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*Research Article*

# Value Distribution for a Class of Small Functions in the Unit Disk

**Paul A. Gunsul**

*Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115, USA*

Correspondence should be addressed to Paul A. Gunsul, paul.gunsul@gmail.com

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If $f$ is a meromorphic function in the complex plane, R. Nevanlinna noted that its characteristic function $T(r,f)$ could be used to categorize $f$ according to its rate of growth as $|z| = r \to \infty$. Later H. Milloux showed for a transcendental meromorphic function in the plane that for each positive integer $k$, $m(r, f^{(k)}/f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set of finite measure where $m$ denotes the proximity function of Nevanlinna theory. If $f$ is a meromorphic function in the unit disk $D = \{z : |z| < 1\}$, analogous results to the previous equation exist when $\limsup_{r\to 1^-}(T(r,f)/\log(1/(1-r))) = +\infty$. In this paper, we consider the class of meromorphic functions $\mathcal{P}$ in $D$ for which $\limsup_{r\to 1^-}(T(r,f)/\log(1/(1-r))) < \infty$, $\lim_{r\to 1^-}T(r,f) = +\infty$, and $m(r, f'/f) = o(T(r,f))$ as $r \to 1$. We explore characteristics of the class and some places where functions in the class behave in a significantly different manner than those for which $\limsup_{r\to 1^-}(T(r,f)/\log(1/(1-r))) = +\infty$ holds. We also explore connections between the class $\mathcal{P}$ and linear differential equations and values of differential polynomials and give an analogue to Nevanlinna's five-value theorem.

## 1. Introduction

This paper uses notation from Nevanlinna theory which is summarized here for the reader's convenience. We denote by $n(r,f)$ the number of poles of $f$ in $|z| \le r < 1$, where each pole is counted according to its multiplicity. Also, $\overline{n}(r,f)$ counts the number of distinct poles of $f$ in $|z| \le r < 1$ disregarding multiplicity. If $x \ge 0$, then $\log^+ x = \max(0, \log x)$. We define the *proximity function* $m(r,f)$, the *counting function* $N(r,f)$, and the Nevanlinna characteristic function $T(r,f)$ as follows:

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(re^{i\theta}\right)\right| d\theta,$$

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r,$$
$$T(r,f) = m(r,f) + N(r,f). \tag{1.1}$$

Also, we have that

$$\overline{N}(r,f) = \int_0^r \frac{\overline{n}(t,f) - \overline{n}(0,f)}{t} dt + \overline{n}(0,f)\log r. \tag{1.2}$$

A meromorphic function $f$ in the unit disk $D = \{z : |z| < 1\}$ can be categorized according to the rate of growth of its Nevanlinna characteristic $T(r,f)$ as $|z| = r$ approaches one. If

$$\limsup_{r\to 1} \frac{T(r,f)}{-\log(1-r)} = +\infty, \tag{1.3}$$

many value distribution theorems analogous to those for transcendental meromorphic functions in the complex plane can be derived. In particular, results useful in studying solutions of linear differential equations which are analogous to theorems of H. Milloux can be shown—namely, for each positive integer $k$,

$$m\left(r, \frac{f^{(k)}}{f}\right) = o(T(r,f)) \tag{1.4}$$

as $r$ approaches one, possibly outside a set of finite measure where $m$ denotes the proximity function of Nevanlinna theory. For meromorphic functions of lesser growth than (1.3), analogous theorems need not hold. If $\mathcal{F}$ consists of those meromorphic functions $f$ in $D$ for which

$$\limsup_{r\to 1} \frac{T(r,f)}{-\log(1-r)} = \alpha(f) < \infty, \tag{1.5}$$

Shea and Sons [1] showed that $f'$ is in $\mathcal{F}$ for each $f$ in $\mathcal{F}$, but also that there exist functions $f$ in $\mathcal{F}$ with unbounded characteristic for which

$$m\left(r, \frac{f'}{f}\right) > \log\frac{1}{1-r} + \log\log\frac{1}{1-r} \tag{1.6}$$

on a sequence of $r$ approaching one. Thus all functions in $\mathcal{F}$ with unbounded characteristic need not satisfy (1.4) for $k = 1$.

In this paper, we denote by $\mathcal{D}$ those functions $f$ in $\mathcal{F}$ for which $T(r,f)$ is unbounded as $r$ approaches one and for which (1.4) *does* hold for $k = 1$. We derive striking properties of class $\mathcal{D}$ and make some connections between functions in class $\mathcal{D}$ and solutions of linear differential equations defined in $D$. For functions in $\mathcal{F}$ with $\alpha(f) > 0$, we develop an interesting theorem analogous to Nevanlinna's five-value theorem for functions in the plane. Further, we prove a value distribution theorem for differential polynomials

$$\Phi = \sum_{k=0}^{n} a_k f^{(k)}, \tag{1.7}$$

where $f$ is in $\mathcal{P}$, the $a_k$ are meromorphic functions in $D$, and $T(r, a_k) = o(T(r, f))$, as $r \to 1$.

Our paper proceeds as follows. In Section 2, we note examples of functions in $\mathcal{P}$ and properties of the class. In Section 3, we prove a uniqueness theorem for functions in class $\mathcal{F}$ (and hence in class $\mathcal{P}$). In Section 4, we look at differential equations for which functions in $\mathcal{P}$ are either coefficients or solutions, and in Section 5 we consider differential polynomials.

Much of the research reported here was part of the author's Ph.D. dissertation written at Northern Illinois University [2].

## 2. Properties and Examples of Functions in Class $\mathcal{P}$

First we note that $\mathcal{P}$ is not empty. For $\beta > 0$, the function $f$ defined in $D$ by

$$f(z) = \exp\left(\frac{\beta i}{1 - z}\right) \tag{2.1}$$

is in class $\mathcal{P}$, since

$$T(r, f) = \frac{\beta}{2\pi} \log\left(\frac{1 + r}{1 - r}\right) \tag{2.2}$$

by a calculation in Benbourenane [3] and by properties of $m$ and a lemma of Tsuji ([4, page 226]),

$$m\left(r, \frac{f'}{f}\right) = O\left(\log\log\frac{1}{1 - r}\right), \quad (r \longrightarrow 1). \tag{2.3}$$

Clearly $\alpha(f)$ as defined in (1.5) is $\beta/2\pi$ for this function.

The following proposition gives some simple closure properties of $\mathcal{P}$.

**Proposition 2.1.** *If $f$ and $g$ are in $\mathcal{P}$ and $c$ is a nonzero complex number, we have*

(i) *$cf$ is in $\mathcal{P}$;*

(ii) *$1/f$ is in $\mathcal{P}$;*

(iii) *$f^n$ is in $\mathcal{P}$ for each positive integer $n$;*

(iv) *$fg$ may not be in $\mathcal{P}$;*

(v) *$f + g$ may not be in $\mathcal{P}$.*

The proof of (i), (ii), and (iii) in Proposition 2.1 follows by easy calculation. To see (iv), let $g = 1/f$, and to see (v), let $g = -f$.

The complicated nature of class $\mathcal{P}$ is demonstrated by the following theorem whereby some sums and products are in $\mathcal{P}$.

