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Research Article

Multiplication Operators between Lipschitz-Type Spaces on a Tree

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Let $\mathcal{L}$ be the space of complex-valued functions f on the set of vertices T of an infinite tree rooted at o such that the difference of the values of f at neighboring vertices remains bounded throughout the tree, and let $\mathcal{L}_{\mathbf{w}}$ be the set of functions $f \in \mathcal{L}$ such that $|f(v) - f(v^-)| = O(|v|^{-1})$, where $|v|$ is the distance between o and v and v^- is the neighbor of v closest to o. In this paper, we characterize the bounded and the compact multiplication operators between $\mathcal{L}$ and $\mathcal{L}_{\mathbf{w}}$ and provide operator norm and essential norm estimates. Furthermore, we characterize the bounded and compact multiplication operators between $\mathcal{L}_{\mathbf{w}}$ and the space L^∞ of bounded functions on T and determine their operator norm and their essential norm. We establish that there are no isometries among the multiplication operators between these spaces.

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Banach spaces of functions defined on a set Ω. For a complex-valued function ψ defined on Ω, the *multiplication operator with symbol ψ from $\mathcal{X}$ to $\mathcal{Y}$* is defined as

$$M_\psi f = \psi f, \quad \forall f \in \mathcal{X}. \tag{1.1}$$

A fundamental objective in the study of the operators with symbol is to tie the properties of the operator to the function theoretic properties of the symbol.

When Ω is taken to be the open unit disk $\mathbb{D}$ in the complex plane, an important space of functions to study is the *Bloch space*, defined as the set $\mathcal{B}$ of analytic functions $f : \mathbb{D} \to \mathbb{C}$ for which

$$\beta_f = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) \left|f'(z)\right| < \infty. \tag{1.2}$$

The Bloch space can also be described as the set consisting of the Lipschitz functions between metric spaces from $\mathbb{D}$ endowed with the Poincaré distance ρ to $\mathbb{C}$ endowed with the Euclidean distance, a fact that was proved by the second author in [1] (see also [2]). In fact, $f \in \mathcal{B}$ if and only if there exist $\beta > 0$ such that for all $z, w \in \mathbb{D}$

$$\begin{aligned} \left|f(z) - f(w)\right| &\leq \beta\rho(z,w), \\ \beta_f &= \sup_{z\neq w} \frac{\left|f(z) - f(w)\right|}{\rho(z,w)}. \end{aligned} \tag{1.3}$$

More recently, considerable research has been carried out in the field of operator theory when the set Ω is taken to be a discrete structure, such as a discrete group or a graph. In this paper, we consider the case when Ω is taken to be an infinite tree.

By a *tree* T we mean a locally finite, connected, and simplyconnected graph, which, as a set, we identify with the collection of its vertices. Two vertices u and v are called *neighbors* if there is an edge connecting them, and we use the notation $u \sim v$. A vertex is called *terminal* if it has a unique neighbor. A *path* is a finite or infinite sequence of vertices $[v_0, v_1, \ldots]$ such that $v_k \sim v_{k+1}$ and $v_{k-1} \neq v_{k+1}$, for all k.

Given a tree T rooted at o and a vertex $u \in T$, a vertex v is called a *descendant* of u if u lies in the unique path from o to v. The vertex u is then called an *ancestor* of v. Given a vertex $v \neq o$, we denote by v^- the unique neighbor which is an ancestor of v. For $v \in T$, the set S_v consisting of v and all its descendants is called the *sector* determined by v.

Define the *length* of a finite path $[u = u_0, u_1, \ldots, v = u_n]$ (with $u_k \sim u_{k+1}$ for $k = 0, \ldots, n$) to be the number n of edges connecting u to v. The *distance*, $d(u, v)$, between vertices u and v is the length of the path connecting u to v. The tree T is a metric space under the distance d. Fixing o as the root of the tree, we define the *length* of a vertex v by $|v| = d(o, v)$. By a *function on a tree* we mean a complex-valued function on the set of its vertices.

In this paper, the tree will be assumed to be rooted at a vertex o and without terminal vertices (and hence infinite).

Infinite trees are discrete structures which exhibit significant geometric and potential theoretic characteristics that are present in the Poincaré disk $\mathbb{D}$. For instance, they have a boundary, which is defined as the set of equivalence classes of paths which differ by finitely many vertices. The union of the boundary with the tree yields a compact space. A useful resource for the potential theory on trees illustrating the commonalities with the disk is [3]. In [4] it was shown that, if the tree has the property that all its vertices have the same number of neighbors, then there is a natural embedding of the tree in the unit disk such that the edges of the tree are arcs of geodesics in $\mathbb{D}$ with the same hyperbolic length and the set of cluster points of the vertices is the entire unit circle.

In [5], the last two authors defined the *Lipschitz space* $\mathcal{L}$ on a tree T as the set consisting of the functions $f : T \to \mathbb{C}$ which are Lipschitz with respect to the distance d on T and the Euclidean distance on $\mathbb{C}$. For this reason, the Lipschitz space $\mathcal{L}$ can be viewed as a discrete

analogue of the Bloch space $\mathcal{B}$. It was also shown that the Lipschitz functions on T are precisely the functions for which

$$\|Df\|_\infty = \sup_{v\in T^*} Df(v) < \infty, \tag{1.4}$$

where $Df(v) = |f(v) - f(v^-)|$ and $T^* = T \setminus \{o\}$. Under the norm

$$\|f\|_{\mathcal{L}} = |f(o)| + \|Df\|_\infty, \tag{1.5}$$

$\mathcal{L}$ is a Banach space containing the space L^∞ of the bounded functions on T. Furthermore, for $f \in L^\infty$, $\|f\|_{\mathcal{L}} \le 2\|f\|_\infty$.

The *little Lipschitz space* is defined as

$$\mathcal{L}_0 = \left\{ f \in \mathcal{L} : \lim_{|v|\to\infty} Df(v) = 0 \right\} \tag{1.6}$$

and was proven to be a separable closed subspace of $\mathcal{L}$. We state the following results that will be useful in the present paper.

Lemma 1.1 (see [5, Lemma 3.4]). (a) *If $f \in \mathcal{L}$ and $v \in T$, then*

$$|f(v)| \le |f(o)| + |v|\|Df\|_\infty. \tag{1.7}$$

In particular, if $\|f\|_{\mathcal{L}} \le 1$, then $|f(v)| \le |v|$ for each $v \in T^*$.

(b) *If $f \in \mathcal{L}_0$, then*

$$\lim_{|v|\to\infty} \frac{f(v)}{|v|} = 0. \tag{1.8}$$

Lemma 1.2 (see [5, Proposition 2.4]). *Let $\{f_n\}$ be a sequence of functions in $\mathcal{L}_0$ converging to 0 pointwise in T such that $\{\|f_n\|_{\mathcal{L}}\}$ is bounded. Then $f_n \to 0$ weakly in $\mathcal{L}_0$.*

In [6], we introduced the *weighted Lipschitz space* on a tree T as the set $\mathcal{L}_{\mathbf{w}}$ of the functions $f : T \to \mathbb{C}$ such that

$$\sup_{v\in T^*} |v| Df(v) < \infty. \tag{1.9}$$

The interest in this space is due to its connection to the bounded multiplication operators on $\mathcal{L}$. Specifically, it was shown in [5] that the bounded multiplication operators on $\mathcal{L}$ are precisely those operators M_ψ whose symbol ψ is a bounded function in $\mathcal{L}_{\mathbf{w}}$. The space $\mathcal{L}_{\mathbf{w}}$ was shown to be a Banach space under the norm

$$\|f\|_{\mathbf{w}} = |f(o)| + \sup_{v\in T^*} |v| Df(v). \tag{1.10}$$

The *little weighted Lipschitz space* was defined as

$$\mathcal{L}_{\mathbf{w},0} = \left\{ f \in \mathcal{L}_{\mathbf{w}} : \lim_{|v|\to\infty} |v| Df(v) = 0 \right\} \tag{1.11}$$

and was shown to be a closed separable subspace of $\mathcal{L}_{\mathbf{w}}$.

In this paper, we will make repeated use of the following results proved in [6].

Lemma 1.3 (see [6, Propositions 2.1 and 2.6]). (a) *If $f \in \mathcal{L}_{\mathbf{w}}$, and $v \in T^*$, then*

$$\left|f(v)\right| \leq (1 + \log|v|)\left\|f\right\|_{\mathbf{w}}. \tag{1.12}$$

(b) *If $f \in \mathcal{L}_{\mathbf{w},0}$, then*

$$\lim_{|v|\to\infty} \frac{f(v)}{\log|v|} = 0. \tag{1.13}$$

Lemma 1.4 (see [6, Proposition 2.7]). *Let $\{f_n\}$ be a sequence of functions in $\mathcal{L}_{\mathbf{w},0}$ converging to 0 pointwise in T such that $\{\|f_n\|_{\mathbf{w}}\}$ is bounded. Then $f_n \to 0$ weakly in $\mathcal{L}_{\mathbf{w},0}$.*

In this paper, we consider the multiplication operators between $\mathcal{L}$ and $\mathcal{L}_{\mathbf{w}}$, as well as between $\mathcal{L}_{\mathbf{w}}$ and L^∞. The multiplication operators between $\mathcal{L}$ and L^∞ were studied by the last two authors in [7].

1.1. Organization of the Paper

In Sections 2 and 3, we study the multiplication operators between $\mathcal{L}_{\mathbf{w}}$ and $\mathcal{L}$. We characterize the bounded and the compact operators and give estimates on their operator norm and their essential norm. We also prove that no isometric multiplication operators exist between the respective spaces.

In Section 4, we characterize the bounded operators and the compact operators from $\mathcal{L}_{\mathbf{w}}$ to L^∞ and determine their operator norm and their essential norm. As was the case in Sections 2 and 3, we show that no isometries exist amongst such operators. In addition, we characterize the multiplication operators that are bounded from below.

Finally, in Section 5, we characterize the bounded and the compact multiplication operators from L^∞ to $\mathcal{L}_{\mathbf{w}}$. We also determine their operator norm and their essential norm. As with all the other cases, we show that there are no isometries amongst such operators.

2. Multiplication Operators from $\mathcal{L}_{\mathbf{w}}$ to $\mathcal{L}$

We begin the section with the study of the bounded multiplication operators $M_\varphi : \mathcal{L}_{\mathbf{w}} \to \mathcal{L}$ and $M_\varphi : \mathcal{L}_{\mathbf{w},0} \to \mathcal{L}_0$.

2.1. Boundedness and Operator Norm Estimates

Let ψ be a function on the tree T. Define

$$\begin{aligned} \tau_\psi &= \sup_{v\in T^*} D\psi(v)\log(1+|v|), \\ \sigma_\psi &= \sup_{v\in T} \frac{|\psi(v)|}{|v|+1}. \end{aligned} \tag{2.1}$$

In the following theorem, we give a boundedness criterion in terms of the quantities τ_ψ and σ_ψ.

Theorem 2.1. *For a function ψ on T, the following statements are equivalent:*

(a) $M_\psi : \mathcal{L}_{\mathbf{w}} \to \mathcal{L}$ *is bounded.*

(b) $M_\psi : \mathcal{L}_{\mathbf{w},0} \to \mathcal{L}_0$ *is bounded.*

(c) τ_ψ *and* σ_ψ *are finite.*

Furthermore, under these conditions, we have

$$\max\{\tau_\psi, \sigma_\psi\} \le \|M_\psi\| \le \tau_\psi + \sigma_\psi. \tag{2.2}$$

Proof. (a)⇒(c) Assume $M_\psi : \mathcal{L}_{\mathbf{w}} \to \mathcal{L}$ is bounded. Applying M_ψ to the constant function 1, we have $\psi \in \mathcal{L}$, so that, by Lemma 1.1, we have $\sigma_\psi < \infty$. Next, consider the function f on T defined by $f(v) = \log(1+|v|)$. Then $f(o) = 0$; for $v \neq o$, a straightforward calculation shows that

$$|v|Df(v) = |v|\big(\log(1+|v|) - \log|v|\big) \le 1 \tag{2.3}$$

and $\lim_{|v|\to\infty} |v|Df(v) = 1$. Thus, $\|f\|_{\mathbf{w}} = 1$ and so $\|M_\psi f\|_{\mathcal{L}} \le \|M_\psi\|$. Therefore, for $v \in T^*$, noting that

$$D(\psi f)(v) = D\psi(v) f(v) + \psi(v^-) Df(v), \tag{2.4}$$

one has

$$\begin{aligned} D\psi(v)\big|f(v)\big| &\le D(\psi f)(v) + \big|\psi(v^-)\big| Df(v) \\ &\le \big\|M_\psi f\big\|_{\mathcal{L}} + \sigma_\psi |v| Df(v) \le \big\|M_\psi\big\| + \sigma_\psi. \end{aligned} \tag{2.5}$$

Hence $\tau_\psi < \infty$.

(c)⇒(a) Assume τ_ψ and σ_ψ are finite. Then, by Lemma 1.3, for $f \in \mathcal{L}_{\mathbf{w}}$ and $v \in T^*$, we have

$$\begin{aligned} D(\psi f)(v) &\le D\psi(v)\big|f(v)\big| + \big|\psi(v^-)\big| Df(v) \\ &\le D\psi(v)\big(1+\log|v|\big)\big\|f\big\|_{\mathbf{w}} + |v|\sigma_\psi Df(v) \\ &\le \tau_\psi \big\|f\big\|_{\mathbf{w}} + \sigma_\psi\big(\big\|f\big\|_{\mathbf{w}} - \big|f(o)\big|\big). \end{aligned} \tag{2.6}$$

Since $|\psi(o)| \le \sigma_\psi$, we obtain

$$\begin{aligned}\|M_\psi f\|_{\mathcal{L}} &\le |\psi(o)||f(o)| + \tau_\psi \|f\|_{\mathbf{w}} + \sigma_\psi(\|f\|_{\mathbf{w}} - |f(o)|)\\ &= (\tau_\psi + \sigma_\psi)\|f\|_{\mathbf{w}} + (|\psi(o)| - \sigma_\psi)|f(o)|\\ &\le (\tau_\psi + \sigma_\psi)\|f\|_{\mathbf{w}},\end{aligned} \tag{2.7}$$

proving the boundedness of $M_\psi : \mathcal{L}_{\mathbf{w}} \to \mathcal{L}$ and the upper estimate.

