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Research Article

Weighted Composition Operators and Supercyclicity Criterion

Bahmann Yousefi[1] and Javad Izadi[2]

[1] *Department of Mathematics, Payame Noor University, P.O. Box: 71955-1368 Shiraz, Iran*
[2] *Department of Mathematics, Payame Noor University, P.O. Box: 19395-4697 Tehran, Iran*

Correspondence should be addressed to Bahmann Yousefi, b_yousefi@pnu.ac.ir

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We consider an equivalent condition to the property of Supercyclicity Criterion, and then we investigate this property for the adjoint of weighted composition operators acting on Hilbert spaces of analytic functions.

1. Introduction

Let T be a bounded linear operator on H. For $x \in H$, the orbit of x under T is the set of images of x under the successive iterates of T:

$$\operatorname{orb}(T, x) = \left\{x, Tx, T^2x, \dots\right\}. \tag{1.1}$$

The vector x is called supercyclic for T if $\mathbb{C}$ $\operatorname{orb}(T, x)$ is dense in H. Also a supercyclic operator is one that has a supercyclic vector. For some sources on these topics, see [1–16].

Let H be a separable Hilbert space of functions analytic on a plane domain G such that, for each λ in G, the linear functional of evaluation at λ given by $f \to f(\lambda)$ is a bounded linear functional on H. By the Riesz representation theorem, there is a vector K_λ in H such that $f(\lambda) = \langle f, K_\lambda\rangle$. We call K_λ the reproducing kernel at λ.

A complex-valued function φ on G is called a multiplier of H if $\varphi H \subset H$. The operator of multiplication by φ is denoted by M_φ and is given by $f \to \varphi f$.

If φ is a multiplier of H and ψ is a mapping from G into G, then $C_{\varphi,\psi} : H \to H$ by

$$C_{\varphi,\psi}(f)(z) = \varphi(z) f(\psi(z)) \tag{1.2}$$

for every $f \in H$ and $z \in G$ is called a weighted composition operators.

The holomorphic self-maps of the open unit disk $\mathbb{D}$ are divided into classes of elliptic and nonelliptic. The elliptic type is an automorphism and has a fixed point in $\mathbb{D}$. It is well known that this map is conjugate to a rotation $z \to \lambda z$ for some complex number λ with $|\lambda| = 1$. The maps of those which are not elliptic are called of non-elliptic type. The iterate of a non-elliptic map can be characterized by the Denjoy-Wolff Iteration theorem.

2. Main Results

We will investigate the property of Hypercyclicity Criterion for a linear operator and in the special case, we will give sufficient conditions for the adjoint of a weighted composition operator associated with elliptic composition function which satisfies the Supercyclicity Criterion.

Theorem 2.1 (Supercyclicity Criterion). *Let H be a separable Hilbert space and T is a continuous linear mapping on H. Suppose that there exist two dense subsets Y and Z in H, a sequence $\{n_k\}$ of positive integers, and also there exist mappings $S_{n_k} : Z \to H$ such that*

(1) $T^{n_k} S_{n_k} z \to z$ *for every* $z \in Z$,

(2) $\|T^{n_k} y\| \|S_{n_k} z\| \to 0$ *for every* $y \in Y$ *and every* $z \in Z$.

Then, T is supercyclic.

If an operator T holds in the assumptions of Theorem 2.1, then one says that T satisfies the Supercyclicity Criterion.

Definition 2.2. Let T be a bounded linear operator on a Hilbert space H. We refer to $\bigcup_{n\geq 1} \operatorname{Ker}(T^n)$ as the generalized kernel of T.