**Theorem 2.2.** *Let $f$ be a meromorphic function in class $\mathcal{P}$.*

(i) *If $c$ is a nonzero complex number for which*

$$\liminf_{r\to 1}\frac{N(r,-c)}{T(r,f)} = 0, \tag{2.4}$$

*then $f + c$ is in $\mathcal{P}$.*

(ii) *If $g$ is a meromorphic function in $D$ which is not identically zero and such that $T(r,g) = o(T(r,f))$, $(r \to 1)$, and $m(r,g'/g) = o(T(r,f))$, $(r \to 1)$, then $fg$ is in $\mathcal{P}$.*

(iii) *There exists a Blaschke product $B$ such that $Bf$ is not in $\mathcal{P}$.*

(iv) *There exists a Blaschke product $B$ such that $Bf$ is in $\mathcal{P}$.*

*Remark 2.3.* In Nevanlinna theory, the Valiron deficiency of a complex value $c$ for a meromorphic function $f$ in $D$ is defined by

$$\Delta(c,f) = \liminf_{r\to 1}\frac{N(r,c)}{T(r,f)}. \tag{2.5}$$

It is known (cf. Theorem 2.20 on page 210 in [5]) that if

$$\lim_{r\to 1} T(r,f) = +\infty, \tag{2.6}$$

then

$$\lim_{r\to 1}\frac{N(r,a)}{T(r,f)} = 1 \tag{2.7}$$

except for at most a set of $a$-values of vanishing inner capacity. This fact enables us to show that the function $F$ in $\mathcal{P}$ defined by

$$F(z) = \exp\left(\frac{i}{1-z}\right) \tag{2.8}$$

has $F + c$ in $\mathcal{P}$ for all complex numbers $c$, because

$$\Delta(-c,F) = 0 \tag{2.9}$$

for all $c \neq 0$.

*Remark 2.4.* The following example illustrates part (ii) of Theorem 2.2.

*Example 2.5.* Let $f(z) = e^{i/(1-z)}$ and $g(z) = e^{(1+z)/(1-z)}$. Then $g$ is not identically zero, and it is well known that $T(r,g) = O(1)$. Therefore, $T(r,g) = o(T(r,f))$ as $r \to 1$. Also we have that

$$m\left(r, \frac{g'}{g}\right) \le T\left(r, \frac{2}{(1-z)^2}\right) = O(1) \quad \text{as } r \longrightarrow 1, \tag{2.10}$$

since $2/(1-z)^2$ is the quotient of two bounded, analytic functions in the unit disk. And so we have that $fg = e^{(i+1+z)/(1-z)} \in \mathcal{P}$.

We turn to the proof of Theorem 2.2.

*Proof of Part (i).* Let $g = f + c$. Then $g' = f'$. We will show that $g \in \mathcal{P}$.

First, by calculation and properties of the Nevanlinna characteristic, note that

$$T(r,g) = T(r,f) + O(1) \quad \text{as } r \longrightarrow 1. \tag{2.11}$$

Also, by calculation and properties of the proximity function, we get

$$\begin{aligned} m\left(r, \frac{g'}{g}\right) &= m\left(r, \frac{f'}{f+c}\right) \\ &\le m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f}{f+c}\right) \\ &\le m\left(r, \frac{f'}{f}\right) + m\left(r, 1 + \frac{c}{f+c}\right) \\ &\le m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{1}{f+c}\right) + O(1) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{2.12}$$

Now, since $f \in \mathcal{P}$,

$$\begin{aligned} m\left(r, \frac{f'}{f}\right) &= o(T(r,f)) \quad \text{as } r \longrightarrow 1, \\ &= o(T(r,g)) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{2.13}$$

Also, since $\Delta(-c,f) = 0$,

$$\limsup_{r \to 1} \frac{m(r, 1/(f+c))}{T(r,f)} = 0, \tag{2.14}$$

and so

$$m\left(r,\frac{1}{f+c}\right)=o(T(r,f))\quad\text{as } r\longrightarrow 1,$$
$$=o(T(r,g))\quad\text{as } r\longrightarrow 1. \tag{2.15}$$

Therefore, $g\in\mathcal{P}$ since $m(r,g'/g)=o(T(r,g))$ as $r\to 1$, and $T(r,g)$ is unbounded as $r\to 1$. □

*Proof of Part (ii).* First, $T(r,fg)$ is unbounded, since it can be shown that

$$T(r,fg)=T(r,f)+O(1)\quad\text{as } r\longrightarrow 1, \tag{2.16}$$

since $T(r,f)$ is unbounded.

Now note that $(fg)'/(fg)=g'/g+f'/f$. Therefore,

$$m\left(r,\frac{(fg)'}{fg}\right)\le m\left(r,\frac{g'}{g}\right)+m\left(r,\frac{f'}{f}\right),$$
$$=o(T(r,f))\quad\text{as } r\longrightarrow 1,$$
$$=o(T(r,fg))\quad\text{as } r\longrightarrow 1. \tag{2.17}$$

Thus $fg\in\mathcal{P}$. □

*Proof of Part (iii).* Let $B$ be the Blaschke product defined in [6, Proposition 6.1, page 273]. This Blaschke product has the feature that for any $\epsilon>0$ there exists an exceptional set $E_1\subset[0,1)$ satisfying

$$\int_{E_1}\frac{dr}{1-r}<\infty, \tag{2.18}$$

such that

$$\left|\frac{B'(z)}{B(z)}\right|=O\left(\left(\frac{1}{1-|z|}\right)^{1.5+\epsilon}\right),\qquad |z|\notin E_1, \tag{2.19}$$

and there exists a set $F_1\in[0,1)$, satisfying

$$\int_{F_1}\frac{dr}{1-r}=\infty, \tag{2.20}$$

and a constant $C>0$, such that

$$\left|\frac{B'(x)}{B(x)}\right|\ge\frac{C}{(1-x)^{1.5}}\log\frac{1}{1-x},\qquad x\in F_1\setminus E_1. \tag{2.21}$$

Let $g(z) = i/(1-z)$. Then $e^{g(z)} \in \mathcal{P}$. Now define $q(z) = B(z)e^{g(z)}$. We will now show that $q(z) \notin \mathcal{P}$.

First, it is easily shown that $T(r, e^{g(z)}) = T(r, q) + O(1)$ as $r \to 1$.

Note that since $(e^{g(z)})' = g' e^{g(z)}$, we have $g' = (e^{g(z)})'/e^{g(z)}$. Therefore, since $e^{g(z)} \in \mathcal{P}$,

$$m(r, g') = o\left(T\left(r, e^{g(z)}\right)\right) \quad \text{as } r \longrightarrow 1. \tag{2.22}$$

And so

$$m\left(r, \frac{B'}{B}\right) = m\left(r, \frac{q'}{q} - g'\right) \leq m\left(r, \frac{q'}{q}\right) + m(r, -g) + \log 2. \tag{2.23}$$

Therefore,

$$m\left(r, \frac{q'}{q}\right) \geq m\left(r, \frac{B'}{B}\right) - m(r, -g) - \log 2. \tag{2.24}$$

Using (2.21), we have on a small exceptional set with $|z| = r$ for $r \in F_1 \setminus E_1$,

$$\left|\frac{B'}{B}\right| \geq \frac{C}{(1-r)^{1.5}} \log \frac{1}{1-r}. \tag{2.25}$$

Taking the $\log^+$ of both sides, we get

$$\begin{aligned} \log^+ \left|\frac{B'}{B}\right| &\geq \log^+ \left(\frac{C}{(1-r)^{1.5}} \log \frac{1}{1-r}\right) \\ &\geq \log \frac{C}{(1-r)^{1.5}} \\ &\geq \log C + 1.5 \log \frac{1}{1-r}. \end{aligned} \tag{2.26}$$

So, calculating the following ratio yields

$$\frac{m(r, q'/q)}{T(r, q)} \geq \frac{m(r, B'/B)}{T(r, e^{g(z)})} - \frac{m(r, g')}{T(r, e^{g(z)})} - \frac{\log 2}{T(r, e^{g(z)})} \longrightarrow 3\pi > 0 \quad \text{as } r \longrightarrow 1. \tag{2.27}$$

Therefore, $q \notin \mathcal{P}$. □

*Proof of Part (iv).* Let $B$ be a Blaschke product with zeros $\{z_n\}$ such that $|z_n| = 1 - 1/n^5$ for all integers $n \geq 2$. Theorem B in Heittokangas [6] shows that $B'$ is in $H^p$ for $p$ in $(0, 3/4)$, so $B'/B$ is of bounded characteristic. Hence $Bf$ is in $\mathcal{P}$ for $f$ in $\mathcal{P}$. □

*Remark 2.6.* Since a Blaschke product is a bounded, analytic function, we see from parts (iii) and (iv) above that multiplication by such functions may or may not yield a function in $\mathcal{P}$.