(b)⇒(c) Suppose $M_\psi : \mathcal{L}_{\mathbf{w},0} \to \mathcal{L}_0$ is bounded. The finiteness of σ_ψ follows again from the fact that $\psi = M_\psi 1 \in \mathcal{L}_0$ and from Lemma 1.1. To prove that $\tau_\psi < \infty$, let $0 < \alpha < 1$ and, for $v \in T$, define $f_\alpha(v) = (\log(1 + |v|))^\alpha$. Then $f_\alpha(o) = 0$ and $|v|Df_\alpha(v) \to 0$ as $|v| \to \infty$; so $f_\alpha \in \mathcal{L}_{\mathbf{w},0}$. Since for $0 < \alpha < 1$, the function $x \mapsto x - x^\alpha$ is increasing for $x \ge 1$, the function $Df_\alpha(v)$ is increasing in α, and $Df_\alpha(v) \le Df(v)$ for $v \in T^*$, where $f(v) = \log(1+|v|)$, for $v \in T$. Thus, $\|f_\alpha\|_{\mathbf{w}} \le \|f\|_{\mathbf{w}} = 1$. Therefore, for $v \in T^*$, we have

$$\begin{aligned}D\psi(v)|f_\alpha(v)| &\le D(\psi f_\alpha)(v) + |\psi(v^-)|Df_\alpha(v)\\ &\le \|M_\psi f_\alpha\| + \sigma_\psi |v| Df_\alpha(v) \le \|M_\psi\| + \sigma_\psi.\end{aligned} \tag{2.8}$$

Letting $\alpha \to 1$, we obtain

$$D\psi(v)\log(1 + |v|) \le \|M_\psi\| + \sigma_\psi. \tag{2.9}$$

Hence $\tau_\psi < \infty$.

(c)⇒(b) Assume σ_ψ and τ_ψ are finite, and let $f \in \mathcal{L}_{\mathbf{w},0}$. Then, by Lemma 1.3, for $v \in T^*$, we have

$$\begin{aligned}D(\psi f)(v) &\le D\psi(v)|f(v)| + |\psi(v^-)|Df(v)\\ &\le D\psi(v)\log(1 + |v|)\frac{|f(v)|}{\log(1 + |v|)} + \frac{|\psi(v^-)|}{|v|}|v|Df(v)\\ &\le \tau_\psi \frac{|f(v)|}{\log(1 + |v|)} + \sigma_\psi |v| Df(v) \longrightarrow 0\end{aligned} \tag{2.10}$$

as $|v| \to \infty$. Thus, $\psi f \in \mathcal{L}_0$. The boundedness of M_ψ and the estimate $\|M_\psi f\|_{\mathcal{L}} \le \tau_\psi + \sigma_\psi$ can be shown as in the proof of (c)⇒(a).

Finally we show that, under boundedness assumptions on M_ψ, $\|M_\psi\| \ge \max\{\tau_\psi, \sigma_\psi\}$. For $v \in T^*$, let $f_v = 1/(|v| + 1)\chi_v$, where χ_v denotes the characteristic function of $\{v\}$. Then $\|f_v\|_{\mathbf{w}} = 1$ and

$$\|\psi f_v\|_{\mathcal{L}} = \frac{|\psi(v)|}{|v| + 1}. \tag{2.11}$$

Furthermore, letting $f_o = (1/2)\chi_o$, we see that $\|f_o\|_{\mathbf{w}} = 1$ and $\|\psi f_o\|_{\mathcal{L}} = |\psi(o)|$. Therefore, we deduce that $\|M_\psi\| \ge \sigma_\psi$.

Next, fix $v \in T^*$ and for $w \in T$, define

$$g_v(w) = \begin{cases} \log(1+|w|) & \text{if } |w| < |v|, \\ \log(1+|v|) & \text{if } |w| \geq |v|. \end{cases} \tag{2.12}$$

Then, $g_v \in \mathcal{L}_{\mathbf{w}}$ and

$$\lim_{|v|\to\infty} \|g_v\|_{\mathbf{w}} = \lim_{|v|\to\infty} |v|\left[\log(1+|v|) - \log|v|\right] = 1. \tag{2.13}$$

Observe that, for $w \in T^*$, we have

$$D(\psi g_v)(w) = \begin{cases} \left|\psi(w)\log(1+|w|) - \psi(w^-)\log|w|\right| & \text{if } |w| < |v|, \\ D\psi(w)\log(1+|v|) & \text{if } |w| \geq |v|. \end{cases} \tag{2.14}$$

Hence

$$\sup_{w\in T^*} D(\psi g_v)(w) \geq \sup_{|w|\geq|v|} D\psi(w)\log(1+|v|) \geq D\psi(v)\log(1+|v|). \tag{2.15}$$

Define $f_v = g_v/\|g_v\|_{\mathbf{w}}$. Then $\|f_v\|_{\mathbf{w}} = 1$ and

$$\|M_\psi\| \geq \|M_\psi f_v\|_{\mathcal{L}} = \frac{\|D(\psi g_v)\|_\infty}{\|g_v\|_{\mathbf{w}}} \geq \frac{D\psi(v)\log(1+|v|)}{\|g_v\|_{\mathbf{w}}}. \tag{2.16}$$

Taking the limit as $|v| \to \infty$, we obtain $\|M_\psi\| \geq \tau_\psi$. Therefore, $\|M_\psi\| \geq \max\{\tau_\psi, \sigma_\psi\}$. □

2.2. Isometries

In this section, we show there are no isometric multiplication operators M_ψ from the spaces $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ to the spaces $\mathcal{L}$ or $\mathcal{L}_0$, respectively.

Assume $M_\psi : \mathcal{L}_{\mathbf{w}} \to \mathcal{L}$ is an isometry. Then $\|\psi\|_{\mathcal{L}} = \|M_\psi 1\|_{\mathcal{L}} = 1$. On the other hand, $|\psi(o)| = (1/2)\|M_\psi \chi_o\|_{\mathcal{L}} = (1/2)\|\chi_o\|_{\mathbf{w}} = 1$. Thus $\sup_{v\in T^*} D\psi(v) = \|\psi\|_{\mathcal{L}} - |\psi(o)| = 0$, which implies that ψ is a constant of modulus 1. Yet, for $v \in T^*$, letting $f_v = (1/(|v|+1))\chi_v$, we see that

$$1 = \|f_v\|_{\mathbf{w}} = \|M_\psi f_v\|_{\mathcal{L}} = \frac{1}{|v|+1}, \tag{2.17}$$

which yields a contradiction. Therefore, we obtain the following result.

Theorem 2.2. *There are no isometries M_ψ from $\mathcal{L}_{\mathbf{w}}$ to $\mathcal{L}$ or $\mathcal{L}_{\mathbf{w},0}$ to $\mathcal{L}_0$, respectively.*

2.3. Compactness and Essential Norm Estimates

In this section, we characterize the compact multiplication operators. As with many classical spaces, the characterization of the compact operators is a "little-oh" condition corresponding the the "big-oh" condition for boundedness. We first collect some useful results about compact operators from $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ to $\mathcal{L}$.

Lemma 2.3. *A bounded multiplication operator M_ψ from $\mathcal{L}_{\mathbf{w}}$ to $\mathcal{L}$ is compact if and only if for every bounded sequence $\{f_n\}$ in $\mathcal{L}_{\mathbf{w}}$ converging to 0 pointwise, the sequence $\{\|\psi f_n\|_{\mathcal{L}}\} \to 0$ as $n \to \infty$.*

Proof. Assume M_ψ is compact, and let $\{f_n\}$ be a bounded sequence in $\mathcal{L}_{\mathbf{w}}$ converging to 0 pointwise. Without loss of generality, we may assume $\|f_n\|_{\mathbf{w}} \leq 1$ for all $n \in \mathbb{N}$. Then the sequence $\{M_\psi f_n\} = \{\psi f_n\}$ has a subsequence $\{\psi f_{n_k}\}$ which converges in the $\mathcal{L}$-norm to some function $f \in \mathcal{L}$. Clearly $\psi(o) f_{n_k}(o) \to \psi(o) f(o)$, and by part (a) of Lemma 1.1, for $v \in T^*$, we have

$$\begin{aligned} \left|\psi(v) f_{n_k}(v) - f(v)\right| &\leq \left|\psi(o) f_{n_k}(o) - f(o)\right| + |v| \left\|D\left(\psi f_{n_k} - f\right)\right\|_\infty \\ &\leq (1 + |v|) \left\|\psi f_{n_k} - f\right\|_{\mathcal{L}}. \end{aligned} \tag{2.18}$$

Thus, $\psi f_{n_k} \to f$ pointwise on T. Since $f_n \to 0$ pointwise, it follows that f must be identically 0, which implies that $\|\psi f_{n_k}\|_{\mathcal{L}} \to 0$. With 0 being the only limit point of $\{\psi f_n\}$ in $\mathcal{L}$, it follows that $\|\psi f_n\|_{\mathcal{L}} \to 0$ as $n \to \infty$.

Conversely, assume every bounded sequence $\{f_n\}$ in $\mathcal{L}_{\mathbf{w}}$ converging to 0 pointwise has the property that $\|\psi f_n\|_{\mathcal{L}} \to 0$ as $n \to \infty$. Let $\{g_n\}$ be a sequence in $\mathcal{L}_{\mathbf{w}}$ with $\|g_n\|_{\mathbf{w}} \leq 1$ for all $n \in \mathbb{N}$. Then $|g_n(o)| \leq 1$ for all $n \in \mathbb{N}$, and by part (a) of Lemma 1.2, for $v \in T^*$, we obtain

$$\left|g_n(v)\right| \leq (1 + \log|v|) \left\|g_n\right\|_{\mathbf{w}} \leq 1 + \log|v|. \tag{2.19}$$

Thus, $\{g_n\}$ is uniformly bounded on finite subsets of T. So some subsequence $\{g_{n_k}\}$ converges pointwise to some function g. Fix $v \in T^*$ and $\varepsilon > 0$. Then for k sufficiently large, we have

$$\left|g(v) - g_{n_k}(v)\right| < \frac{\varepsilon}{2|v|}, \qquad \left|g_{n_k}(v^-) - g(v^-)\right| < \frac{\varepsilon}{2|v|}. \tag{2.20}$$

We deduce

$$\begin{aligned} |v| Dg(v) &\leq |v| \left|g(v) - g_{n_k}(v) + g_{n_k}(v^-) - g(v^-)\right| + |v| Dg_{n_k}(v) \\ &\leq |v| \left|g(v) - g_{n_k}(v)\right| + |v| \left|g_{n_k}(v^-) - g(v^-)\right| + |v| Dg_{n_k}(v) \\ &< \varepsilon + |v| Dg_{n_k}(v) \leq \varepsilon + 1, \end{aligned} \tag{2.21}$$

for all k sufficiently large. So $g \in \mathcal{L}_{\mathbf{w}}$. The sequence defined by $f_k = g_{n_k} - g$ is bounded in $\mathcal{L}_{\mathbf{w}}$ and converges to 0 pointwise. Thus by hypothesis, we obtain $\|\psi f_k\|_{\mathcal{L}} \to 0$ as $k \to \infty$. It follows that $M_\psi g_{n_k} = \psi g_{n_k} \to \psi g$ in the $\mathcal{L}$-norm, thus proving the compactness of M_ψ. □

By an analogous argument, we obtain the corresponding compactness criterion for M_ψ from $\mathcal{L}_{\mathbf{w},0}$ to $\mathcal{L}_0$.

Lemma 2.4. *A bounded multiplication operator M_ψ from $\mathcal{L}_{\mathbf{w},0}$ to $\mathcal{L}_0$ is compact if and only if for every bounded sequence $\{f_n\}$ in $\mathcal{L}_{\mathbf{w},0}$ converging to 0 pointwise, the sequence $\{\|\psi f_n\|_{\mathcal{L}}\} \to 0$ as $n \to \infty$.*

The following result is a variant of Lemma 1.3(a), which will be needed to prove a characterization of the compact multiplication operators from $\mathcal{L}_{\mathbf{w}}$ to $\mathcal{L}$ and from $\mathcal{L}_{\mathbf{w},0}$ to $\mathcal{L}_0$ (Theorem 2.6).

Lemma 2.5. *For $f \in \mathcal{L}_{\mathbf{w}}$ and $v \in T$*

$$\left|f(v)\right| \le \left|f(o)\right| + 2\log(1+|v|)s_{\mathbf{w}}(f), \tag{2.22}$$

where $s_{\mathbf{w}}(f) = \sup_{w\in T^*}|w|Df(w)$.

Proof. Fix $v \in T$ and argue by induction on $n = |v|$. For $n = 0$, inequality (2.22) is obvious. So assume $|v| = n > 0$ and $|f(u)| \le |f(o)| + 2\log(1+|u|)s_{\mathbf{w}}(f)$ for all vertices u such that $|u| < n$. Then

$$\begin{aligned}\left|f(v)\right| &\le \left|f(v) - f(v^-)\right| + \left|f(v^-)\right| \\ &\le \frac{1}{|v|}s_{\mathbf{w}}(f) + \left|f(o)\right| + 2\log|v|s_{\mathbf{w}}(f) \\ &= \left|f(o)\right| + \left(\frac{1}{|v|} + 2\log|v|\right)s_{\mathbf{w}}(f).\end{aligned} \tag{2.23}$$

Next, observe that $1/(|v|+1) \le \log((|v|+1)/|v|)$, so

$$\frac{1}{|v|} \le \frac{2}{|v|+1} \le 2\log\left(\frac{|v|+1}{|v|}\right). \tag{2.24}$$

Hence

$$\frac{1}{|v|} + 2\log|v| \le 2\log(|v|+1). \tag{2.25}$$

Inequality (2.22) now follows immediately from (2.23) and (2.25). □

Theorem 2.6. *Let M_ψ be a bounded multiplication operator from $\mathcal{L}_{\mathbf{w}}$ to $\mathcal{L}$ (or equivalently from $\mathcal{L}_{\mathbf{w},0}$ to $\mathcal{L}_0$). Then the following statements are equivalent:*

(a) *$M_\psi : \mathcal{L}_{\mathbf{w}} \to \mathcal{L}$ is compact.*

(b) *$M_\psi : \mathcal{L}_{\mathbf{w},0} \to \mathcal{L}_0$ is compact.*

(c) *$\lim_{|v|\to\infty}|\psi(v)|/(|v|+1) = 0$ and $\lim_{|v|\to\infty}D\psi(v)\log|v| = 0$.*

Proof. We first prove (a)⇒(c). Assume $M_\psi : \mathcal{L}_{\mathbf{w}} \to \mathcal{L}$ is compact. It suffices to show that, for any sequence $\{v_n\}$ in T such that $2 \le |v_n| \to \infty$, we have $\lim_{n\to\infty}|\psi(v_n)|/(|v_n|+1) = 0$ and $\lim_{n\to\infty} D\psi(v_n)\log|v_n| = 0$. Let $\{v_n\}$ be such a sequence, and for each $n \in \mathbb{N}$, define $f_n = (1/(|v_n|+1))\chi_{v_n}$. Then $f_n(o) = 0$, $f_n \to 0$ pointwise as $n \to \infty$, and $\|f_n\|_{\mathbf{w}} = 1$. By Lemma 2.3, it follows that $\|\psi f_n\|_{\mathcal{L}} \to 0$ as $n \to \infty$. Furthermore

$$\|\psi f_n\|_{\mathcal{L}} = \sup_{v\in T^*}\left|\psi(v)f_n(v) - \psi(v^-)f_n(v^-)\right| = \left|\psi(v_n)f_n(v_n)\right| = \frac{|\psi(v_n)|}{|v_n|+1}. \tag{2.26}$$

Thus $\lim_{n\to\infty}|\psi(v_n)|/(|v_n|+1) = 0$.

