\|\nabla f(x_{n+1}) - \nabla f(z_n)\right\| - (1-\alpha_n)\left\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\right\| \\
&\quad = \alpha_n\left\|\nabla f(x_{n+1}) - \left(\beta_n \nabla f(x_0) + \left(1-\beta_n\right)\nabla f(T_n(x_n + e_n))\right)\right\| \\
&\qquad - (1-\alpha_n)\left\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\right\| \\
&\quad \ge \alpha_n(1-\beta_n)\left\|\nabla f(x_{n+1}) - \nabla f(T_n(x_n + e_n))\right\| - \alpha_n\beta_n\left\|\nabla f(x_{n+1}) - \nabla f(x_0)\right\| \\
&\qquad - (1-\alpha_n)\left\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\right\|.
\end{aligned} \tag{3.39}$$

That is,

$$\begin{aligned}
&\alpha_n(1-\beta_n)\|\nabla f(x_{n+1}) - \nabla f(T(x_n + e_n))\| \\
&\quad \le \left\|\nabla f(x_{n+1}) - \nabla f\left(y_n\right)\right\| + \alpha_n\beta_n\left\|\nabla f(x_{n+1}) - \nabla f(x_0)\right\| \\
&\qquad + (1-\alpha_n)\left\|\nabla f(x_{n+1}) - \nabla f(x_n + e_n)\right\|.
\end{aligned} \tag{3.40}$$

Together with $\liminf_{n\to\infty}\alpha_n > 0$, $\lim_{n\to\infty}\beta_n = 0$, and (3.37), this yields that

$$\lim_{n\to\infty}\left\|\nabla f(x_{n+1}) - \nabla f(T(x_n + e_n))\right\| = 0. \tag{3.41}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, from Lemma 2.12, ∇f is norm-to-norm uniformly continuous on bounded subsets of E and so is ∇f^*. Then, by (3.41), we get

$$\lim_{n\to\infty}\|x_{n+1} - T(x_n + e_n)\| = 0. \tag{3.42}$$

From $\|(x_n+e_n)-T(x_n+e_n)\| \leq \|(x_n+e_n)-(x_{n+1}+e_{n+1})\|+\|(x_{n+1}+e_{n+1})-T(x_n+e_n)\|$, it follows that $\lim_{n\to\infty}\|(x_n+e_n)-T(x_{n+1}+e_{n+1})\| = 0$. So, $\overline{\omega} \in \widetilde{F}(T) = F(T)$. By the same argument of Theorem 3.1, we know that $\{x_n\}$ and $\{y_n\}$ converge strongly to $\text{proj}^f_{F(T)}(x_0)$. This completes the proof. □

If $\alpha_n \equiv 1$ and $e_n \equiv 0$ for all $n \geq 0$, then from Theorem 3.5, we have the following result.

Corollary 3.6. *Let C be a nonempty closed convex subset of a real reflexive Banach space E and $f : E \to R$ a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subset of E, and let T be a weak Bregman relatively nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$
\begin{gathered}
x_0 \in C, \quad Q_0 = C, \\
y_n = \nabla f^*\left(\beta_n \nabla f(x_0) + (1-\beta_n)\nabla f(Tx_n)\right), \\
C_n = \left\{z \in C_{n-1} \cap Q_{n-1} : D_f(z, y_n) \leq (1-\beta_n)D_f(z, x_n) + \beta_n D_f(z, x_0)\right\}, \\
C_0 = \left\{z \in C : D_f(z, y_0) \leq D_f(z, x_0)\right\}, \\
Q_n = \left\{z \in C_{n-1} \cap Q_{n-1} : \left\langle \nabla f(x_0) - \nabla f(x_n), z - x_n \right\rangle \leq 0\right\}, \\
x_{n+1} = \text{proj}^f_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{gathered}
\tag{3.43}
$$

where $\{\beta_n\} \subset [0,1]$ such that $\lim_{n\to\infty}\beta_n = 0$ and $\{e_n\}$ is an error sequence in E with $e_n \to 0$ as $n \to \infty$. Then, the sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to the point $\text{proj}^f_{F(T)}(x_0)$, where $\text{proj}^f_{F(T)}(x_0)$ is the Bregman projection of C onto $F(T)$.

Now, we develop a strong convergence theorem for a Bregman relatively nonexpansive mapping.