Further study of examples in class $\mathcal{D}$ shows that the function $f$ defined by $f(z) = \exp(i/1 - z)$ has (1.4) holding for all $k$. However, there are also $f$ in $\mathcal{D}$ for which (1.4) does not hold for $k = 2$. We have the following theorem.

**Theorem 2.7.** *There exists an analytic function $h$ in $\mathcal{D}$ such that $m(r, h''/h) \neq o(T(r,h))$, as $r \to 1$.*

*Proof.* First, we begin by constructing a function which has unbounded characteristic as $r \to 1$, but its derivative is of bounded characteristic. This construction is from [7, page 557].

Let $a$ be an integer greater than or equal to 2. Define for $m \geq 1$,

$$Q_m = \sum_{k=1}^{m} a^k = \frac{a}{a-1}(a^m - 1),$$
$$t_m = 1 - \frac{\gamma}{Q_m}, \tag{2.28}$$

where $\gamma$ is a constant such that $0 < \gamma < 1$. Now define

$$F(z) = \int_0^z f(w)dw, \tag{2.29}$$

where

$$f(z) = \prod_{m=1}^{\infty}\left(1 + \left(\frac{z}{t_m}\right)^{a^m}\right). \tag{2.30}$$

Shea showed in [7] that $f$ is analytic in the unit disk and satisfies

$$\alpha \log\left(\frac{1}{1-r}\right) < T(r,f) \leq \log M(r,f) < \beta \log\left(\frac{1}{1-r}\right) \quad \text{as } r \longrightarrow 1, \tag{2.31}$$

where $\alpha$ and $\beta$ are constants such that

$$0 < \alpha < \frac{\gamma}{a}, \qquad \beta > \gamma + \frac{\log 2}{\log a}, \tag{2.32}$$

and $M(r,f)$ is the maximum modulus function for $f$. Choose $\gamma$ and $a$ such that $\gamma + \log 2/\log a < 1$, so we can take $\beta < 1$. This implies that $F$ is bounded in the unit disk by the following argument: for $z = re^{i\theta}$,

$$|F(z)| = \left|\int_0^z f(w)dw\right| \leq \int_0^r \left|f\left(\rho e^{i\theta}\right)\right| d\rho \leq \int_0^r M(\rho, f)d\rho. \tag{2.33}$$

By (2.31), we have that

$$M(\rho, f) < \frac{1}{(1-\rho)^{\beta}}. \tag{2.34}$$

Therefore, since $\beta < 1$, we have by a simple integration that

$$|F(z)| < \int_0^r \frac{1}{(1-\rho)^{\beta}} d\rho = O(1) \quad \text{as } r \longrightarrow 1. \tag{2.35}$$

Let $K(z) = \int_0^z F(w)dw$. Define $h(z) = K(z)e^{g(z)}$ where $g(z) = i/(1-z)$. Recall that $e^{g(z)} \in \mathcal{D}$. We now show that $h(z) \in \mathcal{D}$. First, we see that $T(r, K) = O(1)$ as $r \to 1$, by the following:

$$\begin{aligned} T(r,K) &= m(r,K) \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ \int_0^r \left|F\left(\rho e^{i\theta}\right)\right| d\rho\, d\theta, \end{aligned} \tag{2.36}$$

and since $F$ is bounded, there exists a constant $M$ such that

$$T(r,K) \leq \frac{1}{2\pi}\int_0^{2\pi} \log^+ \int_0^r M d\rho\, d\theta \leq \log M. \tag{2.37}$$

Thus, $T(r,h) \leq T(r,K) + T(r,e^{g(z)}) \leq T(r,e^{g(z)}) + O(1)$, as $r \to 1$. On the other hand,

$$\begin{aligned} T\left(r, e^{g(z)}\right) &= T\left(r, \frac{h}{K}\right) \\ &\leq T(r,h) + O(1) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{2.38}$$

Therefore, $T(r,h) \sim T(r,e^{g(z)})$ as $r \to 1$.

Now, we bound $m(r, h'/h)$ from above by using properties of the Nevanlinna characteristic:

$$\begin{aligned} m\left(r, \frac{h'}{h}\right) &= m\left(r, \frac{Kg'e^g + K'e^g}{Ke^g}\right) \\ &\leq m\left(r, \frac{K'}{K}\right) + m(r,g') + \log 2 \\ &\leq m(r,g') + O(1) \quad \text{as } r \longrightarrow 1 \\ &= o\left(T\left(r, e^{g(z)}\right)\right) \quad \text{as } r \longrightarrow 1 \\ &= o(T(r,h)) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{2.39}$$

And so we have $m(r, h'/h) = o(T(r,h))$ as $r \to 1$ and thus $h \in \mathcal{D}$.

Now we show that $m(r, h''/h) \neq o(T(r,h))$ as $r \to 1$. By a quick calculation, we have

$$\begin{aligned} m\left(r, \frac{K''}{K}\right) &\leq m\left(r, \frac{h''}{h} - 2g'\frac{K'}{K} - g'' - (g')^2\right) \\ &\leq m\left(r, \frac{h''}{h}\right) + m(r, 2) + m(r, g') + m\left(r, \frac{K'}{K}\right) \\ &\quad + m(r, g'') + 2m(r, g') + \log 4 \\ &\leq m\left(r, \frac{h''}{h}\right) + o(T(r,h)) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{2.40}$$

Also we have from the above construction $K'' = f$ so there exists an $\alpha > 0$ such that $m(r, K'') > \alpha \log(1/(1-r))$. And so,

$$\begin{aligned} m\left(r, \frac{K''}{K}\right) &\geq m(r, K'') - m(r, K) \\ &\geq \alpha \log\left(\frac{1}{1-r}\right) + O(1) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{2.41}$$

Combining (2.40) and (2.41), we have

$$\frac{m(r, h''/h)}{T(r,h)} \geq 2\pi\alpha \neq 0 \quad \text{as } r \longrightarrow 1. \tag{2.42}$$

□

*Remark 2.8.* If we define $A$ to be the set of functions in $F$ such that (1.4) holds for all positive integers $k$, Theorem 2.7 shows $A$ is properly contained in $F$. Further, we note that in the proof of Theorem 2.7 above $h = Ke^g$ can be replaced with $h = Kp$ where $p \in A$. Also the idea of the proof of Theorem 2.7 can be used to show that for $k > 1$ there exist functions $h$ in $F$ for which

$$m\left(r, \frac{h^{(j)}}{h}\right) = o(T(r,h)) \quad \text{as } r \longrightarrow 1 \tag{2.43}$$

for all integers $1 < j \leq k$, but

$$m\left(r, \frac{h^{(k+1)}}{h}\right) \neq o(T(r,h)) \quad \text{as } r \longrightarrow 1. \tag{2.44}$$

The function $h$ in the proof of Theorem 2.7 provides us with further information about $\mathcal{D}$.