Next, for each $n \in \mathbb{N}$ and $v \in T$, define

$$g_n(v) = \begin{cases} 0 & \text{if } |v| < \sqrt{|v_n|}, \\ 2\log|v| - \log|v_n| & \text{if } \sqrt{|v_n|} \le |v| < |v_n| - 1, \\ \log|v_n| & \text{if } |v| \ge |v_n| - 1. \end{cases} \tag{2.27}$$

Then $Dg_n(v) = 0$ if $|v| \le \sqrt{|v_n|}$ or $|v| > |v_n| - 1$. In addition, if $\sqrt{|v_n|} < |v| \le |v_n| - 1$, then $|v|Dg_n(v) < 4$. Indeed, there are two possibilities. Either $\sqrt{|v_n|} \le |v| - 1$, in which case

$$|v|Dg_n(v) = 2|v|\big(\log|v| - \log(|v|-1)\big) \le \frac{2|v|}{|v|-1} \le 3, \tag{2.28}$$

or $|v| - 1 < \sqrt{|v_n|} < |v|$, in which case

$$\begin{aligned} |v|Dg_n(v) &= |v|\big(2\log|v| - \log|v_n|\big) \\ &\le \left(\sqrt{|v_n|}+1\right)\log\frac{\left(\sqrt{|v_n|}+1\right)^2}{|v_n|} \\ &\le \frac{2\left(\sqrt{|v_n|}+1\right)}{\sqrt{|v_n|}} \le 2\left(1+\frac{1}{\sqrt{2}}\right) < 4. \end{aligned} \tag{2.29}$$

Thus $\{\|g_n\|_{\mathbf{w}}\}$ is bounded, and $\{g_n\}$ converges to 0 pointwise. By Lemma 2.3, it follows that $\|\psi g_n\|_{\mathcal{L}} \to 0$ as $n \to \infty$. Moreover

$$\|\psi g_n\|_{\mathcal{L}} \ge \left|\psi(v_n)g_n(v_n) - \psi(v_n^-)g_n(v_n^-)\right| = D\psi(v_n)\log|v_n|. \tag{2.30}$$

Therefore $\lim_{n\to\infty} D\psi(v_n)\log|v_n| = 0$.

To prove the implication (c)⇒(a), suppose $\lim_{|v|\to\infty} D\psi(v)\log|v| = 0$ and $\lim_{|v|\to\infty}|\psi(v)|/(|v|+1) = 0$. Clearly, if ψ is identically 0, then M_ψ is compact. So assume $M_\psi : \mathcal{L}_{\mathbf{w}} \to \mathcal{L}$ is bounded with ψ not identically 0. By Lemma 2.3, it suffices to show that if

$\{f_n\}$ is bounded in $\mathcal{L}_{\mathbf{w}}$ converging to 0 pointwise, then $\|\psi f_n\|_{\mathcal{L}} \to 0$ as $n \to \infty$. Let $\{f_n\}$ be such a sequence, $s = \sup_{n\in\mathbb{N}} \|f_n\|_{\mathbf{w}}$, and fix $\varepsilon > 0$. Note that

$$\lim_{|v|\to\infty} D\psi(v)\log(1+|v|) = \lim_{|v|\to\infty} D\psi(v)\log|v|\frac{\log(1+|v|)}{\log|v|} = 0. \tag{2.31}$$

Thus there exists an $M \in \mathbb{N}$ such that

$$\left|f_n(o)\right| < \frac{\varepsilon}{3s\|\psi\|_{\mathcal{L}}}, \qquad D\psi(v)\log(1+|v|) < \frac{\varepsilon}{6s}, \qquad \frac{\left|\psi(v)\right|}{|v|+1} < \frac{\varepsilon}{3s}, \tag{2.32}$$

for $|v| \geq M$. Using Lemma 2.5, for $|v| > M$, we have

$$\begin{aligned} D(\psi f_n)(v) &\leq D\psi(v)\left|f_n(v)\right| + Df_n\left(v^-\right)\left|\psi\left(v^-\right)\right| \\ &\leq D\psi(v)\left(\left|f_n(o)\right| + 2\log(|v|+1)\right)\|f_n\|_{\mathbf{w}} + \|f_n\|_{\mathbf{w}}\frac{\left|\psi(v^-)\right|}{|v|} \\ &\leq \left(\|\psi\|_{\mathcal{L}}\left|f_n(o)\right| + 2D\psi(v)\log(|v|+1) + \frac{\left|\psi(v^-)\right|}{|v|}\right)\|f_n\|_{\mathbf{w}} \\ &< \varepsilon. \end{aligned} \tag{2.33}$$

On the other hand, on the set $B_M = \{v \in T : |v| \leq M\}$, $\{f_n\}$ converges to 0 uniformly, and thus Df_n does as well. Moreover

$$\begin{aligned} D(\psi f_n)(v) &\leq D\psi(v)\left|f_n(v)\right| + \left|\psi\left(v^-\right)\right|Df_n(v) \\ &\leq \|\psi\|_{\mathcal{L}}\left|f_n(v)\right| + \max_{|w|\leq M}\left|\psi(w)\right|Df_n(v) \longrightarrow 0, \end{aligned} \tag{2.34}$$

uniformly on B_M. Therefore $D(\psi f_n) \to 0$ uniformly on T. Furthermore, the sequence $\{(\psi f_n)(o)\}$ converges to 0 as $n \to \infty$. Hence $\|\psi f_n\|_{\mathcal{L}} \to 0$ as $n \to \infty$, proving that M_ψ is compact.

Finally, note that the functions f_n and g_n defined in the proof of (a)⇒(c) are in $\mathcal{L}_{\mathbf{w},0}$. So the equivalence of (b) and (c) is proved analogously. □

Recall the *essential norm* of a bounded operator S between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$\|S\|_e = \inf\{\|S - K\| : K \text{ is compact from } \mathcal{X} \text{ to } \mathcal{Y}\}. \tag{2.35}$$

For ψ a function on T, define the quantities

$$\begin{aligned} A(\psi) &= \lim_{n\to\infty} \sup_{|v|\geq n} \frac{|\psi(v)|}{|v|+1}, \\ B(\psi) &= \lim_{n\to\infty} \sup_{|v|\geq n} D\psi(v)\log|v|. \end{aligned} \tag{2.36}$$

Theorem 2.7. *Let M_ψ be a bounded multiplication operator from $\mathcal{L}_{\mathbf{w}}$ to $\mathcal{L}$. Then*

$$\|M_\psi\|_e \geq \max\{A(\psi), B(\psi)\}. \tag{2.37}$$

Proof. For each $n \in \mathbb{N}$, define $f_n = (1/(n+1))\chi_n$, where χ_n denotes the characteristic function of the set $\{v \in T : |v| = n\}$. Then $f_n \in \mathcal{L}_{\mathbf{w},0}$, $\|f_n\|_{\mathbf{w}} = 1$, and $f_n \to 0$ pointwise. Thus, by Lemma 1.4, $\{f_n\}$ converges to 0 weakly in $\mathcal{L}_{\mathbf{w},0}$. Let $\mathcal{K}$ be the set of compact operators from $\mathcal{L}_{\mathbf{w},0}$ to $\mathcal{L}_0$, and let $K \in \mathcal{K}$. Then K is completely continuous [8], and so $\|Kf_n\|_{\mathcal{L}} \to 0$ as $n \to \infty$. Thus

$$\|M_\psi - K\| \geq \limsup_{n\to\infty} \|(M_\psi - K)f_n\|_{\mathcal{L}} \geq \limsup_{n\to\infty} \|M_\psi f_n\|_{\mathcal{L}}. \tag{2.38}$$

Now note that

$$\|M_\psi f_n\|_{\mathcal{L}} = \sup_{|v|=n} \frac{|\psi(v)|}{n+1}. \tag{2.39}$$

Hence

$$\begin{aligned} \|M_\psi\|_e &\geq \inf\{\|M_\psi - K\| : K \in \mathcal{K}\} \\ &\geq \limsup_{n\to\infty} \|M_\psi f_n\|_{\mathcal{L}} \\ &= \lim_{n\to\infty} \sup_{|v|\geq n} \frac{|\psi(v)|}{|v|+1} \\ &= A(\psi). \end{aligned} \tag{2.40}$$

We will now show that $\|M_\psi\|_e \geq B(\psi)$. This estimate is clearly true if $B(\psi) = 0$. So assume $\{v_n\}$ is a sequence in T such that $2 \leq |v_n| \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} D\psi(v_n)\log|v_n| = B(\psi). \tag{2.41}$$

For $n \in \mathbb{N}$ and $v \in T$, define

$$h_n(v) = \begin{cases} \dfrac{\left[\log(|v|+1)\right]^2}{\log|v_n|} & \text{if } 0 \le |v| < |v_n|, \\ \log|v_n| & \text{if } |v| \ge |v_n|. \end{cases} \tag{2.42}$$

Then $h_n(o) = 0$, $h_n(v_n) = h_n(v_n^-) = \log|v_n|$, and

$$|v|Dh_n(v) = \begin{cases} \dfrac{|v|}{\log|v_n|} \log\left(\dfrac{|v|+1}{|v|}\right) \log[|v|(|v|+1)] & \text{if } 1 \le |v| < |v_n|, \\ 0 & \text{if } |v| \ge |v_n|. \end{cases} \tag{2.43}$$

The supremum of $|v|Dh_n(v)$ is attained at the vertices of length $|v_n|-1$ and is given by

$$s_n = \sup_{v\in T^*} |v|Dh_n(v) = (|v_n|-1)\log\left(\frac{|v_n|}{|v_n|-1}\right)\frac{\log[(|v_n|-1)|v_n|]}{\log|v_n|}. \tag{2.44}$$

Since $(|v_n|-1)\log(|v_n|/(|v_n|-1)) \le 1$, we have

$$\frac{(\log 2)^2}{\log|v_n|} \le \|h_n\|_{\mathbf{w}} = s_n \le \frac{\log[(|v_n|-1)|v_n|]}{\log|v_n|} < 2. \tag{2.45}$$

By letting $g_n = h_n/\|h_n\|_{\mathbf{w}}$, we have $g_n \in \mathcal{L}_{\mathbf{w},0}$, $\|g_n\|_{\mathbf{w}} = 1$, and $g_n \to 0$ pointwise. By Lemma 1.4, the sequence $\{g_n\}$ converges to 0 weakly in $\mathcal{L}_{\mathbf{w},0}$. Thus $\|Kg_n\|_{\mathcal{L}} \to 0$ as $n \to \infty$. Therefore,

$$\|M_\psi - K\| \ge \limsup_{n\to\infty} \|(M_\psi - K)g_n\|_{\mathcal{L}} \ge \limsup_{n\to\infty} \|\psi g_n\|_{\mathcal{L}}. \tag{2.46}$$

For each $n \in \mathbb{N}$, we have $g_n(v_n) = g_n(v_n^-) = \log|v_n|/s_n$. So

$$D(\psi g_n)(v_n) = \frac{1}{s_n} D\psi(v_n)\log|v_n|. \tag{2.47}$$

Since $\lim_{n\to\infty} s_n = 1$, we have

$$\begin{aligned} \|M_\psi\|_e &\ge \inf\{\|M_\psi - K\| : K \in \mathcal{K}\} \\ &\ge \limsup_{n\to\infty} \sup_{v\in T^*} D(\psi g_n)(v) \\ &\ge \lim_{n\to\infty} \frac{1}{s_n} D\psi(v_n)\log|v_n| \\ &= B(\psi). \end{aligned} \tag{2.48}$$

Therefore, $\|M_\psi\|_e \ge \max\{A(\psi), B(\psi)\}$. □

We now derive an upper estimate on the essential norm.

Theorem 2.8. *Let M_ψ be a bounded multiplication operator from $\mathcal{L}_{\mathbf{w}}$ to $\mathcal{L}$. Then*

$$\left\|M_\psi\right\|_e \leq A(\psi) + B(\psi). \tag{2.49}$$

Proof. For $n \in \mathbb{N}$, define the operator K_n on $\mathcal{L}_{\mathbf{w}}$ by

$$\left(K_n f\right)(v) = \begin{cases} f(v) & \text{if } |v| \leq n, \\ f(v_n) & \text{if } |v| > n, \end{cases} \tag{2.50}$$

where $f \in \mathcal{L}_{\mathbf{w}}$ and v_n is the ancestor of v of length n. For $f \in \mathcal{L}_{\mathbf{w}}$, $(K_n f)(o) = f(o)$, and $K_n f \in \mathcal{L}_{\mathbf{w},0}$. Let $B_n = \{v \in T : |v| \leq n\}$, and note that $K_n f$ attains finitely many values, whose number does not exceed the cardinality of B_n. Let $\{g_k\}$ be a sequence in $\mathcal{L}_{\mathbf{w}}$ such that $\|g_k\|_{\mathbf{w}} \leq 1$ for each $k \in \mathbb{N}$. Then $a = \sup_{k\in\mathbb{N}} |g_k(o)| \leq 1$, and $|K_n g_k(o)| \leq a$. Furthermore, by part (a) of Lemma 1.3, for each $v \in T^*$ and for each $k \in \mathbb{N}$, we have $|K_n g_k(v)| \leq 1+\log n$. Thus, some subsequence of $\{K_n g_k\}_{k\in\mathbb{N}}$ must converge to a function g on T attaining constant values on the sectors determined by the vertices of length n. It follows that this subsequence converges to g in $\mathcal{L}_{\mathbf{w}}$ as well, proving that K_n is a compact operator on $\mathcal{L}_{\mathbf{w}}$. Since M_ψ is bounded as an operator from $\mathcal{L}_{\mathbf{w}}$ to $\mathcal{L}$, it follows that $M_\psi K_n : \mathcal{L}_{\mathbf{w}} \to \mathcal{L}$ is compact for all $n \in \mathbb{N}$.

