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Research Article

A New Composite General Iterative Scheme for Nonexpansive Semigroups in Banach Spaces

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We introduce a new general composite iterative scheme for finding a common fixed point of nonexpansive semigroups in the framework of Banach spaces which admit a weakly continuous duality mapping. A strong convergence theorem of the purposed iterative approximation method is established under some certain control conditions. Our results improve and extend announced by many others.

1. Introduction

Throughout this paper we denoted by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and all positive real numbers, respectively. Let X be a real Banach space, and let C be a nonempty closed convex subset of X. A mapping T of C into itself is said to be *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|$ for each $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. We know that $F(T)$ is nonempty if C is bounded; for more detail see [1]. A one-parameter family $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ from C of X into itself is said to be a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s+t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}^+$;

(iii) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous;

(iv) $\|T(t)x - T(t)y\| \leq \|x-y\|$ for all $x, y \in C$ and $t \in \mathbb{R}^+$.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) := \cap_{t\in\mathbb{R}^+} F(T(t)) = \{x \in C : T(t)x = x\}$. We know that $F(\mathcal{S})$ is nonempty if C is bounded; see [2]. Recall that a self-mapping $f : C \to C$ is a *contraction* if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for each $x, y \in C$. As in [3], we use the notation $\prod_C$ to denote the collection of all contractions on C, that is, $\prod_C = \{f : C \to C \text{ a contraction}\}$. Note that each $f \in \prod_C$ has a unique fixed point in C.

In the last ten years, the iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [3–5]. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \ge \overline{\gamma}\|x\|^2 \quad \forall x \in H. \tag{1.1}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F} \frac{1}{2}\langle Ax, x \rangle - \langle x, b \rangle, \tag{1.2}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H.

In 2003, Xu [3] proved that the sequence $\{x_n\}$ generated by

$$\begin{gathered} x_0 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad \forall n \ge 0, \end{gathered} \tag{1.3}$$

converges strongly to the unique solution of the minimization problem (1.2) provided that the sequence $\{\alpha_n\}$ satisfies certain conditions. Using the viscosity approximation method, Moudafi [6] introduced the iterative process for nonexpansive mappings (see [3, 7] for further developments in both Hilbert and Banach spaces) and proved that if H is a real Hilbert space, the sequence $\{x_n\}$ generated by the following algorithm:

$$\begin{gathered} x_0 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad \forall n \ge 0, \end{gathered} \tag{1.4}$$

where $f : C \to C$ is a contraction mapping with constant $\alpha \in (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfies certain conditions, converges strongly to a fixed point of T in C which is unique solution x^* of the variational inequality:

$$\left\langle (f - I)x^*, y - x^* \right\rangle \le 0, \quad \forall y \in F(T). \tag{1.5}$$

In 2006, Marino and Xu [8] combined the iterative method (1.3) with the viscosity approximation method (1.4) considering the following general iterative process:

$$\begin{gathered} x_0 \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \ge 0, \end{gathered} \tag{1.6}$$

where $0 < \gamma < \overline{\gamma}/\alpha$. They proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to a unique solution x^* of the variational inequality:

$$\left\langle (\gamma f - A)x^*, y - x^* \right\rangle \leq 0, \quad \forall y \in F(T), \tag{1.7}$$

which is the optimality condition for the minimization problem:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.8}$$

where C is the fixed point set of a nonexpansive mapping T and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). Kim and Xu [9] studied the sequence generated by the following algorithm:

$$\begin{gathered} x_1 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n u + \left(1-\beta_n\right)y_n, \quad \forall n \geq 0, \end{gathered} \tag{1.9}$$

and proved strong convergence of scheme (1.9) in the framework of uniformly smooth Banach spaces. Later, yao, et al. [10] introduced a new iteration process by combining the modified Mann iteration [9] and the viscosity approximation method introduced by Moudafi [6]. Let C be a closed convex subset of a Banach space, and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$ and $f \in \prod_C$. Define $\{x_n\}$ in the following way:

$$\begin{gathered} x_1 \in C \text{ chosen arbitrarily;} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n; \\ x_{n+1} = \beta_n f(x_n) + \left(1-\beta_n\right)y_n, \quad \forall n \geq 0, \end{gathered} \tag{1.10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. They proved under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ converges strongly to a fixed point of T. Recently, Chen and Song [11] studied the sequence generated by the algorithm in a uniformly convex Banach space, as follows:

$$\begin{gathered} x_1 \in C \text{ chosen arbitrarily;} \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad \forall n \in \mathbb{N}, \end{gathered} \tag{1.11}$$

and they proved that the sequence $\{x_n\}$ defined by (1.11) converges strongly to the unique solution of the variational inequality:

$$\left\langle (f - I)x^*, J\left(y - x^*\right) \right\rangle \leq 0, \quad \forall y \in F(T). \tag{1.12}$$