Theorem 2.3. *Let T be a bounded linear operator on a separable Hilbert space H with dense generalized kernel. Then, the following conditions are equivalent:*

(1) *T has a dense range,*

(2) *T is supercyclic,*

(3) *T satisfies the Supercyclicity Criterion.*

Proof. See [2, Corollary 3.3]. □

Remark 2.4. In [2], for the proof of implication (1) $\to$ (3) of Theorem 2.3, it has been shown that $T \bigoplus T$ is supercyclic which implies (by using Lemma 3.1 in [2]) that T satisfies the Supercyclicity Criterion. This implication can be proved directly without using Lemma 3.1 in [2], as follows: If T is a bounded linear operator on a separable Hilbert space H with dense range and dense generalized kernel, then it follows that T is supercyclic [1, Exercise 1.3]. Now suppose that h_0 is a supercyclic vector of T. Set $X_0 = \mathbb{C}\ \operatorname{orb}(T, h_0)$ and Y_0 = the generalized kernel of T. Since T is supercyclic, there exist sequences $\{n_j\}_j \subset \mathbb{N}$, $\{\alpha_j\}_j \subset \mathbb{C}$ and $\{f_j\}_j \subset H$ such that $f_j \to 0$ and $\alpha_j T^{n_j} f_j \to h_0$. Define $S_{n_k} : X_0 \to H$ by

$$S_{n_k}(\alpha T^m h_0) = \alpha\alpha_k T^m f_k. \tag{2.1}$$

Then, clearly, $T^{n_k}S_{n_k} \to I_{X_0}$ pointwise on X_0 and

$$\|T^{n_k}y\|\|S_{n_k}x\| \longrightarrow 0 \tag{2.2}$$

for every $y \in Y_0$ and every $x \in X_0$. Hence, T satisfies the Supercyclicity Criterion.

From now on let H be a Hilbert space of analytic functions on the open unit disc $\mathbb{D}$ such that H contains constants and the functional of evaluation at λ is bounded for all λ in $\mathbb{D}$. Also let $\varphi : \mathbb{D} \to C$ be a nonconstant multiplier of H and let ψ be an analytic map from $\mathbb{D}$ into $\mathbb{D}$ such that the composition operator C_ψ is bounded on H. We define the iterates $\psi_n = \psi \circ \psi \circ \cdots \circ \psi$ (n times). By ψ_n^{-1} or ψ_{-n} we mean the nth iterate of ψ^{-1}, hence $\psi_n^m = \psi_{mn}$ for $m = -1, 1$.

Definition 2.5. We say that $\{z_n\}_{n\geq 0}$ is a B-sequence for ψ if $\psi(z_k) = z_{k-1}$ for all $k \geq 1$.

Corollary 2.6. *Suppose that $\{z_n\}_{n\geq 0} \subset \mathbb{D}$ is a B -sequence for ψ and has limit point in $\mathbb{D}$. If $\varphi(z_0) = 0$, then $C^*_{\varphi,\psi}$ satisfies the Supercyclicity Criterion.*

Proof. Put $A = C_{\varphi,\psi}$. Since $\varphi(z_0) = 0$, we get $K_{z_i} \in \operatorname{Ker}(A^*)^n$ for all $i = 0, \dots, n-1$. Hence A^* has dense generalized kernel. Now let $\langle f, A^*K_{z_n}\rangle = 0$ for all n, thus $\varphi(z_n) \cdot f \circ \psi(z_n) = 0$ for all n. This implies that f is the zero constant function, because φ is nonconstant and $\{z_n\}_{n\geq 0}$ has limit point in $\mathbb{D}$. Thus, A^* has dense range and, by Theorem 2.3, the proof is complete. $\square$

Example 2.7. Let $\psi(z) = e^{-\sqrt{2}\pi i}z$, $\varphi(z) = z - (1/2)$, and define $z_n = (1/2)e^{\sqrt{2}n\pi i}$ for all $n \geq 0$. Now by Corollary 2.6, the operator $C^*_{\varphi,\psi}$ satisfies the Supercyclicity Criterion.