Define the operator $J_n = I - K_n$, where I denotes the identity operator on $\mathcal{L}_{\mathbf{w}}$. Then $J_n f(o) = 0$, and for $v \in T^*$, we have

$$|v|D\left(J_n f\right)(v) = |v|\left|\left(J_n f\right)(v) - \left(J_n f\right)\left(v^-\right)\right| \leq |v|Df(v) \leq \left\|f\right\|_{\mathbf{w}}. \tag{2.51}$$

By part (a) of Lemma 1.3, we see that

$$\left|\left(J_n f\right)(v)\right| \leq \left(1 + \log|v|\right)\left\|f\right\|_{\mathbf{w}}. \tag{2.52}$$

Using (2.51) and (2.52), we obtain

$$\begin{aligned}
\left\|(M_\psi - M_\psi K_n)f\right\|_{\mathcal{L}} &= \left\|(\psi J_n f)\right\|_{\mathcal{L}} \\
&= \sup_{|v|>n}\left|\psi(v)(J_n f)(v) - \psi(v^-)(J_n f)(v^-)\right| \\
&\le \sup_{|v|>n}\left[\left|(J_n f)(v)\right|D\psi(v) + \left|\psi(v^-)\right|D(J_n f)(v)\right] \\
&= \sup_{|v|>n}\left[\left|(J_n f)(v)\right|D\psi(v) + \frac{\left|\psi(v^-)\right|}{|v|}|v|D(J_n f)(v)\right] \\
&\le \sup_{|v|\ge n}\left[(1+\log|v|)D\psi(v) + \frac{\left|\psi(v)\right|}{|v|+1}\right]\|f\|_{\mathbf{w}} \\
&\le \sup_{|v|\ge n}\left[\log|v|D\psi(v)\frac{1+\log|v|}{\log|v|} + \frac{\left|\psi(v)\right|}{|v|+1}\right]\|f\|_{\mathbf{w}} \\
&\le \left[\sup_{|v|\ge n}\log|v|D\psi(v)\frac{1+\log n}{\log n} + \sup_{|v|\ge n}\frac{\left|\psi(v)\right|}{|v|+1}\right]\|f\|_{\mathbf{w}}.
\end{aligned} \tag{2.53}$$

Since

$$\left\|M_\psi\right\|_e \le \limsup_{n\to\infty}\left\|M_\psi - M_\psi K_n\right\| = \limsup_{n\to\infty}\sup_{\|f\|_{\mathbf{w}}=1}\left\|(M_\psi - M_\psi K_n)f\right\|_{\mathcal{L}}, \tag{2.54}$$

taking the limit as $n \to \infty$, we obtain

$$\left\|M_\psi\right\|_e \le B(\psi) + A(\psi), \tag{2.55}$$

as desired. □

3. Multiplication Operators from $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$

We begin this section with a boundedness criterion for the multiplication operators from $M_\psi : \mathcal{L} \to \mathcal{L}_{\mathbf{w}}$ and $M_\psi : \mathcal{L}_0 \to \mathcal{L}_{\mathbf{w},0}$.

3.1. Boundedness and Operator Norm Estimates

Let ψ be a function on the tree T. Define the quantities

$$\begin{aligned}
\theta_\psi &= \sup_{v\in T^*}|v|^2 D\psi(v), \\
\omega_\psi &= \sup_{v\in T}(|v|+1)\left|\psi(v)\right|.
\end{aligned} \tag{3.1}$$

Theorem 3.1. *For a function φ on T, the following statements are equivalent:*

(a) $M_\varphi : \mathcal{L} \to \mathcal{L}_{\mathbf{w}}$ *is bounded.*

(b) $M_\varphi : \mathcal{L}_0 \to \mathcal{L}_{\mathbf{w},0}$ *is bounded.*

(c) θ_φ *and* ω_φ *are finite.*

Furthermore, under the above conditions, one has

$$\max\{\theta_\varphi, \omega_\varphi\} \le \|M_\varphi\| \le \theta_\varphi + \omega_\varphi. \tag{3.2}$$

Proof. (a)⇒(c) Assume M_φ is bounded from $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$. The function $f_o = (1/2)\chi_o \in \mathcal{L}$ and $\|f_o\|_{\mathcal{L}} = 1$. Thus

$$\left|\varphi(o)\right| = \|\varphi f_o\|_{\mathbf{w}} \le \|M_\varphi\|. \tag{3.3}$$

Next, fix $v \in T^*$. Then $\chi_v \in \mathcal{L}$ and $\|\chi_v\|_{\mathcal{L}} = 1$; so

$$(|v| + 1)\left|\varphi(v)\right| = \left\|\varphi\chi_v\right\|_{\mathbf{w}} \le \left\|M_\varphi\right\|. \tag{3.4}$$

Taking the supremum over all $v \in T$, from (3.3) and (3.4) we see that ω_φ is finite and

$$\omega_\varphi \le \left\|M_\varphi\right\|. \tag{3.5}$$

With $v \in T^*$, we now define

$$f_v(w) = \begin{cases} |w| & \text{if } |w| < |v|, \\ |v| & \text{if } |w| \ge |v|. \end{cases} \tag{3.6}$$

Then $f_v \in \mathcal{L}$, $f_v(o) = 0$ and $\|f_v\|_{\mathcal{L}} = 1$. By the boundedness of M_φ we obtain

$$\begin{aligned} \left\|M_\varphi\right\| &\ge \left\|M_\varphi f_v\right\|_{\mathbf{w}} \\ &\ge \sup_{1\le|w|\le|v|} |w|\left|\varphi(w)|w| - \varphi(w^-)(|w| - 1)\right| \\ &\ge \sup_{1\le|w|\le|v|} |w|^2 D\varphi(w) - \sup_{1\le|w|\le|v|} |w|\left|\varphi(w^-)\right|. \end{aligned} \tag{3.7}$$

Therefore,

$$|v|^2 D\psi(v) \le \sup_{1\le|w|\le|v|} |w|^2 D\psi(w) \le \|M_\psi\| + \omega_\psi. \tag{3.8}$$

Taking the supremum over all $v \in T^*$, we obtain $\theta_\psi < \infty$. From this and (3.5), we deduce the lower estimate

$$\|M_\psi\| \ge \max\{\theta_\psi, \omega_\psi\}. \tag{3.9}$$

(c)⇒(a) Assume θ_ψ and ω_ψ are finite. Then, $\psi \in \mathcal{L}_{\mathbf{w}}$, and by Lemma 1.1, for $f \in \mathcal{L}$ with $\|f\|_{\mathcal{L}} = 1$ and $v \in T^*$, we have

$$\begin{aligned} |v|D(\psi f)(v) &\le |v|D\psi(v)\big|f(v)\big| + |v|\big|\psi(v^-)\big|Df(v) \\ &\le |v|D\psi(v)\big|f(o)\big| + |v|^2 D\psi(v)\big\|Df\big\|_\infty + \omega_\psi\big\|Df\big\|_\infty \\ &\le |v|D\psi(v)\big|f(o)\big| + (\theta_\psi + \omega_\psi)\big\|Df\big\|_\infty. \end{aligned} \tag{3.10}$$

Thus, $\psi f \in \mathcal{L}_{\mathbf{w}}$. Note that $|f(o)| + \|Df\|_\infty = 1$ and

$$\big\|\psi\big\|_{\mathbf{w}} = \big|\psi(o)\big| + \sup_{v\in T^*}|v|D\psi(v) \le \omega_\psi + \sup_{v\in T^*}|v|^2 D\psi(v) = \omega_\psi + \theta_\psi. \tag{3.11}$$

From this, we have

$$\big\|\psi f\big\|_{\mathbf{w}} \le \big\|\psi\big\|_{\mathbf{w}}\big|f(o)\big| + (\theta_\psi + \omega_\psi)\big\|Df\big\|_\infty \le \theta_\psi + \omega_\psi, \tag{3.12}$$

proving the boundedness of $M_\psi : \mathcal{L} \to \mathcal{L}_{\mathbf{w}}$ and the upper estimate

$$\|M_\psi\| \le \theta_\psi + \omega_\psi. \tag{3.13}$$

(b)⇒(c) The proof is the same as for (a)⇒(c); since for $v \in T^*$, the functions χ_v and f_v used there belong to $\mathcal{L}_0$.

(c)⇒(b) Assume θ_ψ and ω_ψ are finite, and let $f \in \mathcal{L}_0$. Then, by Lemma 1.1, for $v \in T^*$, we have

$$\begin{aligned} |v|D(\psi f)(v) &\le |v|D\psi(v)\big|f(v)\big| + |v|\big|\psi(v^-)\big|Df(v) \\ &\le |v|^2 D\psi(v)\frac{\big|f(v)\big|}{|v|} + |v|\big|\psi(v^-)\big|Df(v) \\ &\le \theta_\psi\frac{\big|f(v)\big|}{|v|} + \omega_\psi Df(v) \longrightarrow 0 \end{aligned} \tag{3.14}$$

as $|v| \to \infty$. Thus, $\psi f \in \mathcal{L}_{\mathbf{w},0}$. The proof of the boundedness of M_ψ is similar to that in (c)⇒(a). □

3.2. Isometries

In this section, we show there are no isometric multiplication operators M_ψ from the space $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$ or from $\mathcal{L}_0$ to $\mathcal{L}_{\mathbf{w},0}$.

Suppose $M_\psi : \mathcal{L} \to \mathcal{L}_{\mathbf{w}}$ is an isometry. Then $\|\psi\|_{\mathbf{w}} = \|M_\psi 1\|_{\mathbf{w}} = 1$. On the other hand,

$$|\psi(o)| = \frac{1}{2}\|\psi\chi_o\|_{\mathbf{w}} = \frac{1}{2}\|\chi_o\|_{\mathcal{L}} = 1. \tag{3.15}$$

Thus $\sup_{v\in T^*}|v|D\psi(v) = \|\psi\|_{\mathbf{w}} - |\psi(o)| = 0$, which implies that ψ is a constant of modulus 1. Now observe that, for $v \in T^*$, we have

$$1 = \|\chi_v\|_{\mathcal{L}} = \|M_\psi\chi\|_{\mathbf{w}} = (|v|+1)|\psi(v)| = |v|+1, \tag{3.16}$$

which is a contradiction. Since $\chi_v \in \mathcal{L}_0$ for all $v \in T$, if $M_\psi : \mathcal{L}_0 \to \mathcal{L}_{\mathbf{w},0}$ is an isometry, then the above argument yields again a contradiction. Thus, we proved the following result.

Theorem 3.2. *There are no isometries M_ψ from $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$ or from $\mathcal{L}_0$ to $\mathcal{L}_{\mathbf{w},0}$.*

3.3. Compactness and Essential Norm

We now characterize the compact multiplication operators, but first we give a useful compactness criterion for multiplication operators from $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$ or from $\mathcal{L}_0$ to $\mathcal{L}_{\mathbf{w},0}$.

Lemma 3.3. *A bounded multiplication operator M_ψ from $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$ ($\mathcal{L}_0$ to $\mathcal{L}_{\mathbf{w},0}$) is compact if and only if for every bounded sequence $\{f_n\}$ in $\mathcal{L}$ ($\mathcal{L}_0$) converging to 0 pointwise, the sequence $\|\psi f_n\|_{\mathbf{w}}$ converges to 0 as $n \to \infty$.*

Proof. Suppose M_ψ is compact from $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$ and $\{f_n\}$ is a bounded sequence in $\mathcal{L}$ converging to 0 pointwise. Without loss of generality, we may assume $\|f_n\|_{\mathcal{L}} \le 1$ for all $n \in \mathbb{N}$. Since M_ψ is compact, the sequence $\{\psi f_n\}$ has a subsequence $\{\psi f_{n_k}\}$ that converges in the $\mathcal{L}_{\mathbf{w}}$-norm to some function $f \in \mathcal{L}_{\mathbf{w}}$.

By Lemma 1.3, for $v \in T^*$ we have

$$|\psi(v)f_{n_k}(v) - f(v)| \le (1+\log|v|)\|\psi f_{n_k} - f\|_{\mathbf{w}}. \tag{3.17}$$

Thus, $\psi f_{n_k} \to f$ pointwise on T^*. Furthermore, since $|\psi(o)f_{n_k}(o) - f(o)| \le \|\psi f_{n_k} - f\|_{\mathbf{w}}$, $\psi(0)f_{n_k}(0) \to f(0)$ as $k \to \infty$. Thus $\psi f_{n_k} \to f$ pointwise on T. Since by assumption, $f_n \to 0$ pointwise, it follows that f is identically 0, and thus $\|\psi f_{n_k}\|_{\mathbf{w}} \to 0$. Since 0 is the only limit point in $\mathcal{L}_{\mathbf{w}}$ of the sequence $\{\psi f_n\}$, we deduce that $\|\psi f_n\|_{\mathbf{w}} \to 0$ as $n \to \infty$.

Conversely, suppose that every bounded sequence $\{f_n\}$ in $\mathcal{L}$ that converges to 0 pointwise has the property that $\|\psi f_n\|_{\mathbf{w}} \to 0$ as $n \to \infty$. Let $\{g_n\}$ be a sequence in $\mathcal{L}$ such that $\|g_n\|_{\mathcal{L}} \le 1$ for all $n \in \mathbb{N}$. Then $|g_n(o)| \le 1$, and by part (a) of Lemma 1.1, for $v \in T^*$ we have $|g_n(v)| \le |v|$. So $\{g_n\}$ is uniformly bounded on finite subsets of T. Thus there is a subsequence $\{g_{n_k}\}$, which converges pointwise to some function g.