In 2010, Sunthrayuth and Kumam [12] introduced the a general iterative scheme generated by

$$x_0 \in C \text{ chosen arbitrarily;}$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad \forall n \geq 0, \tag{1.13}$$

for the approximation of common fixed point of a one-parameter nonexpansive semigroup in a Banach space under some appropriate control conditions. They proved strong convergence theorems of the iterative scheme which solve some variational inequality. Very recently, Kumam and Wattanawitoon [13] studied and introduced a new composite explicit viscosity iteration method of fixed point solutions of variational inequalities for nonexpansive semigroups in Hilbert spaces. They proved strong convergence theorems of the composite iterative schemes which solve some variational inequalities under some appropriate conditions. In the same year, Sunthrayuth et al. [14] introduced a general composite iterative scheme for nonexpansive semigroups in Banach spaces. They established some strong convergence theorems of the general iteration scheme under different control conditions.

In this paper, motivated by Yao et al. [10], Sunthrayuth, and Kumam [12] and Kumam and Wattanawitoon [13] we introduce a new general iterative algorithm (3.23) for finding a common point of the set of solution of some variational inequality for nonexpansive semigroups in Banach spaces which admit a weakly continuous duality mapping and then proved the strong convergence theorem generated by the proposed iterative scheme. The results presented in this paper improve and extend some others from Hilbert spaces to Banach spaces and some others as special cases.

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ (resp., $x_n \rightharpoonup^* x$) to indicate that the sequence $\{x_n\}$ weakly (resp., weak*) converges to x; as usual $x_n \to x$ will symbolize strong convergence; also, a mapping I denote the identity mapping. Let X be a real Banach space, and let X^* be its dual space. Let $U = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be *uniformly convex* if, for each $\epsilon \in (0,2]$, there exists a $\delta > 0$ such that for each $x, y \in U$, $\|x-y\| \geq \epsilon$ implies $\|x+y\|/2 \leq 1-\delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex (see also [15]). A Banach space is said to be *smooth* if the limit $\lim_{t\to 0}\|x+ty\| - \|x\|/t$ exists for each $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$.

Let $\varphi : [0,\infty) := \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called a *gauge function* . The duality mapping $J_\varphi : X \to 2^{X^*}$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \left\{f^* \in X^* : \langle x, f^*\rangle = \|x\|\varphi(\|x\|), \left\|f^*\right\| = \varphi(\|x\|), \ \forall x \in X\right\}, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality paring. In particular, the duality mapping with the gauge function $\varphi(t) = t$, denoted by J, is referred to as the normalized duality mapping. Clearly, there holds the relation $J_\varphi(x) = (\varphi(\|x\|)/\|x\|)J(x)$ for each $x \neq 0$ (see [16]).

Browder [16] initiated the study of certain classes of nonlinear operators by means of the duality mapping J_φ. Following Browder [16], we say that Banach space X has a *weakly continuous duality mapping* if there exists a gauge function φ for which the duality mapping $J_\varphi(x)$ is single-valued and continuous from the weak topology to the weak* topology; that is, for each $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J(x_n)\}$ converges weakly* to $J_\varphi(x)$. It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set $\Phi(t) = \int_0^t \varphi(\tau)d\tau$, for all $t \geq 0$; then $J_\varphi(x) = \partial\Phi(\|x\|)$, where ∂ denotes the subdifferential in the sense of convex analysis (recall that the subdifferential of the convex function $\phi : X \to \mathbb{R}$ at $x \in X$ is the set $\partial\phi(x) = \{x^* \in X; \phi(y) \geq \phi(x) + \langle x^*, y - x\rangle$, for all $y \in X\})$.

In a Banach space having a weakly continuous duality mapping J_φ with a gauge function φ, we defined an operator A is to be strongly positive (see [17]) if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, J_\varphi(x)\rangle \geq \overline{\gamma}\|x\|\varphi(\|x\|), \tag{2.2}$$

$$\|aI - bA\| = \sup_{\|x\|\leq 1} \left|\langle (aI - bA)x, J_\varphi(x)\rangle\right|, \quad a \in [0,1],\ b \in [-1,1]. \tag{2.3}$$

If $X := H$ is a real Hilbert space, then the inequality (2.2) reduces to (1.1).

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [18].