**Theorem 2.9.** *There exists a function $h$ in $\mathcal{D}$ such that $h'$ is not in $\mathcal{D}$.*

*Proof.* The function $h = Ke^g$ of Theorem 2.7 is in $\mathcal{D}$. Using the Nevanlinna calculus and properties of $K$, one can show $m(r, h''/h') \neq o(T(r,h'))$ as $r \to 1$. We omit the details here (cf. [2]). □

For functions $f$ in class $\mathcal{F}$, we may call $\alpha(f)$ defined in (1.5) the index of $f$. In [1], Shea and Sons showed for $f$ in $\mathcal{F}$ that

$$\alpha(f') \leq \alpha(f)(1 + k(f)) + 1, \tag{2.45}$$

where

$$k(f) = \limsup_{r \to 1} \frac{\overline{N}(r, f)}{T(r, f) + 1}, \tag{2.46}$$

and this inequality is best possible. For analytic functions in class $\mathcal{D}$, we get

**Theorem 2.10.** *If $f$ is an analytic function in class $\mathcal{D}$, then*

(i) $T(r, f') \leq T(r, f) + o(T(r, f))$ *as* $r \to 1$;

(ii) $\alpha(f') \leq \alpha(f)$;

(iii) *if* $N(r, 1/f) = o(\log 1/(1 - r))$ *as* $r \to 1$, *then* $\alpha(f) = \alpha(f')$.

*Proof.* For part (i) since $f$ is analytic in $\mathcal{D}$, we have

$$\begin{aligned} T(r, f') = m(r, f') &\leq m\left(r, \frac{f'}{f}\right) + m(r, f) \\ &= m\left(r, \frac{f'}{f}\right) + T(r, f) \end{aligned} \tag{2.47}$$

from which the result follows. Using the definition of the index of $f'$ and of $f$, part (ii) comes from (i).

To see (iii), we observe

$$T(r, f) \leq T(r, f') + T\left(r, \frac{f}{f'}\right) = T(r, f') + T\left(r, \frac{f'}{f}\right) + O(1) \quad \text{as } r \longrightarrow 1. \tag{2.48}$$

Thus,

$$T(r, f) \leq T(r, f') + m\left(r, \frac{f'}{f}\right) + N(r, f') + N\left(r, \frac{1}{f}\right) + O(1) \quad \text{as } r \longrightarrow 1. \tag{2.49}$$

Dividing both sides of (2.49) by $\log 1/(1 - r)$ and taking the limit superior as $r$ approaches one, we get $\alpha(f) \leq \alpha(f')$. □

## 3. Connections between Class $\mathcal{P}$ and Differential Equations

We discuss some relationships between the coefficients of the linear differential equation and its solutions and how the coefficients and solutions relate to class $\mathcal{P}$. We consider the complex linear differential equation

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z)f = 0 \tag{3.1}$$

in the unit disk, with analytic coefficients.

There has been a tremendous amount of recent research on the relationship between the growth of the solutions of (3.1) and the growth of the analytic coefficients in the unit disk. Some recent papers include [8–10]. We now quote some of the important results that have a connection with class $\mathcal{F}$ and, therefore, class $\mathcal{P}$. The theorems use the definitions of the weighted Hardy space and weighted Bergman space which are stated below for convenience.

*Definition 3.1.* We say that an analytic function $f$ in the unit disk is in the weighted Hardy space $H_q^p$ for $0 < p < \infty$ and $0 \le q < \infty$ if

$$\sup_{0\le r<1}\left(1-r^2\right)^q\left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p\,d\theta\right)^{1/p} < \infty. \tag{3.2}$$

We say that $f$ is in $H_q^\infty$ if

$$\sup_{z\in D}\left(1-|z|^2\right)^q\,|f(z)| < \infty. \tag{3.3}$$

*Definition 3.2.* We say that an analytic function $f$ in the unit disk, $D$, is in the weighted Bergman space $A_q^p$ if the area integral over $D$ satisfies

$$\left(\int_D |f(z)|^p\left(1-|z|^2\right)^q d\sigma\right)^{1/p} < \infty \tag{3.4}$$

for $0 < p < \infty$ and $-1 < q < \infty$.

The theorems below also mention the Nevanlinna class $N$, the meromorphic functions of bounded characteristic in $D$. If a function is in $N$, then it is not in $\mathcal{P}$, since $\mathcal{P}$ only has functions of unbounded characteristic.

**Theorem 3.3** (see [10, page 320]). *Let f be a nontrivial solution of* (3.1) *with analytic coefficients* $a_j$, $j = 0,\ldots,n-1$, *in the unit disk. Then we have that*

(i) *if* $-1 < \alpha < 0$ *and* $a_j \in H^{1/(n-j)}_{(\alpha+1)(n-j)}$ *for all* $j = 0,\ldots,n-1$, *then* $f \in N$;

(ii) *if* $a_j \in A^{1/(n-j)}$ *for all* $j = 0,\ldots,n-1$, *or* $a_j \in A^1_{(n-j-1)}$ *for all* $j = 0,\ldots,n-1$, *then* $f \in N$;

(iii) *if* $a_j \in H^{1/(n-j)}_{n-j}$ *for all* $j = 0,\ldots,n-1$, *then* $f \in \mathcal{F}$.

**Theorem 3.4** (see [10, page 320]). *We have that*

(i) *if all nontrivial solutions* $f \in N$*, then the coefficients* $a_j \in \bigcap_{0<p<1/(n-j)} A^p$ *for all* $j = 0, \ldots, n-1$;

(ii) *if all nontrivial solutions* $f \in \mathcal{F}$*, then the coefficients* $a_j \in \bigcap_{0<p<1/(n-j)} H^p_{1/p}$ *for all* $j = 0, \ldots, n-1$.

We also have the following characterization, which uses the *order of growth* of $f$ in the unit disk defined as

$$\rho(f) = \limsup_{r \to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}. \tag{3.5}$$

**Theorem 3.5** (see [9, page 44]). *All solutions* $f$ *of* (3.1)*, where* $a_j$ *is analytic in* $D$ *for all* $j = 0, \ldots, k-1$*, satisfy* $\rho(f) = 0$ *if and only if* $a_j \in \bigcap_{0<p<1/(n-j)} A^p$ *for all* $j = 0, \ldots, n-1$.

When $n = 1$ in (3.1), we observe using Theorem 3.3, if $f'/f = -a_0 \in H^1_{(\alpha+1)}$ with $-1 < \alpha < 0$, then $f \in N$ and, therefore, $f \notin \mathcal{D}$. We can also conclude that if $f'/f = -a_0 \in A^1$, then $f \in N$ and so $f \notin \mathcal{D}$. Also, if $f'/f = -a_0 \in H^1_1$, then $f \in \mathcal{F}$, which means $f$ may be in $\mathcal{D}$. On the other hand, using Theorem 3.4, we have if $f \in \mathcal{D}$, then $f'/f = -a_0 \in \bigcap_{0<p<1} H^p_{1/p}$.

If $a_0(z) = -\beta i/(1-z)^2$, then $f(z) = e^{\beta i/(1-z)}$ is a solution of the differential equation. However, if $a_0 = \beta i/(1-z)^k$ for an integer $k \geq 3$, then $a_0$ is of bounded characteristic, but the solution

$$f = Ce^{\beta i/(1-z)^{k-1}} \tag{3.6}$$

has order $\rho = k - 2 > 0$ and, therefore, $f \notin \mathcal{F}$. This shows the delicate nature between the growth of the coefficient and the solution; that is, a subtle change in growth of the coefficient can result in a solution that is no longer considered slow growth.

When $n = 2$ in (3.1), Theorems 3.3 and 3.4 have the following corollary.