Fix $\varepsilon > 0$ and $v \in T^*$. Then $|g_{n_k}(v) - g(v)| < \varepsilon/2$ as well as $|g_{n_k}(v^-) - g(v^-)| < \varepsilon/2$ for k sufficiently large. Therefore, for all k sufficiently large, we have

$$Dg(v) \le \left|g(v) - g_{n_k}(v)\right| + \left|g_{n_k}(v^-) - g(v^-)\right| + Dg_{n_k}(v) < \varepsilon + Dg_{n_k}(v). \tag{3.18}$$

Thus $g \in \mathcal{L}$. The sequence $f_{n_k} = g_{n_k} - g$ is bounded in $\mathcal{L}$ and converges to 0 pointwise. So $\|\psi f_{n_k}\|_{\mathbf{w}} \to 0$ as $k \to \infty$. Thus $\psi g_{n_k} \to \psi g$ in the $\mathcal{L}_{\mathbf{w}}$-norm. Therefore, M_ψ is compact.

The proof for the case of $M_\psi : \mathcal{L}_0 \to \mathcal{L}_{\mathbf{w},0}$ is similar. □

Theorem 3.4. *Let M_ψ be a bounded multiplication operator from $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$ (or equivalently from $\mathcal{L}_0$ to $\mathcal{L}_{\mathbf{w},0}$). Then the following are equivalent:*

(a) *$M_\psi : \mathcal{L} \to \mathcal{L}_{\mathbf{w}}$ is compact.*

(b) *$M_\psi : \mathcal{L}_0 \to \mathcal{L}_{\mathbf{w},0}$ is compact.*

(c) *$\lim_{|v|\to\infty} |v|^2 D\psi(v) = 0$ and $\lim_{|v|\to\infty} (|v| + 1)|\psi(v)| = 0$.*

Proof. (a)⇒(c) Suppose $M_\psi : \mathcal{L} \to \mathcal{L}_{\mathbf{w}}$ is compact. We need to show that if $\{v_n\}$ is a sequence in T such that $2 \le |v_n|$ increasing unboundedly, then $\lim_{n\to\infty} |v_n|^2 D\psi(v_n) = 0$ and $\lim_{n\to\infty} (|v_n| + 1)|\psi(v_n)| = 0$. Let $\{v_n\}$ be such a sequence, and for $n \in \mathbb{N}$ define $f_n = ((|v_n| + 1)/|v_n|)\chi_{v_n}$. Clearly $f_n \to 0$ pointwise, and $\|f_n\|_{\mathcal{L}} \le 3/2$. Using Lemma 3.3, we see that

$$\left\|\psi f_n\right\|_{\mathbf{w}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.19}$$

On the other hand, since $f_n(o) = 0$ for all $n \in \mathbb{N}$, we have

$$\left\|\psi f_n\right\|_{\mathbf{w}} = \sup_{v\in T^*} |v| D(\psi f_n)(v) \ge |v_n| \left(\frac{|v_n| + 1}{|v_n|}\right) \left|\psi(v_n)\right| = (|v_n| + 1)\left|\psi(v_n)\right|. \tag{3.20}$$

Hence $\lim_{n\to\infty} (|v_n| + 1)|\psi(v_n)| = 0$.

Next, for $n \in \mathbb{N}$, define

$$g_n(v) = \begin{cases} 0 & \text{if } |v| < \left\lfloor \dfrac{|v_n|}{2} \right\rfloor, \\ 2|v| - |v_n| + 2 & \text{if } \left\lfloor \dfrac{|v_n|}{2} \right\rfloor \le |v| < |v_n|, \\ |v_n| & \text{if } |v| \ge |v_n|, \end{cases} \tag{3.21}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Then $g_n \to 0$ pointwise, and $\|g_n\|_{\mathcal{L}} = 2$. Since $g_n(v_n) = g_n(v_n^-) = |v_n|$, we have

$$\left\|\psi g_n\right\|_w \ge |v_n| \left|\psi(v_n) g_n(v_n) - \psi(v_n^-) g_n(v_n^-)\right| = |v_n|^2 D\psi(v_n). \tag{3.22}$$

By Lemma 3.3 we obtain $\lim_{n\to\infty} |v_n|^2 D\psi(v_n) \le \lim_{n\to\infty} \|\psi g_n\|_{\mathbf{w}} = 0$.

(c)⇒(a) Suppose $\lim_{|v|\to\infty} |v|^2 D\psi(v) = 0$ and $\lim_{|v|\to\infty} (|v| + 1)|\psi(v)| = 0$. Assume ψ is not identically zero, otherwise M_ψ is trivially compact. By Lemma 3.3, to prove that M_ψ is

compact, it suffices to show that if $\{f_n\}$ is a bounded sequence in $\mathcal{L}$ converging to 0 pointwise, then $\|\psi f_n\|_{\mathbf{w}} \to 0$ as $n \to \infty$. Let $\{f_n\}$ be such a bounded sequence, $s = \sup_{n\in\mathbb{N}} \|f_n\|_{\mathcal{L}}$, and fix $\varepsilon > 0$. There exists $M \in \mathbb{N}$ such that $(|v|+1)|\psi(v)| < \varepsilon/2s$ and $|v|^2 D\psi(v) < \varepsilon/2s$ for $|v| \geq M$. For $v \in T^*$ and by Lemma 1.1, we have

$$\begin{aligned} |v|D(\psi f_n)(v) &\leq |v|\left|\psi(v)\right|Df_n(v) + |v|D\psi(v)\left|f_n(v^-)\right| \\ &\leq |v|\left|\psi(v)\right|Df_n(v) + |v|D\psi(v)\left(\left|f_n(o)\right| + |v|\left\|Df_n\right\|_\infty\right) \\ &\leq (|v|+1)\left|\psi(v)\right|Df_n(v) + |v|^2 D\psi(v)\left(\left|f_n(o)\right| + \left\|Df_n\right\|_\infty\right) \\ &= (|v|+1)\left|\psi(v)\right|Df_n(v) + |v|^2 D\psi(v)\left\|f_n\right\|_{\mathcal{L}}. \end{aligned} \tag{3.23}$$

Since $f_n \to 0$ uniformly on $\{v \in T : |v| \leq M\}$ as $n \to \infty$, so does Df_n. So, on the set $\{v \in T : |v| \leq M\}$, $|v|D(\psi f_n)(v) \to 0$ as $n \to \infty$. On the other hand, on $\{v \in T : |v| \geq M\}$, we have

$$|v|D(\psi f_n)(v) \leq (|v|+1)\left|\psi(v)\right|Df_n(v) + |v|^2 D\psi(v)\left\|f_n\right\|_{\mathcal{L}} < \varepsilon. \tag{3.24}$$

So $|v|D(\psi f_n)(v) \to 0$ as $n \to \infty$. Since $f_n \to 0$ pointwise, $\psi(o)f_n(o) \to 0$ as $n \to \infty$. Thus $\|\psi f_n\|_{\mathbf{w}} \to 0$ as $n \to \infty$. The compactness of M_ψ follows at once from Lemma 3.3.

The proof of the equivalence of (b) and (c) is analogous. □

For ψ a function on T, define

$$\begin{aligned} \mathcal{A}(\psi) &= \limsup_{n\to\infty} {}_{|v|\geq n} |v|\left|\psi(v)\right|, \\ \mathcal{B}(\psi) &= \limsup_{n\to\infty} {}_{|v|\geq n} |v|^2 D\psi(v). \end{aligned} \tag{3.25}$$

Theorem 3.5. *Let M_ψ be a bounded multiplication operator from $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$. Then*

$$\left\|M_\psi\right\|_e \geq \max\left\{\mathcal{A}(\psi), \frac{1}{2}\mathcal{B}(\psi)\right\}. \tag{3.26}$$

Proof. Fix $k \in \mathbb{N}$, and for each $n \in \mathbb{N}$, consider the sets

$$\begin{aligned} E_{n,k} &= \{v \in T : n \leq |v| \leq kn,\ |v| \text{ even}\}, \\ O_{n,k} &= \{v \in T : n \leq |v| \leq kn,\ |v| \text{ odd}\}. \end{aligned} \tag{3.27}$$

Define the functions $f_{n,k} = \chi_{E_{n,k}}$ and $g_{n,k} = \chi_{O_{n,k}}$. Then $f_{n,k}, g_{n,k} \in \mathcal{L}_0$, $\|f_{n,k}\|_{\mathcal{L}} = \|g_{n,k}\|_{\mathcal{L}} = 1$, and $f_{n,k}$ and $g_{n,k}$ approach 0 pointwise as $n \to \infty$. By Lemma 1.2, the sequences $\{f_{n,k}\}$ and $\{g_{n,k}\}$ approach 0 weakly in $\mathcal{L}_0$ as $n \to \infty$. Let $\mathcal{K}_0$ be the set of compact operators from $\mathcal{L}_0$ to $\mathcal{L}_{\mathbf{w},0}$, and note that every operator in $\mathcal{K}_0$ is completely continuous. Thus, if $K \in \mathcal{K}_0$, then $\|Kf_{n,k}\|_{\mathbf{w}} \to 0$ and $\|Kg_{n,k}\|_{\mathbf{w}} \to 0$, as $n \to \infty$.

Therefore, if $K \in \mathcal{K}_0$, then

$$\begin{aligned}\left\|M_\psi - K\right\| &\geq \limsup_{n\to\infty}\left\|\left(M_\psi - K\right)f_{n,k}\right\|_{\mathbf{w}} \\ &\geq \limsup_{n\to\infty}\left\|M_\psi f_{n,k}\right\|_{\mathbf{w}} \\ &\geq \limsup_{n\to\infty}\sup_{v\in E_{n,k}}(|v|+1)\left|\psi(v)\right|.\end{aligned} \tag{3.28}$$

Similarly,

$$\left\|M_\psi - K\right\| \geq \limsup_{n\to\infty}\sup_{v\in O_{n,k}}(|v|+1)\left|\psi(v)\right|. \tag{3.29}$$

Therefore, combining (3.28) and (3.29), we obtain

$$\begin{aligned}\left\|M_\psi\right\|_e &= \inf\left\{\left\|M_\psi - K\right\| : K \in \mathcal{K}_0\right\} \\ &\geq \limsup_{n\to\infty}\sup_{kn\geq|v|\geq n}(|v|+1)\left|\psi(v)\right| \\ &\geq \limsup_{n\to\infty}\sup_{kn\geq|v|\geq n}|v|\left|\psi(v)\right|.\end{aligned} \tag{3.30}$$

Letting $k \to \infty$, we obtain $\|M_\psi\|_e \geq \mathcal{A}(\psi)$.

Next, we wish to show that $\|M_\psi\|_e \geq (1/2)\mathcal{B}(\psi)$. The result is clearly true if $\mathcal{B}(\psi) = 0$. So assume there exists a sequence $\{v_n\}$ in T such that $2 < |v_n| \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty}|v_n|^2 D\psi(v_n) = \mathcal{B}(\psi). \tag{3.31}$$

For $n \in \mathbb{N}$, define

$$h_n(v) = \begin{cases} 0 & \text{if } v = o, \\ \dfrac{(|v|+1)^2}{|v_n|} & \text{if } 1 \leq |v| < |v_n|, \\ |v_n| & \text{if } |v| \geq |v_n|. \end{cases} \tag{3.32}$$

Clearly, $h_n(o) = 0$, $h_n(v_n) = h_n(v_n^-) = |v_n|$, and

$$Dh_n(v) = \begin{cases} \dfrac{4}{|v_n|} & \text{if } |v| = 1, \\ \dfrac{2|v|+1}{|v_n|} & \text{if } 1 < |v| < |v_n|, \\ 0 & \text{if } |v| \geq |v_n|. \end{cases} \tag{3.33}$$

The supremum of $Dh_n(v)$ is attained on the set $\{v \in T : |v| = |v_n| - 1\}$. Thus $\|h_n\|_{\mathcal{L}} = (2|v_n| - 1)/|v_n| < 2$. Define $g_n = h_n/\|h_n\|_{\mathcal{L}}$, and observe that $g_n \in \mathcal{L}_0$, $\|g_n\|_{\mathcal{L}} = 1$, and $g_n \to 0$ pointwise on T. By Lemma 1.2, $g_n \to 0$ weakly in $\mathcal{L}_0$. Thus $\|Kg_n\|_{\mathbf{w}} \to 0$ as $n \to \infty$ for any $K \in \mathcal{K}_0$.

For each $n \in \mathbb{N}$, $g_n(v_n) = g_n(v_n^-) = |v_n|^2/(2|v_n| - 1)$. Thus

$$\begin{aligned} |v_n| D(\psi g_n)(v_n) &= |v_n| \left| \psi(v_n) g_n(v_n) - \psi(v_n^-) g_n(v_n^-) \right| \\ &= \frac{|v_n|}{2|v_n| - 1} |v_n|^2 D\psi(v_n). \end{aligned} \tag{3.34}$$

We deduce that

$$\begin{aligned} \|M_\psi\|_e &= \inf\{\|M_\psi - K\| : K \in \mathcal{K}_0\} \\ &\geq \limsup_{n\to\infty} \|(M_\psi - K) g_n\|_{\mathbf{w}} \\ &\geq \limsup_{n\to\infty} \|M_\psi g_n\|_{\mathbf{w}} \\ &\geq \lim_{n\to\infty} \sup_{v\in T^*} |v| D(\psi g_n)(v) \\ &\geq \lim_{n\to\infty} |v_n| D(\psi g_n)(v_n) \\ &= \lim_{n\to\infty} \frac{|v_n|}{2|v_n| - 1} |v_n|^2 D\psi(v_n) \geq \frac{1}{2} \mathcal{B}(\psi). \end{aligned} \tag{3.35}$$

Therefore,

$$\|M_\psi\|_e \geq \max\left\{ \mathcal{A}(\psi), \frac{1}{2} \mathcal{B}(\psi) \right\}. \tag{3.36}$$

□

We next derive an upper estimate on the essential norm.

Theorem 3.6. *Let M_ψ be a bounded multiplication operator from $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$. Then*

$$\|M_\psi\|_e \leq \mathcal{A}(\psi) + \mathcal{B}(\psi). \tag{3.37}$$

Proof. For each $n \in \mathbb{N}$, consider the operator K_n defined by

$$(K_n f)(v) = \begin{cases} f(v) & \text{if } |v| \leq n, \\ f(v_n) & \text{if } |v| > n, \end{cases} \tag{3.38}$$

for $f \in \mathcal{L}$, where v_n is the ancestor of v of length n. Then $(K_n f)(o) = f(o)$, and $K_n f \in \mathcal{L}_{\mathbf{w},0}$. Arguing as in the proof of Theorem 2.8, by the boundedness of M_ψ, it follows that $M_\psi K_n$ is a compact operator from $\mathcal{L}$ to $\mathcal{L}_{\mathbf{w}}$.