Lemma 2.1 (see [18]). *Assume that a Banach space X has a weakly continuous duality mapping J_φ with gauge φ.*

(i) *For all $x, y \in X$, the following inequality holds:*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle. \tag{2.4}$$

In particular, for all $x, y \in X$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle. \tag{2.5}$$

(ii) *Assume that a sequence $\{x_n\}$ in X converges weakly to a point $x \in X$. Then the following identity holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall x, y \in X. \tag{2.6}$$

Lemma 2.2 (see [17]). *Assume that a Banach space X has a weakly continuous duality mapping J_φ with gauge φ. Let A be a strongly positive linear bounded operator on X with a coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \varphi(1)(1 - \rho\overline{\gamma})$.*

Lemma 2.3 (see [11]). *Let C be a closed convex subset of a uniformly convex Banach space X and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Then, for each $r > 0$ and $h \geq 0$,*

$$\lim_{t\to\infty} \sup_{x\in C\cap B_r} \left\| \frac{1}{t}\int_0^t T(s)x\,ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x\,ds\right) \right\| = 0. \tag{2.7}$$

Lemma 2.4 (see [19]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\mu_n)a_n + \delta_n, \tag{2.8}$$

where $\{\mu_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \mu_n = \infty$;

(ii) $\limsup_{n\to\infty}(\delta_n/\mu_n) \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Let X be a Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$, and let C be a nonempty closed convex subset of X such that $C \pm C \subset C$. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup from C into itself, let f be a contraction mapping with a coefficient $\alpha \in (0,1)$, let A be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < \overline{\gamma}\varphi(1)/\alpha$, and let $t \in (0,1)$ such that $t \leq \varphi(1)\|A\|^{-1}$ which satisfies $t \to 0$. Define the mapping $T_t^f : C \to C$ by

$$T_t^f := t\gamma f + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)ds \tag{3.1}$$

to be a contraction mapping. Indeed, for each $x, y \in C$,

$$\begin{aligned} \left\|T_t^f x - T_t^f y\right\| &= \left\| t\gamma(f(x) - f(y)) + (I - tA)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t}(T(s)x - T(s)y)ds\right)\right\| \\ &\leq t\gamma\|f(x) - f(y)\| + \|I - tA\|\left(\frac{1}{\lambda_t}\int_0^{\lambda_t}\|T(s)x - T(s)y\|ds\right) \\ &\leq t\gamma\alpha\|x - y\| + \varphi(1)(1 - t\overline{\gamma})\|x - y\| \\ &\leq (1 - t(\varphi(1)\overline{\gamma} - \gamma\alpha))\|x - y\|. \end{aligned} \tag{3.2}$$

Thus, by Banach contraction mapping principle, there exists a unique fixed point $x_t \in C$, that is,

$$x_t = t\gamma f(x_t) + (I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\,ds. \tag{3.3}$$

Remark 3.1. We note that space l_p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. This shows that φ is invariant on $[0,1]$.

Lemma 3.2. *Let X be a uniformly convex Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$, and let C be a nonempty closed convex subset of X such that $C \pm C \subset C$. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup from C into itself such that $F(\mathcal{S}) \neq \emptyset$, let f be a contraction mapping with a coefficient $\alpha \in (0,1)$, let A be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < \overline{\gamma}\varphi(1)/\alpha$, and let $t \in (0,1)$ such that $t \leq \varphi(1)\|A\|^{-1}$ which satisfies $t \to 0$. Then the net $\{x_t\}$ defined by (3.3) with $\{\lambda_t\}_{0<t<1}$ is a positive real divergent sequence, converges strongly as $t \to 0$ to a common fixed point x^*, in which $x^* \in F(\mathcal{S})$, and is the unique solution of the variational inequality:*

$$\left\langle \gamma f(x^*) - Ax^*, J_\varphi(x - x^*) \right\rangle \leq 0, \quad \forall x \in F(\mathcal{S}). \tag{3.4}$$

Proof. Firstly, we show the uniqueness of a solution of the variational inequality (3.4). Suppose that $\tilde{x}, x^* \in F(\mathcal{S})$ are solutions of (3.4); then

$$\begin{aligned} &\left\langle \gamma f(x^*) - Ax^*, J_\varphi(\tilde{x} - x^*) \right\rangle \leq 0, \\ &\left\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x^* - \tilde{x}) \right\rangle \leq 0. \end{aligned} \tag{3.5}$$