**Corollary 3.6.** *Let* $f$ *be a non-trivial solution of* (3.1) *with analytic coefficients* $a_0$ *and* $a_1$ *in the unit disk. Then*

(i) *if* $a_1 \in H^1_{(\alpha+1)}$ *and* $a_0 \in H^{1/2}_{2(\alpha+1)}$ *for* $-1 < \alpha < 0$*, then* $f \in N$ *and* $f \notin \mathcal{D}$;

(ii) *if* $a_1 \in A^1$ *and* $a_0 \in A^{1/2}$ *or* $a_0$ *and* $a_1 \in A^1_1$*, then* $f \in N$ *and* $f \notin \mathcal{D}$;

(iii) *if* $a_1 \in H^1_1$ *and* $a_0 \in H^{1/2}_2$*, then* $f \in \mathcal{F}$ *and, therefore, could be in* $\mathcal{D}$;

(iv) *if all non-trivial solutions* $f \in \mathcal{F}$*, then* $a_1 \in \bigcap_{0<p<1} H^p_{1/p}$ *and* $a_0 \in \bigcap_{0<p<1/2} H^p_{1/p}$.

The function $w = e^{\beta i/(1-z)}$ is a solution to the equation

$$w'' - \frac{\beta i}{(1-z)^2} w' - \frac{2\beta i}{(1-z)^3} w = 0. \tag{3.7}$$

Since $w$ has no zeros, another solution of the above second-order differential equation that is linearly independent of $w = f_1(z)$ is

$$f_2(z) = f_1(z)\int \frac{dz}{\left(f_1(z)\right)^2} = e^{\beta i/(1-z)}\int e^{-2\beta i/(1-z)}dz. \tag{3.8}$$

Computation shows

$$f_2(z) = e^{\beta i/(1-z)}\left(-(1-z) - 2\beta i \log\left(\frac{-2\beta i}{1-z}\right) + \sum_{n=1}^{\infty}\frac{-2\beta i}{n(n+1)!}\left(\frac{-2\beta i}{1-z}\right)^n\right). \tag{3.9}$$

We know that $f_1 \in \mathcal{D}$, but what can be said about $f_2$? The above form of $f_2$ makes it difficult to calculate the growth, but we do know that, by (iii) in Corollary 3.6, if $a_0 \in H_2^{1/2}$ and $a_1 \in H_1^1$, then $f_2 \in \mathcal{F}$.

*Example 3.7.* We show for $a_0 = -2\beta i/(1-z)^3$ and $a_1 = -\beta i/(1-z)^2$, $a_0 \in H_2^{1/2}$ and $a_1 \in H_1^1$. To see $a_0 \in H_2^{1/2}$, we first note

$$|a_0|^{1/2} = \left|\frac{-2\beta i}{(1-z)^3}\right|^{1/2} = \frac{\sqrt{2\beta}}{|1-z|^{3/2}}. \tag{3.10}$$

We integrate and apply a lemma from Tsuji [4, page 226] which states that, in particular,

$$\int_0^{2\pi}\frac{d\theta}{\left|1-re^{i\theta}\right|^{3/2}} = O\left(\frac{1}{(1-r)^{1/2}}\right), \tag{3.11}$$

and get that there exists a constant $M$ such that

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{\sqrt{2\beta}}{|1-z|^{3/2}}d\theta = \frac{\sqrt{2\beta}}{2\pi}\int_0^{2\pi}\frac{1}{|1-z|^{3/2}}d\theta \le \frac{\sqrt{2\beta}}{2\pi}\frac{M}{(1-r)^{1/2}}. \tag{3.12}$$

And so

$$\left(1-r^2\right)^2\left(\frac{1}{2\pi}\int_0^{2\pi}|a_0|^{1/2}d\theta\right)^2 \le (1+r)^2(1-r)^2\left(\frac{\sqrt{2\beta}M}{2\pi(1-r)^{1/2}}\right)^2 = (1+r)^2(1-r)C, \tag{3.13}$$

which goes to zero as $r \to 1$. Therefore, $a_0 \in H_2^{1/2}$.

A similar calculation shows that $a_1 \in H_1^1$.

The question as to whether $f_2 \in \mathcal{D}$ is not a trivial question as there exist examples, such as Example 3.9 below, where at least one solution is in class $\mathcal{D}$ and at least one solution is not in class $\mathcal{D}$.

*Example 3.8.* The function $h$ in Theorem 2.7 is a solution of (3.1) with $n = 2$ when $a_0 = -g'' - (g')^2$ and $a_1 = -(K'(2g') + K'')/(Kg' + K')$. Then since

$$\frac{h''}{h} = -a_1\frac{h'}{h} - a_0, \tag{3.14}$$

we have that

$$m\left(r, \frac{h''}{h}\right) \le m(r, -a_1) + m\left(r, \frac{h'}{h}\right) + m(r, -a_0) + \log 2. \tag{3.15}$$

Now, recall from the proof of Theorem 2.7 that

$$\alpha \log\frac{1}{1-r} \le m\left(r, \frac{h''}{h}\right) + o(T(r,h)) \quad \text{as } r \longrightarrow 1, \tag{3.16}$$

and since $h \in \mathcal{D}$, by (3.15), we conclude that at least one of $a_0$ or $a_1$ has index greater than or equal to $\alpha$. Therefore, as a consequence of Theorem 2.7, we have a growth estimate for the coefficients of this differential equation.

It can also be shown that $a_0 \in H_2^{1/2}$. However, $a_1$ is not in $H_1^1$, and thus the converse of Corollary 3.6(iii) is not true.

For differential equations of the form (3.1) where $n \ge 3$, we first quote two examples. The first example has some solutions of (3.1) in $\mathcal{D}$ and some not.

*Example 3.9* (see [9, Example 10, page 52]). The functions

$$f_1(z) = e^{i((1+z)/(1-z))} - e^{-i((1+z)/(1-z))}, \qquad f_2(z) = e^{i((1+z)/(1-z))}, \qquad f_3(z) = \frac{1+z}{1-z} \tag{3.17}$$

are linearly independent solutions of

$$f''' + a_1(z)f'' + a_1(z)f' + a_0(z)f = 0, \tag{3.18}$$

where

$$\begin{aligned} a_0(z) &= \frac{-8}{(1+z)(1-z)^5}, \\ a_1(z) &= \frac{4}{(1-z)^4} + 2\frac{3z^2+8z+5}{(1+z)^2(1-z)^2}, \\ a_2(z) &= -2\frac{3z+4}{(1+z)(1-z)}. \end{aligned} \tag{3.19}$$

It can be shown that $f_2 \in \mathcal{D}$. However, $f_3$ is of bounded characteristic and, therefore, $f_3 \notin \mathcal{D}$. (It is known that $f_1$ has order zero but unknown if $f_1 \in \mathcal{D}$.) Also, according to [9], we have that $a_0 \in \bigcap_{0<p<1/3} A^p_{-1/3}$, $a_1 \in \bigcap_{0<p<1/2} A^p_0$, and $a_2 \in \bigcap_{0<p<1} A^p_0$.

This next example is also from [9].

*Example 3.10* (see [9, Example 11, page 53]). The functions

$$f_1(z) = e^{i((1+z)/(1-z))} - e^{-i((1+z)/(1-z))}, \qquad f_2(z) = e^{i((1+z)/(1-z))},$$
$$f_{3,4}(z) = \left(\frac{1+z}{1-z}\right) \times e^{\pm i((1+z)/(1-z))} \tag{3.20}$$

are linearly independent solutions of

$$f^{(4)} + a_3(z)f^{(3)} + a_2(z)f'' + a_1(z)f' + a_0(z)f = 0, \tag{3.21}$$

where

$$\begin{aligned} a_0(z) &= \frac{16}{(1-z)^8}, \\ a_1(z) &= \frac{16(1+z^2)}{(1+z)(1-z)^5} - \frac{3+9z+9z^2+3z^3}{(1+z)^3(1-z)^3}, \\ a_2(z) &= \frac{8}{(1-z)^4} + 4\frac{9+18z+9z^2}{(1+z)^2(1-z)^2}, \\ a_3(z) &= -12\frac{1+z}{(1+z)(1-z)}. \end{aligned} \tag{3.22}$$

It is known that $f_2$, $f_3$, and $f_4$ are all in $\mathcal{D}$, and it is known that $f_1$ has order zero. Also, it can be shown that $a_0 \in \bigcap_{0<p<1/4} A_0^p$, $a_1 \in \bigcap_{0<p<1/3} A_{-1/3}^p$, $a_2 \in \bigcap_{0<p<1/2} A_0^p$, and $a_3 \in \bigcap_{0<p<1} A_0^1$.