Define the operator $J_n = I - K_n$, where I is the identity operator I on $\mathcal{L}$. Then,

$$D(J_n f)(v) \le Df(v) \le \|f\|_{\mathcal{L}}. \tag{3.39}$$

Since $(J_n f)(v) = 0$ for $|v| \le n$, by Lemma 1.1, we obtain

$$|(J_n f)(v)| \le |v| \|f\|_{\mathcal{L}}. \tag{3.40}$$

From these two estimates, we arrive at

$$\begin{aligned} \|M_\psi J_n f\|_{\mathbf{w}} &= \sup_{|v|>n} |v| |\psi(v)(J_n f)(v) - \psi(v^-)(J_n f)(v^-)| \\ &\le \sup_{|v|>n} [|v| D\psi(v) |(J_n f)(v)| + |v| |\psi(v^-)| D(J_n f)(v)] \\ &\le \sup_{|v|>n} |v|^2 D\psi(v) \frac{|(J_n f)(v)|}{|v|} + \sup_{|v|>n} |v| |\psi(v^-)| D(J_n f)(v) \\ &\le \sup_{|v|>n} |v|^2 D\psi(v) \|f\|_{\mathcal{L}} + \sup_{|v|>n} |v| |\psi(v^-)| \|f\|_{\mathcal{L}}. \end{aligned} \tag{3.41}$$

Since

$$\begin{aligned} \|M_\psi\|_e &\le \limsup_{n\to\infty} \|M_\psi - M_\psi K_n\| \\ &= \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{L}}=1} \|(M_\psi - M_\psi K_n) f\|_{\mathbf{w}} \\ &= \limsup_{n\to\infty} \sup_{\|f\|_{\mathcal{L}}=1} \|M_\psi J_n f\|_{\mathbf{w}}, \end{aligned} \tag{3.42}$$

from (3.41), taking the limit as $n \to \infty$, we obtain

$$\|M_\psi\|_e \le \mathcal{B}(\psi) + \mathcal{A}(\psi), \tag{3.43}$$

as desired. □

4. Multiplication Operators from $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ to L^∞

In this section, we study the multiplication operators M_ψ from the weighted Lipschitz space or the little weighted Lipschitz space into L^∞. We begin by characterizing the bounded operators and determining their operator norm. In addition, we characterize the bounded operators that are bounded from below and show that there are no isometries among them. Finally, we characterize the compact multiplication operators and determine the essential norm.

4.1. Boundedness and Operator Norm

For a function ψ on T, define

$$\gamma_\psi = \max\left\{ \left|\psi(o)\right|, \sup_{v\in T^*} (1+\log|v|)\left|\psi(v)\right| \right\}. \tag{4.1}$$

Theorem 4.1. *For a function ψ on T, the following statements are equivalent:*

(a) $M_\psi : \mathcal{L}_{\mathbf{w}} \to L^\infty$ *is bounded.*

(b) $M_\psi : \mathcal{L}_{\mathbf{w},0} \to L^\infty$ *is bounded.*

(c) $\sup_{v\in T^*} \log|v|\|\psi(v)|$ *is finite.*

Furthermore, under the above conditions, one has $\|M_\psi\| = \gamma_\psi$.

Proof. The implication (a)⇒(b) is obvious.

(b)⇒(a) We begin by showing that, for each $f \in \mathcal{L}_{\mathbf{w}}$, the function ψf is bounded. Since M_ψ is bounded on $\mathcal{L}_{\mathbf{w},0}$, $\psi = M_\psi 1 \in L^\infty$. Thus, if f is constant, then $\psi f \in L^\infty$. Fix $f \in \mathcal{L}_{\mathbf{w}}$, f nonconstant, $v \in$ T, and set $n = |v|$. For $w \in T$, define

$$f_n(w) = \begin{cases} f(w) & \text{if } |w| \le n, \\ f(w_n) & \text{if } |w| > n, \end{cases} \tag{4.2}$$

where w_n is the ancestor of w of length n. Then $f_n \in \mathcal{L}_{\mathbf{w},0}$, and $\|f_n\|_{\mathbf{w}} \le \|f\|_{\mathbf{w}}$. Thus, $\psi f_n \in L^\infty$ and

$$\left\|\psi f_n\right\|_\infty \le \left\|M_\psi\right\| \left\|f\right\|_{\mathbf{w}}. \tag{4.3}$$

So $|\psi(v)f(v)| = |\psi(v)f_n(v)| \le \|M_\psi\|\|f\|_{\mathbf{w}}$. Therefore, $\psi f \in L^\infty$ and

$$\left\|\psi f\right\|_\infty \le \left\|M_\psi\right\| \left\|f\right\|_{\mathbf{w}}, \tag{4.4}$$

proving the boundedness of M_ψ as an operator from $\mathcal{L}_{\mathbf{w}}$ to L^∞.

(a)⇒(c) Assume $M_\psi : \mathcal{L}_{\mathbf{w}} \to L^\infty$ is bounded. Then $\psi = M_\psi 1 \in L^\infty$, and

$$\left\|M_\psi\right\| \ge \left\|\psi\right\|_\infty \ge \left|\psi(o)\right|. \tag{4.5}$$

For $v \in T$, define $f(v) = \log(1+|v|)$. Then $f(o) = 0$, and since for $x \ge 1$ the function $x \mapsto x\log((x+1)/x)$ is increasing and has limit 1 as $x \to \infty$, $f \in \mathcal{L}_{\mathbf{w}}$ and $\|f\|_{\mathbf{w}} = 1$. Thus

$$\left\|M_\psi\right\| \ge \left\|\psi f\right\|_\infty = \sup_{v\in T^*} \log(1+|v|)\left|\psi(v)\right|, \tag{4.6}$$

proving (c). Furthermore, from (4.5) and (4.6), we obtain

$$\left\|M_\psi\right\| \ge \gamma_\psi. \tag{4.7}$$

(c)⇒(a) Assume $\sup_{v\in T^*} \log|v||\psi(v)| < \infty$. Let $f \in \mathcal{L}_{\mathbf{w}}$ such that $\|f\|_{\mathbf{w}} = 1$. Then $|\psi(o)f(o)| \le |\psi(o)|$, and by Lemma 1.3, for $v \in T^*$, we have

$$\left|\psi(v)f(v)\right| \le \left(1+\log|v|\right)\left|\psi(v)\right| \le \gamma_\psi. \tag{4.8}$$

Thus, $\psi f \in L^\infty$ and

$$\left\|\psi f\right\|_\infty \le \gamma_\psi, \tag{4.9}$$

proving the boundedness of M_ψ as an operator from $\mathcal{L}_{\mathbf{w}}$ to L^∞. Taking the supremum over all functions $f \in \mathcal{L}_{\mathbf{w}}$ such that $\|f\|_{\mathbf{w}} = 1$, from (4.9) we obtain $\|M_\psi\| \le \gamma_\psi$. Therefore, from (4.7) we conclude that $\|M_\psi\| = \gamma_\psi$. □

4.2. Boundedness from Below

Recall that an operator S from a Banach space $\mathcal{X}$ to a Banach space $\mathcal{Y}$ is *bounded below* if there exists a constant $C > 0$ such that for all $x \in X$

$$\|Sx\| \ge C\|x\|. \tag{4.10}$$

Theorem 4.2. *A bounded multiplication operator M_ψ from $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ to L^∞ is bounded below if and only if*

$$\inf_{v\in T} \frac{\left|\psi(v)\right|}{|v|+1} > 0. \tag{4.11}$$

Proof. Assume M_ψ is bounded below, and, arguing by contradiction, assume there exists $v \in T$ such that $\psi(v) = 0$. Then $M_\psi \chi_v$ is identically 0. Since operators that are bounded below are necessarily injective [8], it follows that M_ψ is not bounded below. Therefore, if M_ψ is bounded below, then ψ is nonvanishing.

Next assume ψ is nonvanishing and $\inf_{v\in T}|\psi(v)|/(|v|+1) = 0$. Then, there exists a sequence $\{v_n\}$ in T with $1 \le |v_n| \to \infty$, such that $|\psi(v_n)|/(|v_n|+1) \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$, define $f_n = (1/(|v_n|+1))\chi_{v_n}$. Then $\|f_n\|_{\mathbf{w}} = 1$, but

$$\|\psi f_n\|_\infty = \frac{\left|\psi(v_n)\right|}{|v_n|+1} \longrightarrow 0. \tag{4.12}$$

Thus, M_ψ is not bounded below.

Conversely, assume $\inf_{v\in T}|\psi(v)|/(|v|+1) = c > 0$ and that M_ψ is not bounded below. Then, for each $n \in \mathbb{N}$, there exists $f_n \in \mathcal{L}_{\mathbf{w}}$ such that $\|f_n\|_{\mathbf{w}} = 1$ and $\|\psi f_n\|_\infty < 1/n$. Then, for each $v \in T$, we have

$$c(|v|+1)\left|f_n(v)\right| \le \left|\psi(v)f_n(v)\right| < \frac{1}{n}, \tag{4.13}$$

so that the sequence $\{g_n\}$ defined by $g_n(v) = (|v| + 1) f_n(v)$ converges to 0 uniformly.

On the other hand, for $v \in T^*$, we have

$$\begin{aligned} |v| D f_n(v) &= \left| \frac{|v|}{|v| + 1} g_n(v) - g_n(v^-) \right| \\ &\leq |g_n(v)| + |g_n(v^-)| \longrightarrow 0 \end{aligned} \tag{4.14}$$

uniformly as $n \to \infty$. Since $|\psi(o) f_n(o)| < 1/n$, yet $\|f_n\|_{\mathbf{w}} = 1$, this yields a contradiction. □

4.3. Isometries

In this section, we show there are no isometries among the multiplication operators from the spaces $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ into L^∞.

Suppose M_ψ is an isometry from $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ to L^∞. Then, for $v \in T$ the function $f_v = (1/(|v| + 1))\chi_v$ is in $\mathcal{L}_{\mathbf{w},0}$, $\|f_v\|_{\mathbf{w}} = 1$, and

$$\frac{1}{|v| + 1} |\psi(v)| = \|M_\psi f_v\|_\infty = \|f_v\|_{\mathbf{w}} = 1. \tag{4.15}$$

Thus, $|\psi(v)| = |v| + 1$. On the other hand, since M_ψ is bounded, by Theorem 4.1, we have $\sup_{v \in T^*} \log |v| |\psi(v)| < \infty$; so $\psi(v) \to 0$ as $|v| \to \infty$, which yields a contradiction. Thus, we proved the following result.

Theorem 4.3. *There are no isometric multiplication operators M_ψ from $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ to L^∞.*

4.4. Compactness and Essential Norm

We begin by giving a useful compactness criterion for the bounded operators from $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ into L^∞.

Lemma 4.4. *A bounded multiplication operator M_ψ from $\mathcal{L}_{\mathbf{w}}$ to L^∞ is compact if and only if for every bounded sequence $\{f_n\}$ in $\mathcal{L}_{\mathbf{w}}$ converging to 0 pointwise, the sequence $\|\psi f_n\|_\infty$ approaches 0 as $n \to \infty$.*

Proof. Assume M_ψ is compact on $\mathcal{L}_{\mathbf{w}}$, and let $\{f_n\}$ be a bounded sequence in $\mathcal{L}_{\mathbf{w}}$ converging to 0 pointwise. By rescaling the sequence, if necessary, we may assume $\|f_n\|_{\mathbf{w}} \leq 1$ for all $n \in \mathbb{N}$. By the compactness of M_ψ, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{\psi f_{n_k}\}$ converges in the norm to some function $f \in L^\infty$. In particular, $\psi f_{n_k} \to f$ pointwise. Since by assumption, $f_n \to 0$ pointwise, it follows that f must be identically 0. Thus, the only limit point of sequence $\{\psi f_n\}$ in L^∞ is 0. Hence $\|\psi f_n\|_\infty \to 0$.

Conversely, assume that for every bounded sequence $\{f_n\}$ in $\mathcal{L}_{\mathbf{w}}$ converging to 0 pointwise, the sequence $\|\psi f_n\|_\infty$ approaches 0 as $n \to \infty$. Let $\{g_n\}$ be a sequence in $\mathcal{L}_{\mathbf{w}}$ with $\|g_n\|_{\mathbf{w}} \leq 1$. Fix $w \in T$, and, by replacing g_n with $g_n - g_n(w)$, assume $g_n(w) = 0$ for all $n \in \mathbb{N}$. Then, for each $v \in T$, $|g_n(v)| = |g_n(v) - g_n(w)| \leq d(v, w)$. Therefore, g_n is uniformly bounded on finite subsets of T, and so some subsequence $\{g_{n_k}\}_{k \in \mathbb{N}}$ converges

pointwise to some function g on T. Fix $\varepsilon > 0$ and $v \in T^*$. Then, $|g(o) - g_{n_k}(o)| < \varepsilon/2$, $|g_{n_k}(v) - g(v)| < \varepsilon/(2|v|)$, and $|g_{n_k}(v^-) - g(v^-)| < \varepsilon/(2|v|)$ for all k sufficiently large. Thus,

$$\begin{aligned} |v|Dg(v) &\le |v|\left|g(v) - g(v^-) - \left(g_{n_k}(v) - g_{n_k}(v^-)\right)\right| + |v|Dg_{n_k}(v) \\ &< \varepsilon + |v|Dg_{n_k}(v), \end{aligned} \tag{4.16}$$

for k sufficiently large. Consequently, $g \in \mathcal{L}_{\mathbf{w}}$, we have

$$\begin{aligned} \|g\|_{\mathbf{w}} &= |g(o)| + \sup_{v\in T^*} |v|Dg(v) \\ &\le |g(o) - g_{n_k}(o)| + |g_{n_k}(o)| + \varepsilon + \sup_{v\in T^*} Dg_{n_k}(v) \\ &< 2\varepsilon + \|g_{n_k}\|_{\mathbf{w}} \le 2\varepsilon + 1. \end{aligned} \tag{4.17}$$

Since ε was arbitrary, it follows that $\|g\|_{\mathbf{w}} \le 1$. Therefore, the sequence $\{f_k\}$ defined by $f_k = g_{n_k} - g$ is bounded in $\mathcal{L}_{\mathbf{w}}$ and converges to 0 pointwise, hence, by the hypothesis, $\|\psi f_k\|_\infty \to 0$ as $n \to \infty$. We conclude that $\psi g_{n_k} \to \psi g$ in L^∞, proving the compactness of M_ψ. □

By an analogous argument, we obtain the corresponding compactness criterion for $M_\psi : \mathcal{L}_{\mathbf{w},0} \to L^\infty$.