Adding up (3.5), we obtain

$$\begin{aligned} 0 &\geq \left\langle \left(\gamma f(x^*) - Ax^*\right) - \left(\gamma f(\tilde{x}) - A\tilde{x}\right), J_\varphi(\tilde{x} - x^*) \right\rangle \\ &= \left\langle A(\tilde{x} - x^*), J_\varphi(\tilde{x} - x^*) \right\rangle - \gamma \left\langle f(\tilde{x}) - f(x^*), J_\varphi(\tilde{x} - x^*) \right\rangle \\ &\geq \overline{\gamma}\|\tilde{x} - x^*\|\varphi\|\tilde{x} - x^*\| - \gamma \left\| f(\tilde{x}) - f(x^*) \right\| \left\| J_\varphi(\tilde{x} - x^*) \right\| \\ &\geq \overline{\gamma}\Phi(\|\tilde{x} - x^*\|) - \gamma\alpha\Phi(\|\tilde{x} - x^*\|) \\ &= (\overline{\gamma} - \gamma\alpha)\Phi(\|\tilde{x} - x^*\|) \\ &\geq (\varphi(1)\overline{\gamma} - \gamma\alpha)\Phi(\|\tilde{x} - x^*\|), \end{aligned} \tag{3.6}$$

which is a contradiction, we must have $\tilde{x} = x^*$, and the uniqueness is proved. Here in after, we use $\tilde{x}$ to denote the unique solution of the variational inequality (3.4).

Next, we show that $\{x_t\}$ is bounded. Indeed, for each $p \in F(\mathcal{S})$, we have

$$\begin{aligned} \|x_t - p\| &= \left\| t\left(\gamma f(x_t) - Ap\right) + (I - tA)\left(\frac{1}{\lambda_t} \int_0^{\lambda_t} \left(T(s)x_t - p\right) ds \right) \right\| \\ &\leq t\left\|\gamma f(x_t) - Ap\right\| + \|I - tA\| \frac{1}{\lambda_t} \int_0^{\lambda_t} \|T(s)x_t - p\| ds \\ &\leq t\gamma \left\| f(x_t) - f(p) \right\| + t\left\|\gamma f(p) - Ap\right\| + \varphi(1)(1 - t\overline{\gamma})\left\|x_t - p\right\| \\ &\leq \left(1 - t(\varphi(1)\overline{\gamma} - \gamma\alpha)\right)\left\|x_t - p\right\| + t\left\|\gamma f(p) - Ap\right\|. \end{aligned} \tag{3.7}$$

It follows that

$$\|x_t - p\| \leq \frac{1}{\varphi(1)\overline{\gamma} - \gamma\alpha}\|\gamma f(p) - Ap\|. \tag{3.8}$$

Hence, $\{x_t\}$ is bounded, so are $\{f(x_t)\}$ and $\{A((1/\lambda_t)\int_0^{\lambda_t} T(s)x_t ds)\}$.

Next, we show that $\|x_t - T(h)x_t\| \to 0$ as $t \to 0$. We note that

$$\left\|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds\right\| = t\left\|\gamma f(x_t) - A\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds\right)\right\|. \tag{3.9}$$

Moreover, we note that

$$\begin{aligned}\|x_t - T(h)x_t\| &\leq \left\|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds\right\| + \left\|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds - T(h)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds\right)\right\| \\ &\quad + \left\|T(h)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds\right) - T(h)x_t\right\| \\ &\leq 2\left\|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds\right\| + \left\|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds - T(h)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds\right)\right\|,\end{aligned} \tag{3.10}$$

for all $h \geq 0$. Define the set $K = \{z \in C : \|z - p\| \leq \|\gamma f(p) - Ap\|/(\varphi(1)\overline{\gamma} - \gamma\alpha)\}$; then K is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $h \geq 0$. Since $\{x_t\} \subset K$ and K is bounded, there exists $r > 0$ such that $K \subset B_r$, and it follows by Lemma 2.3 that

$$\lim_{\lambda_t \to \infty}\left\|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds - T(h)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t\, ds\right)\right\| = 0, \tag{3.11}$$

for each $h \geq 0$. From (3.9)-(3.10), letting $t \to 0$ and noting (3.11) then, for each $h \geq 0$, we obtain

$$\|x_t - T(h)x_t\| \longrightarrow 0. \tag{3.12}$$

Assume that $\{t_n\}_{n=1}^{\infty} \subset (0,1)$ is such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $\lambda_n := \lambda_{t_n}$. We will show that $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x} \in F(\mathcal{S})$. Since $\{x_n\}$ is bounded sequence and Banach space X is a uniformly convex, hence it is reflexive, and there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $\tilde{x} \in C$ as $j \to \infty$. Again, since J_φ is weakly sequentially continuous, we have by Lemma 2.1 that

$$\limsup_{j \to \infty} \Phi\left(\left\|x_{n_j} - z\right\|\right) = \limsup_{j \to \infty} \Phi\left(\left\|x_{n_j} - \tilde{x}\right\|\right) + \Phi(\|z - \tilde{x}\|). \tag{3.13}$$

Let $H(z) = \limsup_{j \to \infty}\Phi(\|x_{n_j} - z\|)$, for all $z \in C$.