Proceeding as in the discussion of Example 3.8, we have the following theorem.

**Theorem 3.11.** *If a function $f$ in $\mathcal{D}$ satisfies a differential equation of the form* (3.1) *such that*

$$\begin{aligned} m\left(r, \frac{f^{(n)}}{f}\right) &\neq o(T(r,f)) \quad \text{as } r \longrightarrow 1, \\ m\left(r, \frac{f^{(j)}}{f}\right) &= o(T(r,f)) \quad \text{as } r \longrightarrow 1 \end{aligned} \tag{3.23}$$

*for all integers $j$ such that $1 \leq j < n$, then at least one of the analytic coefficients $a_j$ has index greater than or equal to $\alpha$ as $r \to 1$ for some $\alpha > 0$.*

For nonhomogeneous differential equations of the form

$$f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_1(z)f' + a_0(z)f = a_n(z), \tag{3.24}$$

where $a_j$ is analytic in the disk for all $j = 0, \ldots, n$, we state a result from [9] that applies to class $\mathcal{F}$.

**Theorem 3.12** (see [9, Theorem 7, page 46]). *All solutions $f$ of* (3.24) *satisfy $\rho(f) = 0$ if and only if $\rho(a_n) = 0$ and $a_j \in \bigcap_{0<p<1/(k-j)} A^p$ for all $j = 0, \ldots, n-1$. Therefore, if all solutions $f$ of* (3.24) *are in $\mathcal{D}$, then $\rho(a_n) = 0$ and $a_j \in \bigcap_{0<p<1/(k-j)} A^p$ for all $j = 0, \ldots, n-1$.*

Theorems 3.3 and 3.4 do not tell the whole story regarding class $\mathcal{F}$. Instead of the coefficients being in a certain function class, what can we say about the solutions of (3.1) if we know the coefficients have a certain index in class $\mathcal{F}$? We show the following proposition.

**Proposition 3.13.** *Let $k$ be a positive integer. If $a_0$ is an analytic function in $D$ for which the index of $a_0$ is $\alpha(a_0) > k$, then $f \notin \mathcal{D}$ for $f \in \mathcal{F}$ when $f^{(k)} - a_0 f = 0$.*

*Proof.* Note that since $f^{(k)} - a_0 f = 0$, we have $a_0 = f^{(k)}/f$, and we want to show that

$$\limsup_{r \to 1^-} \frac{m(r, f'/f)}{T(r, f)} \neq 0. \tag{3.25}$$

Since $\alpha(a_0) > k$, there exists a real number $s > k$ such that $T(r, a_0) \geq s \log(1/(1-r))$ on some sequence of $r$'s as $r \to 1$. Also, since $f \in \mathcal{F}$, we have $T(r, f) \leq t \log(1/(1-r))$. Now, since $f \in \mathcal{F}$, $f^{(k)} \in \mathcal{F}$ and so $f^{(i)}/f^{(i-1)} \in \mathcal{F}$ for $1 \leq i \leq k$. By Shea and Sons [1], $m(r, f^{(i)}/f^{(i-1)}) \leq \log(1/(1-r)) + (2 + o(1)) \log\log(1/(1-r))$ for $1 \leq i \leq k$ as $r \to 1$. Now, we have the following:

$$\begin{aligned} m\left(r, \frac{f^{(k)}}{f}\right) &= m\left(r, \frac{f^{(k)}}{f^{(k-1)}} \frac{f^{(k-1)}}{f^{(k-2)}} \cdots \frac{f'}{f}\right) \\ &\leq m\left(r, \frac{f^{(k)}}{f^{(k-1)}}\right) + \cdots + m\left(r, \frac{f''}{f'}\right) + m\left(r, \frac{f'}{f}\right), \\ &\leq k \log\left(\frac{1}{1-r}\right) + k(2 + o(1)) \log\log\left(\frac{1}{1-r}\right) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{3.26}$$

So, since $a_0$ is analytic, we have

$$\begin{aligned} m\left(r, \frac{f'}{f}\right) &\geq m\left(r, \frac{f^{(k)}}{f}\right) - m\left(r, \frac{f^{(k)}}{f^{(k-1)}}\right) - \cdots - m\left(r, \frac{f''}{f'}\right) \\ &\geq s \log\left(\frac{1}{1-r}\right) - k \log\left(\frac{1}{1-r}\right) - k(2 + o(1)) \log\log\left(\frac{1}{1-r}\right) \quad \text{as } r \longrightarrow 1 \\ &= (s-k) \log\left(\frac{1}{1-r}\right) - k(2 + o(1)) \log\log\left(\frac{1}{1-r}\right) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{3.27}$$

Therefore, $m(r, f'/f)/T(r, f) \geq (s-k)/t > 0$ as $r \to 1$. □

With a similar argument as above, Proposition 3.13 is also true if $a_0$ is meromorphic and $N(r, a_0) = o(\log(1/(1-r)))$ as $r \to 1$.

## 4. The Identical Function Theorem

For functions in class $\mathcal{F}$, we have an analogue to the Nevanlinna five-value theorem which we quote as stated in [11].

**Theorem 4.1** (see [11, page 48]). *Suppose that $f_1$ and $f_2$ are meromorphic in the plane and let $E_j(a)$ be the set of points $z$ such that $f_j(z) = a (j = 1, 2)$. Then if $E_1(a) = E_2(a)$ for five distinct values of $a$, $f_1(z) \equiv f_2(z)$ or $f_1$ and $f_2$ are both constant.*

Our analogue and proof follow. The proof has a subtle difference from the direct analogue of the proof of Theorem 4.1 in [11].

**Theorem 4.2.** *Let $f_1(z)$ and $f_2(z)$ be meromorphic functions in class $\mathcal{F}$ such that $\alpha(f_1) \geq \alpha(f_2) > 0$ and let $E_j(a)$ be the set of points $z$ such that $f_j(z) = a$ for $j = 1, 2$. Then if $E_1(a) = E_2(a)$ for $q$ distinct values of $a$ such that $q$ is an integer and $q > 4 + 2/\alpha(f_1)$, then $f_1(z) \equiv f_2(z)$.*

*Proof.* Suppose $f_1$ and $f_2$ are not identical and that $\{a_1, a_2, \ldots, a_q\}$ are $q$ distinct complex numbers such that $E_1(a_\nu)$ and $E_2(a_\nu)$ are identical for $\nu = 1, 2, \ldots, q$ and $q$ is an integer greater than or equal to $4 + 2/\alpha(f_1)$. We write the following notations:

$$N_\nu(r) = \overline{N}\left(r, \frac{1}{f_1(z) - a_\nu}\right) = \overline{N}\left(r, \frac{1}{f_2(z) - a_\nu}\right) \quad (\nu = 1, 2, \ldots, q). \tag{4.1}$$