Lemma 4.5. *A bounded multiplication operator M_ψ from $\mathcal{L}_{\mathbf{w},0}$ to L^∞ is compact if and only if for every bounded sequence $\{f_n\}$ in $\mathcal{L}_{\mathbf{w},0}$ converging to 0 pointwise, the sequence $\|\psi f_n\|_\infty$ approaches 0 as $n \to \infty$.*

Theorem 4.6. *For a bounded operator M_ψ from $\mathcal{L}_{\mathbf{w}}$ to L^∞ (or equivalently from $\mathcal{L}_{\mathbf{w},0}$ to L^∞) the following statements are equivalent:*

(a) *$M_\psi : \mathcal{L}_{\mathbf{w}} \to L^\infty$ is compact.*

(b) *$M_\psi : \mathcal{L}_{\mathbf{w},0} \to L^\infty$ is compact.*

(c) *$\lim_{|v|\to\infty} \log|v||\psi(v)| = 0$.*

Proof. (a)⇒(b) is trivial.

(b)⇒(c): Let $\{v_n\}$ be a sequence of vertices such that $1 \le |v_n| \to \infty$ as $n \to \infty$. We need to show that

$$\lim_{n\to\infty} \log|v_n|\left|\psi(v_n)\right| = 0. \tag{4.18}$$

For $n \in \mathbb{N}$ define

$$f_n(v) = \begin{cases} 0 & \text{if } v = o, \\ \dfrac{(\log|v|)^2}{\log|v_n|} & \text{if } 1 \le |v| < |v_n|, \\ \log|v_n| & \text{if } |v| \ge |v_n|. \end{cases} \tag{4.19}$$

Then $\{f_n\}$ converges to 0 pointwise. Using the fact that $(|v|-1)(\log|v|-\log(|v|-1)) \le 1$ for any choice of v in T^* with $|v|>1$, we have

$$|v|Df_n(v) = \frac{|v|}{|v|-1}\frac{(|v|-1)\left[(\log|v|)^2-(\log(|v|-1))^2\right]}{\log|v_n|} \le \frac{2(\log|v|+\log(|v|-1))}{\log|v_n|} \le 4, \quad (4.20)$$

for $2 \le |v| \le |v_n|$. Moreover, $|v|Df_n(v) = 0$ for $|v| = 1$ and for $|v| > |v_n|$. Thus, $f_n \in \mathcal{L}_{\mathbf{w},0}$, and $\{\|f_n\|_{\mathbf{w}}\}$ is bounded. By the compactness of M_ψ as an operator from $\mathcal{L}_{\mathbf{w},0}$ to L^∞ and by Lemma 4.5, we deduce

$$\log|v_n|\left|\psi(v_n)\right| \le \left\|\psi f_n\right\|_\infty \longrightarrow 0 \quad (4.21)$$

as $n \to \infty$.

(c)⇒(a) Assume $\{f_n\}$ is a sequence in $\mathcal{L}_{\mathbf{w}}$ converging to 0 pointwise and such that $a = \sup_{n\in\mathbb{N}}\|f_n\|_{\mathbf{w}} < \infty$. By Lemma 1.3, for all $v \in T^*$ and all $n \in \mathbb{N}$, we have

$$\left|\psi(v)f_n(v)\right| \le a(1+\log|v|)\left|\psi(v)\right|. \quad (4.22)$$

Fix $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $N \ge 3$, and for $|v| \ge N$, $\log|v||\psi(v)| < \varepsilon/2a$. Thus, for $|v| \ge N$ and for all $n \in \mathbb{N}$, $|\psi(v)f_n(v)| \le 2a\log|v||\psi(v)| < \varepsilon$. On the other hand, since $f_n \to 0$ pointwise, for each vertex v such that $|v| < N$ and $\psi(v) \ne 0$, we obtain $|f_n(v)| < \varepsilon/|\psi(v)|$ for all n sufficiently large. Hence $|\psi(v)f_n(v)| < \varepsilon$ for all $v \in T$ and all n sufficiently large. Therefore, $\|M_\psi f_n\|_\infty \to 0$ as $n \to \infty$, which, by Lemma 4.4, proves the compactness of M_ψ. □

Next, we determine the essential norm of the bounded multiplication operators M_ψ from $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ to L^∞.

Theorem 4.7. *Let M_ψ be a bounded multiplication operator from $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ to L^∞. Then*

$$\left\|M_\psi\right\|_e = \lim_{n\to\infty}\sup_{|v|\ge n}\log|v|\left|\psi(v)\right|. \quad (4.23)$$

Proof. Define $A(\psi) = \lim_{n\to\infty}\sup_{|v|\ge n}\log|v||\psi(v)|$. If $A(\psi) = 0$, then by Theorem 4.6, M_ψ is compact, hence its essential norm is 0. So assume $A(\psi) > 0$. We first show that $\|M_\psi\|_e \ge A(\psi)$. Let $\{v_n\}$ be a sequence in T such that $1 \le |v_n| \to \infty$ as $n \to \infty$ and

$$A(\psi) = \lim_{n\to\infty}\log|v_n|\left|\psi(v_n)\right|. \quad (4.24)$$

Fix $p \in (0,1)$, and for each $n \in \mathbb{N}$, define

$$f_{n,p}(v) = \begin{cases} 0 & \text{if } v = o, \\ \dfrac{(\log|v|)^{p+1}}{(\log|v_n|)^p} & \text{if } 1 \le |v| < |v_n|, \\ \log|v_n| & \text{if } |v| \ge |v_n|. \end{cases} \quad (4.25)$$

Then $\{f_{n,p}\}$ converges to 0 pointwise, $f_{n,p} \in \mathcal{L}_{\mathbf{w},0}$, $f_{n,p}(v_n) = \log|v_n|$, and

$$\begin{aligned}\|f_{n,p}\|_{\mathbf{w}} &= \sup_{2\le|v|\le|v_n|} \frac{|v|}{(\log|v_n|)^p}\left[(\log|v|)^{p+1} - (\log(|v|-1))^{p+1}\right] \\ &= \frac{|v_n|}{(\log|v_n|)^p}\left[(\log|v_n|)^{p+1} - (\log(|v_n|-1))^{p+1}\right] \le p+1.\end{aligned} \tag{4.26}$$

By Lemma 1.4, $\{f_{n,p}\}$ converges to 0 weakly in $\mathcal{L}_{\mathbf{w},0}$. Let K be a compact operator from $\mathcal{L}_{\mathbf{w},0}$ (or equivalently, from $\mathcal{L}_{\mathbf{w}}$) to L^∞. Since compact operators are completely continuous, it follows that $\|Kf_{n,p}\|_\infty \to 0$ as $n \to \infty$. Thus,

$$\begin{aligned}\|M_\psi - K\| &\ge \limsup_{n\to\infty} \frac{\|(M_\psi - K)f_{n,p}\|_\infty}{\|f_{n,p}\|_{\mathbf{w}}} \\ &\ge \frac{1}{p+1}\limsup_{n\to\infty}\|M_\psi f_{n,p}\|_\infty \\ &\ge \frac{1}{p+1}\limsup_{n\to\infty}\log|v_n|\,|\psi(v_n)|.\end{aligned} \tag{4.27}$$

Taking the infimum over all such compact operators K and passing to the limit as p approaches 0, we obtain

$$\|M_\psi\|_e \ge \lim_{n\to\infty}\log|v_n|\,|\psi(v_n)| = A(\psi). \tag{4.28}$$

To prove the estimate $\|M_\psi\|_e \le A(\psi)$, for each $n \in \mathbb{N}$ and for $f \in \mathcal{L}_{\mathbf{w}}$, define

$$K_n f(v) = \begin{cases} f(v) & \text{if } |v| \le n, \\ f(v_n) & \text{if } |v| > n, \end{cases} \tag{4.29}$$

where v_n is the ancestor of v of length n. In the proof of Theorem 2.8, it is was shown that K_n is a compact operator on $\mathcal{L}_{\mathbf{w}}$. Since $M_\psi : \mathcal{L}_{\mathbf{w}} \to L^\infty$ is bounded, it follows that $M_\psi K_n$ is also compact as an operator from $\mathcal{L}_{\mathbf{w}}$ to L^∞.

Let $v \in T$, and let w be a vertex in the path from o to v of length $k \ge 1$. Label the vertices from w to v by v_j, $j = k, \ldots, |v|$. Then for $f \in \mathcal{L}_{\mathbf{w}}$ with $\|f\|_{\mathbf{w}} = 1$, we have

$$|f(v) - f(w)| \le \sum_{j=k+1}^{|v|} |f(v_j) - f(v_{j-1})| \le \sum_{j=k+1}^{|v|} \frac{1}{j} \le \log|v|. \tag{4.30}$$

Thus

$$\|(M_\psi - M_\psi K_n)f\|_\infty = \sup_{|v|>n}|\psi(v)|\,|f(v) - f(v_n)| \le \sup_{|v|>n}\log|v|\,|\psi(v)|. \tag{4.31}$$

We deduce

$$\|M_\varphi\|_e \leq \sup_{\|f\|_{\mathbf{w}}=1} \|(M_\varphi - M_\varphi K_n)f\|_\infty \leq \sup_{|v|>n} \log|v|\,|\varphi(v)|. \tag{4.32}$$

Taking the limit as $n \to \infty$, we obtain $\|M_\varphi\|_e \leq A(\varphi)$. □

5. Multiplication Operators from L^∞ to $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$

In this last section, we study the multiplication operators M_φ from L^∞ into the weighted Lipschitz space or the little weighted Lipschitz space. We first characterize the bounded operators and determine the operator norm. We also show there are no isometries among such operators. Finally, we characterize the compact multiplication operators and determine the essential norm.

5.1. Boundedness and Operator Norm

For a function φ on T, define

$$\eta_\varphi = |\varphi(o)| + \sup_{v\in T^*} |v|\left[|\varphi(v)| + |\varphi(v^-)|\right]. \tag{5.1}$$

Theorem 5.1. *For a function φ on T, the following statements are equivalent:*

(a) $M_\varphi : L^\infty \to \mathcal{L}_w$ *is bounded.*

(b) $\sup_{v\in T}|v||\varphi(v)| < \infty$.

Furthermore, under these conditions, one has

$$\|M_\varphi\| = \eta_\varphi. \tag{5.2}$$

Proof. (a)⇒(b) Assume $M_\varphi : L^\infty \to \mathcal{L}_{\mathbf{w}}$ is bounded. Fix $v \in T^*$. Since $\chi_v \in L^\infty$ and $\|\chi_v\|_\infty = 1$, the function $\varphi\chi_v \in \mathcal{L}_{\mathbf{w}}$, so

$$|v|\,|\varphi(v)| < (|v|+1)|\varphi(v)| = \sup_{w\in T^*} |w|D(\varphi\chi_v)(w) \leq \|M_\varphi\|. \tag{5.3}$$

Thus, $\sup_{v\in T}|v||\varphi(v)|$ is finite.

(b)⇒(a) Suppose $\sup_{v\in T}|v||\varphi(v)| < \infty$. Let $f \in L^\infty$ such that $\|f\|_\infty = 1$. Then

$$\|M_\varphi f\|_{\mathbf{w}} \leq |\varphi(o)| + \sup_{v\in T^*} |v|\left[|\varphi(v)| + |\varphi(v^-)|\right] < \infty. \tag{5.4}$$

Thus, M_φ is bounded and $\|M_\varphi\| \leq \eta_\varphi$.

We next show that $\|M_\psi\| \geq \eta_\psi$. The inequality is obvious if ψ is identically 0. For ψ not identically 0 and for $v \in T$, define

$$f(v) = \begin{cases} 0 & \text{if } \psi(v) = 0, \\ \dfrac{\overline{\psi(v)}}{|\psi(v)|} & \text{if } \psi(v) \neq 0,\ |v| \text{ even}, \\ -\dfrac{\overline{\psi(v)}}{|\psi(v)|} & \text{if } \psi(v) \neq 0,\ |v| \text{ odd}. \end{cases} \tag{5.5}$$

Then $\|f\|_\infty = 1$, and for $v \in T^*$, $D(\psi f)(v) = |\psi(v)| + |\psi(v^-)|$, so that

$$\left\|M_\psi f\right\|_{\mathbf{w}} = \left|\psi(o)\right| + \sup_{v \in T^*} |v| \left[\left|\psi(v)\right| + \left|\psi\left(v^-\right)\right|\right]. \tag{5.6}$$

Thus, $\|M_\psi\| \geq \eta_\psi$, completing the proof. □

In the next result, we characterize the bounded multiplication operators from L^∞ to $\mathcal{L}_{\mathbf{w},0}$.

Theorem 5.2. *For a function ψ on T, the following statements are equivalent:*

(a) $M_\psi : L^\infty \to \mathcal{L}_{\mathbf{w},0}$ *is bounded.*

(b) $\lim_{|v| \to \infty} |v||\psi(v)| = 0$.

Furthermore, under these conditions, one has,

$$\left\|M_\psi\right\| = \eta_\psi. \tag{5.7}$$

Proof. (a)⇒(b) Assume $M_\psi : L^\infty \to \mathcal{L}_{\mathbf{w},0}$ is bounded. Applying M_ψ to the constant function 1, we obtain $\psi = M_\psi 1 \in \mathcal{L}_{\mathbf{w},0}$. On the other hand, if $\mathcal{O} = \{v \in T : |v| \text{ is odd}\}$, then $\psi\chi_{\mathcal{O}} \in \mathcal{L}_{\mathbf{w},0}$, so for $v \in T^*$, we have

$$\begin{aligned} |v|\left|\psi(v)\right| = |v|\left|\psi(v)\right| D\chi_{\mathcal{O}}(v) &\leq |v| D\left(\psi\chi_{\mathcal{O}}\right)(v) + |v| D\psi(v) \left|\chi_{\mathcal{O}}\left(v^-\right)\right| \\ &\leq |v| D\left(\psi\chi_{\mathcal{O}}\right)(v) + |v| D\psi(v) \longrightarrow 0, \end{aligned} \tag{5.8}$$

as $|v| \to \infty$, proving (b).