Theorem 3.7. *Let C be a nonempty closed convex subset of a real reflexive Banach space E and $f : E \to R$ a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subset of E, and let $T : C \to C$ be a Bregman relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$
\begin{gathered}
x_0 \in C, \quad Q_0 = C, \\
z_n = \nabla f^*\left(\beta_n \nabla f(x_0) + (1-\beta_n)\nabla f(T(x_n+e_n))\right), \\
y_n = \nabla f^*\left(\alpha_n \nabla f(z_n) + (1-\alpha_n)\nabla f(x_n+e_n)\right), \\
C_n = \left\{z \in C_{n-1} \cap Q_{n-1} : D_f(z, y_n) \leq (1-\alpha_n\beta_n)D_f(z, x_n+e_n) + \alpha_n\beta_n D_f(z, x_0)\right\}, \\
C_0 = \left\{z \in C : D_f(z, y_0) \leq D_f(z, x_0)\right\}, \\
Q_n = \left\{z \in C_{n-1} \cap Q_{n-1} : \left\langle \nabla f(x_0) - \nabla f(x_n), z - x_n \right\rangle \leq 0\right\}, \\
x_{n+1} = \text{proj}^f_{C_n \cap Q_n} x_0, \quad \forall n \geq 0,
\end{gathered}
\tag{3.44}
$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *such that* $\liminf_{n\to\infty}\alpha_n > 0$ *and* $\lim_{n\to\infty}\beta_n = 0$ *and* $\{e_n\}$ *is an error sequence in* E *with* $e_n \to 0$ *as* $n \to \infty$. *Then, the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converges strongly to the point* $\mathrm{proj}^f_{F(T)}(x_0)$, *where* $\mathrm{proj}^f_{F(T)}(x_0)$ *is the Bregman projection of* C *onto* $F(T)$.

Proof. The proof is similar to Theorem 3.5 and so is omitted. This completes the proof. □

If $\alpha_n \equiv 1$ and $e_n \equiv 0$ for all $n \geq 0$, then from Theorem 3.7, we get the following corollary.

Corollary 3.8. *Let* E *be a real reflexive Banach space and* $f : E \to R$ *a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subset of* E, *and let* $T : E \to E$ *be a Bregman relatively nonexpansive mapping such that* $F(T) \neq \emptyset$. *Assume that* $\{\beta_n\}$ *is a real sequence in* $[0,1]$ *such that* $\lim_{n\to\infty}\beta_n = 0$. *Define a sequence* $\{x_n\}$ *by the following algorithm:*

$$\begin{gathered}
x_0 \in C, \qquad Q_0 = C, \\
y_n = \nabla f^*\left(\beta_n \nabla f(x_0) + (1-\beta_n)\nabla f(Tx_n)\right), \\
C_n = \left\{z \in C_{n-1} \cap Q_{n-1} : D_f(z, y_n) \leq (1-\alpha_n\beta_n)D_f(z, x_n) + \alpha_n\beta_n D_f(z, x_0)\right\}, \\
C_0 = \left\{z \in C : D_f(z, y_0) \leq D_f(z, x_0)\right\}, \\
Q_n = \left\{z \in C_{n-1} \cap Q_{n-1} : \left\langle \nabla f(x_0) - \nabla f(x_n), z - x_n \right\rangle \leq 0\right\}, \\
x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n} x_0, \quad \forall n \geq 0.
\end{gathered} \tag{3.45}$$

Then, the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the point* $\mathrm{proj}^f_{F(T)}(x_0)$, *where* $\mathrm{proj}^f_{F(T)}(x_0)$ *is the Bregman projection of* C *onto* $F(T)$.

In [30], Qin and Su obtained the following.

Theorem QS (see [30, Theorem 2.2]). *Let* C *be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space* E, *and let* $T : C \to C$ *be a relatively nonexpansive mapping such that* $F(T) \neq \emptyset$. *Assume that* $\{\beta_n\}$ *is a real sequence in* $[0,1)$ *such that* $\lim_{n\to\infty}\beta_n = 0$. *Then, the sequence* $\{x_n\}$ *generated by* (1.5) *converges strongly to* $\Pi_{F(T)}x_0$, *where* $\Pi_{F(T)}$ *is the generalized projection (see, e.g., [2, 3]) from* E *onto* $F(T)$.

Remark 3.9. Corollary 3.8 extends Theorems QS [30] from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces.

4. Conclusions

In this paper, we introduce a conception of weak Bregman relatively nonexpansive mapping in reflexive Banach space and give an example to illustrate the existence of weak Bregman relatively nonexpansive mapping and the difference between weak Bregman relatively nonexpansive mapping and Bregman relatively nonexpansive mapping which enlarge the Bregman operator theory. Secondly, by using projection techniques, we construct several modification of Mann-type iterative algorithms with errors and Halpern-type iterative algorithms with errors to find fixed points of weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings in Banach spaces. Thirdly, strong convergence theorems for weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings are derived under some suitable assumptions. By further research, on the one hand, we may apply our algorithms to find zeros of finite families of maximal monotone operators, solutions of system of convex minization problems, solutions of system of variational inequalities, equilibrium, and equation operators (see, e.g., [24]). On the other hand, one may give some numerical experiments to verify the theoretical assertions and show how to compute the generalized projections. These topics will be done in the future.

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