Now using a reformulation of Nevanlinna's inequality for functions in class $\mathcal{F}$ [1], we have the following for all $r < 1$:

$$(q-2)T(r, f_1) \leq \sum_{\nu=1}^{q} N_\nu(r) + \log\frac{1}{1-r} + O\left(\log\log\frac{1}{1-r}\right) \quad \text{as } r \longrightarrow 1, \tag{4.2}$$

$$(q-2)T(r, f_2) \leq \sum_{\nu=1}^{q} N_\nu(r) + \log\frac{1}{1-r} + O\left(\log\log\frac{1}{1-r}\right) \quad \text{as } r \longrightarrow 1. \tag{4.3}$$

Now assume that $\alpha(f_1) \geq \alpha(f_2) > 0$. Then from the definition of index we have that for $0 < \epsilon < \alpha(f_1)$, there exists a sequence $\{r_m\} \to 1$ such that

$$\begin{aligned} &T(r_m, f_1) > (\alpha(f_1) - \epsilon)\log\frac{1}{1-r_m} \quad \forall m \longrightarrow \infty, \\ &\text{or} \quad \log\frac{1}{1-r_m} < \frac{1}{\alpha(f_1) - \epsilon}T(r_m, f_1). \end{aligned} \tag{4.4}$$

So, combining (4.2) and (4.3) and using (4.4), we get

$$\begin{aligned}(q-2)\left(T(r_m,f_1)+T(r_m,f_2)\right) &\le 2\sum_{\nu=1}^{q}N_\nu(r_m)+(2+o(1))\log\frac{1}{1-r_m} \quad \text{as } m\longrightarrow\infty \\ &< 2\sum_{\nu=1}^{q}N_\nu(r_m)+\frac{2+o(1)}{\alpha(f_1)-\epsilon}T(r_m,f_1) \quad \text{as } m\longrightarrow\infty,\end{aligned} \tag{4.5}$$

which leads to

$$\left(q-2-\frac{2+o(1)}{\alpha(f_1)-\epsilon}\right)\left(T(r_m,f_1)+T(r_m,f_2)\right)\le 2\sum_{\nu=1}^{q}N_\nu(r_m). \tag{4.6}$$

Since $f_1$ and $f_2$ are not identical, we have

$$\begin{aligned}T\left(r_m,\frac{1}{(f_1-f_2)}\right) &= T(r_m,f_1-f_2)+O(1) \quad \text{as } m\longrightarrow\infty \\ &\le T(r_m,f_1)+T(r_m,f_2)+O(1) \quad \text{as } m\longrightarrow\infty \\ &\le \frac{2}{q-2-(2+o(1))/(\alpha(f_1)-\epsilon)}\sum_{\nu=1}^{q}N_\nu(r_m)+O(1) \quad \text{as } m\longrightarrow\infty.\end{aligned} \tag{4.7}$$

On the other hand, every common root of the equations $f_\nu(z)=a$ is a pole of $1/(f_1-f_2)$, and so we have

$$\begin{aligned}\sum_{\nu=1}^{q}N_\nu(r_m) &\le \overline{N}\left(r_m,\frac{1}{(f_1-f_2)}\right)+O(1) \\ &\le \frac{2}{q-2-(2+o(1))/(\alpha(f_1)-\epsilon)}\sum_{\nu=1}^{q}N_\nu(r_m)+O(1) \quad \text{as } r\longrightarrow 1,\end{aligned} \tag{4.8}$$

which gives a contradiction since $q>4+2/\alpha(f_1)$ implies $2/(q-2-(2+o(1))/(\alpha(f_1)-\epsilon))<1$ as $r\to 1$, unless $\sum_{\nu=1}^{q}N_\nu(r_m)=O(1)$. This, however, cannot occur since $T(r,f_1)$ and $T(r,f_2)$ are unbounded. Therefore, the result follows. □

*Remark 4.3.* Since $\mathcal{P}\subset\mathcal{F}$, Theorem 4.2 gives conditions for when two functions in $\mathcal{P}$ are identical.

## 5. Values of Differential Polynomials

We now turn our focus on determining values for differential polynomials in the disk as it relates to class $\mathcal{P}$. In a preliminary report by Sons [12], the author explores various results for functions satisfying (1.3) in the disk and their analogues for functions in class $\mathcal{F}$. Some of these results for class $\mathcal{F}$ can be refined further if we restrict the functions to class $\mathcal{P}$. We state a theorem from Sons (without proof) and follow it with a refinement for class $\mathcal{P}$.

**Theorem 5.1** (see [12, Theorem 4]). *Let $f$ be a meromorphic function in $D$ which is in class $\mathcal{F}$ and for which*

$$N\left(r, \frac{1}{f}\right) + N(r, f) = o(T(r, f)), \quad \text{as } r \longrightarrow 1. \tag{5.1}$$

*Let $n$ be a positive integer, and for $k = 0, 1, 2, \ldots, n$ let $a_k$ be a meromorphic function in $D$ for which*

$$T(r, a_k) = o(T(r, f)), \quad \text{as } r \longrightarrow 1. \tag{5.2}$$

*If $\psi$ is defined in $D$ by*

$$\psi = \sum_{k=0}^{n} a_k f^{(k)} \tag{5.3}$$

*and $\psi$ is nonconstant, then $\psi$ assumes every complex number except possibly zero infinitely often provided the index of $f$ is $\alpha > 1 + n(n+1)/2$.*

**Theorem 5.2.** *Let $f$ be a meromorphic function in $D$ which is in class $\mathcal{D}$ and for which*

$$N\left(r, \frac{1}{f}\right) + N(r, f) = o(T(r, f)), \quad \text{as } r \longrightarrow 1. \tag{5.4}$$

*Let $n$ be a positive integer, and for $k = 0, 1, 2, \ldots, n$, let $a_k$ be a meromorphic function in $D$ for which*

$$T(r, a_k) = o(T(r, f)), \quad \text{as } r \longrightarrow 1. \tag{5.5}$$

*Also, define $E$ to be the set $\{k : m(r, f^{(k)}/f) = o(T(r, f)) \text{ as } r \rightarrow 1\}$. If $\psi$ is defined in $D$ by*

$$\psi = \sum_{k=0}^{n} a_k f^{(k)} \tag{5.6}$$

*and $\psi$ is nonconstant, then $\psi$ assumes every complex number except possibly zero infinitely often, provided the index of $f$ is $\alpha > 1 + n(n+1)/2 - \sum E$, where $\sum E$ is the sum of the values of $E$.*

*Proof.* Since class $\mathcal{F}$ is closed under differentiation, addition, and multiplication, we know that $\psi$ is in class $\mathcal{F}$. Therefore, we can apply the reformulation of the Second Fundamental theorem for class $\mathcal{F}$ [1] to $\psi$. Thus, using $0$, $\infty$, and $c$, a nonzero complex number, we get

$$\begin{aligned} T(r, \psi) \leq{} & \overline{N}\left(r, \frac{1}{\psi}\right) + \overline{N}\left(r, \frac{1}{\psi - c}\right) + \overline{N}(r, \psi) + \log\left(\frac{1}{1-r}\right) \\ & + O\left(\log\log\left(\frac{1}{1-r}\right)\right), \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{5.7}$$

Since poles of $\psi$ come from poles of $a_k$ or $f$, we have an upper bound for $\overline{N}(r,\psi)$:

$$\overline{N}(r,\psi) \leq \overline{N}(r,f) + \sum_{k=0}^{n} \overline{N}(r,a_k). \tag{5.8}$$

From the hypothesis, we then have

$$\overline{N}(r,\psi) \leq \overline{N}(r,f) + o(T(r,f)), \quad \text{as } r \longrightarrow 1. \tag{5.9}$$