Theorem 2.8. *Let ψ be an elliptic automorphism with interior fixed point p and $\varphi : \mathbb{D} \to C$ satisfies the inequality $|\varphi(p)| < 1 \leq |\varphi(z)|$ for all z in a neighborhood of the unit circle. Then, the operator $C^*_{\varphi,\psi}$ satisfies the Supercyclicity Criterion.*

Proof. Put $\Psi = \alpha_p \circ \psi \circ \alpha_p$ and $\Phi = \varphi \circ \alpha_p$ where

$$\alpha_p(z) = \frac{p - z}{1 - \overline{p}z}. \tag{2.3}$$

Since Ψ is an automorphism with $\Psi(0) = 0$, thus Ψ is a rotation $z \to e^{i\theta}z$ for some $\theta \in [0, 2\pi]$ and every $z \in U$. Set $T = C^*_{\Phi,\Psi}$ and $S = C^*_{\varphi,\psi}$. Then, clearly $S^* = C_{\alpha_p}T^*C^{-1}_{\alpha_p}$, thus T is similar to S which implies that S satisfies the Supercyclicity Criterion if and only if T satisfies the Supercyclicity Criterion. Since $|\alpha_p(z)| \to 1^-$ when $|z| \to 1^-$, so $|\Phi(0)| < 1 \leq |\Phi(z)|$ for all z in a neighborhood of the unit circle. So, without loss of generality, we suppose that ψ is a rotation $z \to e^{i\theta}z$ and $|\varphi(0)| < 1 \leq |\varphi(z)|$ for all z in a neighborhood of the unit circle. Therefore, there exist a constant λ and a positive number $\delta < 1$ such that $|\varphi(z)| < \lambda < 1$ when $|z| < \delta$, and $|\varphi(z)| \geq 1$ when $|z| > 1 - \delta$. Set $U_1 = \{z : |z| < \delta\}$ and $U_2 = \{z : |z| > 1 - \delta\}$. Also, consider the sets

$$\begin{aligned} H_1 &= \operatorname{span}\{K_z : z \in U_1\}, \\ H_2 &= \operatorname{span}\{K_z : z \in U_2\}, \end{aligned} \tag{2.4}$$

where span $\{\cdot\}$ is the set of finite linear combinations of $\{\cdot\}$. By using the Hahn-Banach theorem, H_1 and H_2 are dense subsets of H. Since ψ is a rotation, the sequence $\{\psi_n^m(\lambda)\}_n$ is a subset of the compact set $\{z : |z| = \lambda\}$ for each λ in $\mathbb{D}$ and $m = -1, 1$. Now by, using the Banach-Steinhaus theorem, the sequence $\{K_{\psi_n^m(\lambda)}\}_n$ is bounded for each λ in $\mathbb{D}$ and $m = -1, 1$. Note that, for each $\mathbb{Z}$, $|z| = |\psi_n(z)|$. So, if $z \in U_1$, then $|\varphi(\psi_i(z))| < \lambda < 1$ and if $z \in U_2$, then $|\varphi(\psi_i^{-1}(z))| \geq 1$ for each positive integer i. Also, note that

$$S^n(K_z) = \left[\prod_{i=0}^{n-1} \overline{\varphi(\psi_i(z))}\right] K_{\psi_n(z)} \tag{2.5}$$

for every positive integer n and $z \in \mathbb{D}$ (see [12]). Now, if $z \in U_1$, then $S^n K_z \to 0$ as $n \to \infty$. Therefore the sequence $\{S^n\}$ converges pointwise to zero on the dense subset H_1. Define a sequence of linear maps $W_n : H_2 \to H_2$ by extending the definition

$$W_n K_z = \left[\prod_{i=1}^{n} \overline{\left(\varphi(\psi_i^{-1}(z))\right)^{-1}}\right] K_{\psi_n^{-1}(z)} \tag{2.6}$$

$(z \in U_2)$ linearly to H_2. Note that, for all $z \in U_2$, the sequence $\{W_n K_z\}_n$ is bounded and $S^n W_n K_z = K_z$ on H_2 which implies that $S^n W_n$ is identity on the dense subset H_2. Hence,

$$\|S^n f\| \|W_n g\| \longrightarrow 0 \tag{2.7}$$

for every $f \in H_1$ and every $g \in H_2$. Now, by Theorem 2.1, the proof is complete. □

Corollary 2.9. *Under the conditions of Theorem 2.8, $C^*_{\varphi,\psi} \bigoplus C^*_{\varphi,\psi}$ is supercyclic.*

Proof. It is clear since $C^*_{\varphi,\psi}$ satisfies the Supercyclicity Criterion. □

Example 2.10. Let $\varphi(z) = (3/2)z$ and $\psi(z) = e^{i\theta}z$. Then, the operator $C^*_{\varphi,\psi}$ satisfies the Supercyclicity Criterion, because 0 is an interior fixed point of ψ, and $\varphi(0) < 1 \leq |\varphi(z)|$ for $|z| > 2/3$.