(b)⇒(a) Suppose $|v||\psi(v)| \to 0$ as $|v| \to \infty$. First observe that

$$\begin{aligned} |v| D\psi(v) &\leq |v|\left|\psi(v)\right| + \frac{|v|}{|v| - 1}(|v| - 1)\left|\psi\left(v^-\right)\right| \\ &\leq |v|\left|\psi(v)\right| + 2(|v| - 1)\left|\psi\left(v^-\right)\right| \longrightarrow 0, \end{aligned} \tag{5.9}$$

as $|v| \to \infty$. Then for $f \in L^\infty$ and $v \in T^*$, we have

$$\begin{aligned} |v| D(\psi f)(v) &\leq |v|\left|\psi(v)\right| Df(v) + |v| D\psi(v) \left|f\left(v^-\right)\right| \\ &\leq \left(2|v|\left|\psi(v)\right| + |v| D\psi(v)\right) \left\|f\right\|_\infty \longrightarrow 0, \end{aligned} \tag{5.10}$$

as $|v| \to \infty$. Thus, $\psi f \in \mathcal{L}_{\mathbf{w},0}$. The proof of the boundedness of M_ψ and of the formula $\|M_\psi\| = \eta_\psi$ is similar to the case when $M_\psi : L^\infty \to \mathcal{L}_{\mathbf{w}}$. □

5.2. Isometries

As for all other multiplication operators in this paper, there are no isometries among the multiplication operators from L^∞ into $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$.

Assume M_ψ is an isometry from L^∞ to $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$. Then, for $v \in T$ the function $f_v = (1/(|v|+1))\chi_v$ is in $\mathcal{L}_{\mathbf{w},0}$ with $\|M_\psi \chi_v\|_{\mathbf{w}} = \|\chi_v\|_\infty = 1$. In particular, it follows that $|\psi(o)| = 1/2$, and for $v \in T^*$,

$$(|v| + 1)\left|\psi(v)\right| = 1. \tag{5.11}$$

Thus, $|\psi(v)| = 1/(|v| + 1)$. On the other hand, taking as a test function f the characteristic function of the set $\{v \in T : |v| \leq 1\}$, we obtain

$$1 = \|f\|_\infty = \|M_\psi f\|_{\mathbf{w}} = |\psi(o)| + \max\left\{\sup_{|v|=1}|\psi(v) - \psi(o)|, \sup_{|v|=1} 2|\psi(v)|\right\} \geq \frac{3}{2}, \tag{5.12}$$

which yields a contradiction. Therefore, we obtain the following result.

Theorem 5.3. *There are no isometric multiplication operators M_ψ from L^∞ to $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$.*

5.3. Compactness and Essential Norm

The following two results are compactness criteria for multiplication operators from L^∞ into $\mathcal{L}_{\mathbf{w}}$ or $\mathcal{L}_{\mathbf{w},0}$ similar to those given in the previous sections.

Lemma 5.4. *A bounded multiplication operator M_ψ from L^∞ to $\mathcal{L}_{\mathbf{w}}$ is compact if and only if for every bounded sequence $\{f_n\}$ in L^∞ converging to 0 pointwise, the sequence $\|\psi f_n\|_{\mathbf{w}}$ approaches 0 as $n \to \infty$.*

Proof. Assume M_ψ is compact, and let $\{f_n\}$ be a bounded sequence in L^∞ converging to 0 pointwise. By rescaling the sequence, if necessary, we may assume $\|f_n\|_\infty \leq 1$ for all $n \in \mathbb{N}$. By the compactness of M_ψ, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{\psi f_{n_k}\}$ converges in the $\mathcal{L}_{\mathbf{w}}$-norm to some function $f \in \mathcal{L}_{\mathbf{w}}$. Since by Lemma 1.3, for $v \in T^*$,

$$\left|\psi(v) f_{n_k}(v) - f(v)\right| \leq \left(1 + \log|v|\right)\left\|\psi f_{n_k} - f\right\|_{\mathbf{w}} \tag{5.13}$$

and $|\psi(o) f_{n_k}(o) - f(o)| \leq \|\psi f_{n_k} - f\|_{\mathbf{w}}$, it follows that $\psi f_{n_k} \to f$ pointwise. Since by assumption, $f_n \to 0$ pointwise, the function f must be identically 0. Thus, the only limit point of the sequence $\{\psi f_n\}$ in $\mathcal{L}_{\mathbf{w}}$ is 0. Hence $\|\psi f_n\|_{\mathbf{w}} \to 0$ as $n \to \infty$.

Conversely, suppose $\|\psi f_n\|_{\mathbf{w}}$ approaches 0 as $n \to \infty$ for every bounded sequence $\{f_n\}$ in L^∞ converging to 0 pointwise. Let $\{g_n\}$ be a sequence in L^∞ with $\|g_n\|_\infty \leq 1$. Then some subsequence $\{g_{n_k}\}$ converges to a bounded function g. Thus, the sequence $f_{n_k} = g_{n_k} - g$

converges to 0 uniformly, and $\|f_n\|_\infty$ is bounded. By the hypothesis, it follows that $\|\psi f_{n_k}\|_{\mathbf{w}} \to 0$ as $k \to \infty$. Thus, $\psi g_{n_k} \to \psi g$ in $\mathcal{L}_{\mathbf{w}}$. Therefore, M_ψ is compact. □

By an analogous argument, we obtain the corresponding result for $M_\psi : L^\infty \to \mathcal{L}_{\mathbf{w},0}$.

Lemma 5.5. *A bounded multiplication operator M_ψ from L^∞ to $\mathcal{L}_{\mathbf{w},0}$ is compact if and only if for every bounded sequence $\{f_n\}$ in L^∞ converging to 0 pointwise, the sequence $\|\psi f_n\|_{\mathbf{w}}$ approaches 0 as $n \to \infty$.*

Theorem 5.6. *For a bounded operator M_ψ from L^∞ to $\mathcal{L}_{\mathbf{w}}$, the following statements are equivalent:*
(a) *M_ψ is compact.*
(b) $\lim_{|v|\to\infty} |v||\psi(v)| = 0$.

Proof. (a)⇒(b) Assume M_ψ is compact. Let $\{v_n\}$ be a sequence in T such that $|v_n| \to \infty$ as $n \to \infty$. For $n \in \mathbb{N}$, let f_n denote the characteristic function of the set $\{w \in T : |w| \geq |v_n|\}$. Then $\|f_n\|_\infty = 1$ and $f_n \to 0$ pointwise. By Lemma 5.4 and the compactness of M_ψ, it follows that

$$|v_n|\left|\psi(v_n)\right| = |v_n| D(\psi f_n)(v_n) \leq \left\|M_\psi f_n\right\|_{\mathbf{w}} \longrightarrow 0 \tag{5.14}$$

as $n \to \infty$.

(b)⇒(a) Assume $\lim_{|v|\to\infty} |v||\psi(v)| = 0$ and that ψ is not identically 0. In particular, ψ is bounded. Let $\{f_n\}$ be a sequence in L^∞ converging pointwise to 0 such that $\|f_n\|_\infty$ is bounded above by some positive constant C. Then corresponding to $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|v||\psi(v)| < \varepsilon/4C$ for all vertices v such that $|v| \geq N$. Therefore, for $|v| > N$ and $n \in \mathbb{N}$, we have

$$|v| D(\psi f_n)(v) \leq |v|\left|\psi(v)\right| Df_n(v) + |v| D\psi(v)\left|f_n\left(v^-\right)\right| < \varepsilon. \tag{5.15}$$

Furthermore, the sequence $\{f_n\}$ converges to 0 uniformly on the set $\{v \in T : |v| \leq N\}$ so that $|f_n(v)| < \varepsilon/4N\|\psi\|_\infty$ for all n sufficiently large. Hence $|v|D(\psi f_n)(v) < \varepsilon$ for all $v \in T^*$ and all n sufficiently large. Consequently, $\|\psi f_n\|_{\mathbf{w}} \to 0$ as $n \to \infty$. Using Lemma 5.4, we deduce that M_ψ is compact. □

Since the above proof is also valid when M_ψ is a bounded operator from L^∞ to $\mathcal{L}_{\mathbf{w},0}$, through the application of Lemma 5.5, from Theorems 5.2 and 5.6 we obtain the following result.

Corollary 5.7. *For a function ψ on T, the following statements are equivalent:*
(a) *$M_\psi : L^\infty \to \mathcal{L}_{\mathbf{w}}$ is compact.*
(b) *$M_\psi : L^\infty \to \mathcal{L}_{\mathbf{w},0}$ is bounded.*
(c) *$M_\psi : L^\infty \to \mathcal{L}_{\mathbf{w},0}$ is compact.*
(d) $\lim_{|v|\to\infty} |v||\psi(v)| = 0$.

We now determine the essential norm of the bounded multiplication operators from L^∞ to $\mathcal{L}_{\mathbf{w}}$.

Theorem 5.8. *Let $M_\psi : L^\infty \to \mathcal{L}_{\mathbf{w}}$ be bounded. Then*

$$\|M_\psi\|_e = \lim_{n\to\infty} \sup_{|v|\geq n} |v|\left[\left|\psi(v)\right| + \left|\psi(v^-)\right|\right]. \tag{5.16}$$

Proof. Set $B(\psi) = \lim_{n\to\infty}\sup_{|v|\geq n}|v|[|\psi(v)| + |\psi(v^-)|]$. In the case $B(\psi) = 0$, then $\lim_{|v|\to\infty}|v|\psi(v) = 0$, so by Theorem 5.6, M_ψ is compact, and thus $\|M_\psi\|_e = 0$. So assume $B(\psi) > 0$. Then there exists a sequence $\{v_n\}$ in T such that $1 \leq |v_n| \to \infty$ as $n \to \infty$ and

$$B(\psi) = \lim_{n\to\infty} |v_n|\left[\left|\psi(v_n)\right| + \left|\psi(v_n^-)\right|\right]. \tag{5.17}$$

For each $n \in \mathbb{N}$ let f_n be the function on T defined by

$$f_n(v) = \begin{cases} 0 & \text{if } |v| < |v_n| \text{ or } \psi(v) = 0, \\ \dfrac{\overline{\psi(v)}}{\left|\psi(v)\right|} & \text{if } |v| \geq |v_n|,\ |v| \text{ is even},\ \psi(v) \neq 0, \\ -\dfrac{\overline{\psi(v)}}{\left|\psi(v)\right|} & \text{otherwise.} \end{cases} \tag{5.18}$$

Then $\|f_n\|_\infty = 1$, and $\{f_n\}$ converges to 0 pointwise. Thus, for any compact operator $K : L^\infty \to \mathcal{L}_{\mathbf{w}}$, there exists a subsequence $\{f_{n_k}\}$ such that $\|Kf_{n_k}\|_{\mathbf{w}} \to 0$ as $k \to \infty$. Thus

$$\begin{aligned} \|M_\psi - K\| &\geq \limsup_{k\to\infty} \left\|(M_\psi - K)f_{n_k}\right\|_{\mathbf{w}} \geq \limsup_{k\to\infty} \left\|\psi f_{n_k}\right\|_{\mathbf{w}} \\ &= \limsup_{k\to\infty} \sup_{|v|\geq|v_{n_k}|} |v|\left[\left|\psi(v)\right| + \left|\psi(v^-)\right|\right] = B(\psi). \end{aligned} \tag{5.19}$$

Therefore, $\|M_\psi\|_e \geq B(\psi)$.

We now show that $\|M_\psi\|_e \leq B(\psi)$. For each $n \in \mathbb{N}$, define the operator K_n on L^∞ by

$$K_n f(v) = \begin{cases} f(v) & \text{if } |v| \leq n, \\ 0 & \text{if } |v| > n. \end{cases} \tag{5.20}$$

Then, for $v \in T^*$, we have

$$|v|D(K_n f)(v) = \begin{cases} |v|Df(v) & \text{for } 1 \leq |v| \leq n, \\ (n+1)\left|f(v^-)\right| & \text{for } |v| = n+1, \\ 0 & \text{for } |v| > n+1. \end{cases} \tag{5.21}$$

Thus, $K_n f \in \mathcal{L}_{\mathbf{w}}$ with $\|K_n f\|_{\mathbf{w}} \leq |f(o)| + 2n\|f\|_\infty$.

Assume $\{f_k\}$ is a sequence in L^∞ with $\|f_k\|_\infty \le 1$. Then there exists a subsequence $\{f_{k_j}\}$ converging pointwise to some function $f \in L^\infty$. Thus,

$$\begin{aligned}
&\left\|K_n f_{k_j} - K_n f\right\|_{\mathbf{w}} \\
&\quad = \left|f_{k_j}(o) - f(o)\right| + \max\left\{\sup_{1\le|v|\le n}|v|D\left(f_{k_j}-f\right)(v), \sup_{|v|=n+1}|v|\left|f_{k_j}(v^-) - f(v^-)\right|\right\} \\
&\quad \le \left|f_{k_j}(o) - f(o)\right| + 2n\max\left\{\sup_{1\le|v|\le n} D\left(f_{k_j}-f\right)(v), \sup_{|v|=n+1}\left|f_{k_j}(v^-) - f(v^-)\right|\right\}.
\end{aligned} \tag{5.22}$$

So $\|K_n f_{k_j} - K_n f\|_{\mathbf{w}} \to 0$ as $j \to \infty$. Therefore, K_n is compact, and thus, since M_ψ is bounded, $M_\psi K_n$ is also compact.

For $f \in L^\infty$, we have

$$\begin{aligned}
\left\|(M_\psi - M_\psi K_n)f\right\|_{\mathbf{w}} &= \sup_{|v|>n}|v|\left|\psi(v)f(v) - \psi(v^-)f(v^-) + \psi(v^-)K_n f(v^-)\right| \\
&= \max\left\{\sup_{|v|=n+1}|v|\left|\psi(v)\right|\left|f(v)\right|, \sup_{|v|>n+1}|v|\left|\psi(v)f(v) - \psi(v^-)f(v^-)\right|\right\} \\
&\le \sup_{|v|>n}|v|\left[\left|\psi(v)\right| + \left|\psi(v^-)\right|\right]\|f\|_\infty.
\end{aligned} \tag{5.23}$$

Therefore, we obtain

$$\begin{aligned}
\left\|M_\psi\right\|_e &\le \limsup_{n\to\infty}\left\|M_\psi - M_\psi K_n\right\| \\
&= \limsup_{n\to\infty}\sup_{\|f\|_\infty=1}\left\|(M_\psi - M_\psi K_n)f\right\|_{\mathbf{w}} \\
&\le B(\psi),
\end{aligned} \tag{5.24}$$

thus completing the proof. □

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