Therefore, using (5.9) and the First Fundamental theorem, we get

$$\begin{aligned}
T(r,\psi) &= m\left(r,\frac{1}{\psi}\right) + N\left(r,\frac{1}{\psi}\right) + O(1), \quad (r \longrightarrow 1) \\
&\leq \overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}\left(r,\frac{1}{\psi - c}\right) + \overline{N}(r,\psi) + \log\left(\frac{1}{1-r}\right) \\
&\quad + O\left(\log\log\left(\frac{1}{1-r}\right)\right), \quad \text{as } r \longrightarrow 1, \\
&\leq \overline{N}\left(r,\frac{1}{\psi}\right) + \overline{N}\left(r,\frac{1}{\psi - c}\right) + \overline{N}(r,f) + \log\left(\frac{1}{1-r}\right) \\
&\quad + O\left(\log\log\left(\frac{1}{1-r}\right)\right), \quad \text{as } r \longrightarrow 1.
\end{aligned} \tag{5.10}$$

Now, solving for $m(r,1/\psi)$ in the above calculation, we have the following inequality:

$$\begin{aligned}
m\left(r,\frac{1}{\psi}\right) &\leq \overline{N}\left(r,\frac{1}{\psi}\right) - N\left(r,\frac{1}{\psi}\right) + \overline{N}\left(r,\frac{1}{\psi - c}\right) + \overline{N}(r,f) \\
&\quad + \log\left(\frac{1}{1-r}\right) + o(T(r,f)) + O\left(\log\log\left(\frac{1}{1-r}\right)\right) \quad \text{as } r \longrightarrow 1.
\end{aligned} \tag{5.11}$$

Since $\overline{N}(r,1/\psi) \leq N(r,1/\psi)$, the $N(r,1/\psi)$ terms cancel, and so we can say that

$$\begin{aligned}
m\left(r,\frac{1}{\psi}\right) &\leq \overline{N}\left(r,\frac{1}{\psi - c}\right) + \overline{N}(r,f) + \log\left(\frac{1}{1-r}\right) + o(T(r,f)) \\
&\quad + O\left(\log\log\left(\frac{1}{1-r}\right)\right) \quad \text{as } r \longrightarrow 1.
\end{aligned} \tag{5.12}$$

So, using the First Fundamental theorem, properties of the proximity function and (5.12) give us the following:

$$\begin{aligned} T(r,f) &= m\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f}\right) + O(1) \\ &\le m\left(1,\frac{1}{\psi}\frac{\psi}{f}\right) + N\left(r,\frac{1}{f}\right) + O(1) \\ &\le m\left(r,\frac{1}{\psi}\right) + m\left(r,\frac{\psi}{f}\right) + N\left(r,\frac{1}{f}\right) + O(1) \\ &\le \overline{N}\left(r,\frac{1}{\psi-c}\right) + \overline{N}(r,f) + \log\left(\frac{1}{1-r}\right) + m\left(r,\frac{\psi}{f}\right) \\ &\quad + N\left(r,\frac{1}{f}\right) + o(T(r,f)) + O\left(\log\log\left(\frac{1}{1-r}\right)\right) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{5.13}$$

Noticing the fact that $\overline{N}(r,f) \le N(r,f)$ and using the hypothesis that

$$N(r,f) + N\left(r,\frac{1}{f}\right) = o(T(r,f)) \quad \text{as } r \longrightarrow 1, \tag{5.14}$$

we can say that

$$\begin{aligned} T(r,f) \le \overline{N}\left(r,\frac{1}{\psi-c}\right) + m\left(r,\frac{\psi}{f}\right) + \log\left(\frac{1}{1-r}\right) + o(T(r,f)) \\ + O\left(\log\log\left(\frac{1}{1-r}\right)\right) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{5.15}$$

We now estimate $m(r,\psi/f)$. By using properties of the proximity function, we get

$$\begin{aligned} m\left(r,\frac{\psi}{f}\right) &= m\left(r,\frac{\sum_{k=0}^{n} a_k f^{(k)}}{f}\right) = m\left(r,\sum_{k=0}^{n} a_k \frac{f^{(k)}}{f}\right) \\ &\le \sum_{k=0}^{n} m\left(r, a_k \frac{f^{(k)}}{f}\right) + \log(n+1) \\ &\le \sum_{k=0}^{n} m(r,a_k) + \sum_{k=1}^{n} m\left(r,\frac{f^{(k)}}{f}\right) + \log(n+1). \end{aligned} \tag{5.16}$$

Recall the set $E = \{k : m(r, f^{(k)}/f) = o(T(r,f)) \text{ as } r \to 1\}$. Notice that since $f \in \mathcal{D}$, $E$ is not empty. The set $E$ also allows us to split the following sum into two pieces. Indeed,

$$\sum_{k=1}^{n} m\left(r,\frac{f^{(k)}}{f}\right) = \sum_{k\in E} m\left(r,\frac{f^{(k)}}{f}\right) + \sum_{k\notin E} m\left(r,\frac{f^{(k)}}{f}\right). \tag{5.17}$$

But now we can say that

$$\sum_{k\in E} m\left(r,\frac{f^{(k)}}{f}\right) = o(T(r,f)) \quad \text{as } r \longrightarrow 1, \tag{5.18}$$

since this is true for each $k \in E$. Therefore, using (5.18) and the hypothesis

$$T(r,a_k) = o(T(r,f)), \quad \text{as } r \longrightarrow 1, \tag{5.19}$$

we can update (5.16) to say that

$$m\left(r,\frac{\varphi}{f}\right) \leq \sum_{k\notin E} m\left(r,\frac{f^{(k)}}{f}\right) + o(T(r,f)) \quad \text{as } r \longrightarrow 1. \tag{5.20}$$

We use (3.26) to say that

$$m\left(r,\frac{f^{(k)}}{f}\right) \leq k\log\frac{1}{1-r} + o(T(r,f)) \quad \text{as } r \longrightarrow 1, \tag{5.21}$$

and, thus, (5.20) becomes

$$m\left(r,\frac{\varphi}{f}\right) \leq \sum_{k\notin E} k\log\frac{1}{1-r} + o(T(r,f)) \quad \text{as } r \longrightarrow 1. \tag{5.22}$$

We can calculate $\sum_{k\notin E} k\log(1/(1-r))$ by noting that

$$\sum_{k\notin E} k\log\frac{1}{1-r} = \log\frac{1}{1-r}\sum_{k\notin E} k = \left(\frac{n(n+1)}{2} - \sum E\right)\log\frac{1}{1-r}, \tag{5.23}$$

where $\sum E$ is the sum of the elements in $E$. Note that $n(n+1)/2 - \sum E = \beta \geq 0$. Thus, (5.20) becomes

$$m\left(r,\frac{\varphi}{f}\right) \leq \left(\frac{n(n+1)}{2} - \sum E\right)\log\frac{1}{1-r} + o(T(r,f)) \quad \text{as } r \longrightarrow 1. \tag{5.24}$$

Therefore, we can now update (5.15) to say

$$\begin{aligned} T(r,f) \leq{}& \overline{N}\left(r,\frac{1}{\varphi - c}\right) + \left(\frac{n(n+1)}{2} - \sum E + 1\right)\log\frac{1}{1-r} \\ &+ o(T(r,f)) + O\left(\log\log\left(\frac{1}{1-r}\right)\right) \quad \text{as } r \longrightarrow 1. \end{aligned} \tag{5.25}$$

Since the index of $f$ is equal to $\alpha > n(n+1)/2 - \sum E + 1$, we have that

$$\gamma T(r,f) \leq \overline{N}\left(r, \frac{1}{\psi - c}\right) + o(T(r,f)) \quad \text{as } r \longrightarrow 1, \tag{5.26}$$

where $\gamma > 0$. Since $T(r,f)$ is unbounded, we have proved the claim that $\psi$ assumes every complex number except possibly zero infinitely often. □

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