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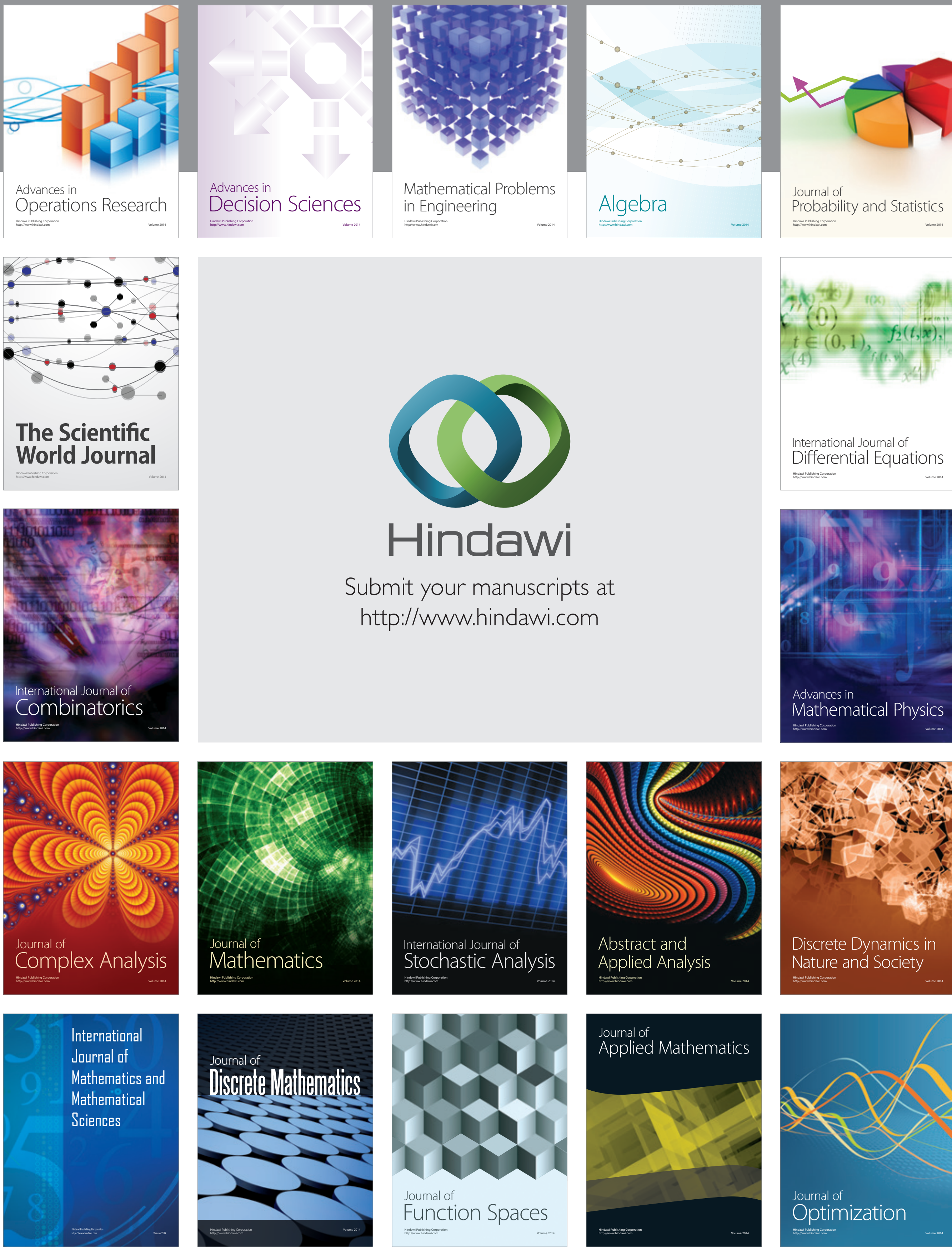

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