It follows that $H(z) = H(\tilde{x}) + \Phi(\|z - \tilde{x}\|)$, for all $z \in C$. From (3.12), we have

$$\begin{aligned} H(T(h)\tilde{x}) &= \limsup_{j\to\infty} \Phi\left(\left\|x_{n_j} - T(h)\tilde{x}\right\|\right) \\ &= \limsup_{j\to\infty} \Phi\left(\left\|T(h)x_{n_j} - T(h)\tilde{x}\right\|\right) \\ &\le \limsup_{j\to\infty} \Phi\left(\left\|x_{n_j} - \tilde{x}\right\|\right) = H(\tilde{x}). \end{aligned} \tag{3.14}$$

On the other hand, we note that

$$\begin{aligned} H(T(h)\tilde{x}) &= \limsup_{j\to\infty} \Phi\left(\left\|x_{n_j} - \tilde{x}\right\|\right) + \Phi(\|T(h)\tilde{x} - \tilde{x}\|) \\ &= H(\tilde{x}) + \Phi(\|T(h)\tilde{x} - \tilde{x}\|). \end{aligned} \tag{3.15}$$

Combining (3.14) with (3.15), we obtain $\Phi(\|T(h)\tilde{x} - \tilde{x}\|) \le 0$. This implies that $T(h)\tilde{x} = \tilde{x}$, that is, $\tilde{x} \in F(\mathcal{S})$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau$, for all $t \ge 0$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is the gauge function, then for $1 \ge k \ge 0$, $\varphi(ky) \le \varphi(y)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(ky)dy \le k\int_0^t \varphi(y)dy = k\Phi(t). \tag{3.16}$$

By Lemma 2.1, we have

$$\begin{aligned} \Phi(\|x_n - \tilde{x}\|) &= \Phi\left(\left\|t_n(\gamma f(x_n) - A\tilde{x}) + (I - tA)\left(\frac{1}{\lambda_n}\int_0^{\lambda_t} T(s)x_n\,ds - \tilde{x}\right)\right\|\right) \\ &\le \Phi\left(\left\|(I - t_nA)\left(\frac{1}{\lambda_n}\int_0^{\lambda_n} (T(s)x_n - \tilde{x})ds\right)\right\|\right) + t_n\langle \gamma f(x_n) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \\ &\le \Phi\big(\varphi(1)(1 - t_n\overline{\gamma})(\|x_n - \tilde{x}\|)\big) + t_n\langle \gamma f(x_n) - \gamma f(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\ &\quad + t_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \\ &\le \varphi(1)(1 - t_n\overline{\gamma})\Phi(\|x_n - \tilde{x}\|) + t_n\gamma\alpha\|x_n - \tilde{x}\|\left\|J_\varphi(x_n - \tilde{x})\right\| \\ &\quad + t_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \\ &= \varphi(1)(1 - t_n\overline{\gamma})\Phi(\|x_n - \tilde{x}\|) + t_n\gamma\alpha\Phi(\|x_n - \tilde{x}\|) + t_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \\ &\le \big(1 - t_n(\varphi(1)\overline{\gamma} - \gamma\alpha)\big)\Phi(\|x_n - \tilde{x}\|) + t_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle. \end{aligned} \tag{3.17}$$

This implies that

$$\Phi(\|x_n - \tilde{x}\|) \leq \frac{1}{\varphi(1)\overline{\gamma} - \gamma\alpha} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_{\varphi}(x_n - \tilde{x}) \rangle. \tag{3.18}$$

In particular, we have

$$\Phi\left(\left\|x_{n_j} - \tilde{x}\right\|\right) \leq \frac{1}{\varphi(1)\overline{\gamma} - \gamma\alpha} \left\langle \gamma f(\tilde{x}) - A\tilde{x}, J_{\varphi}\left(x_{n_j} - \tilde{x}\right) \right\rangle. \tag{3.19}$$

Since the mapping J_{φ} is single-valued and weakly continuous, it follows from (3.19) that $\Phi(\|x_{n_j} - \tilde{x}\|) \to 0$ as $j \to \infty$. This implies that $x_{n_j} \to \tilde{x}$ as $j \to \infty$.

Next, we show that $\tilde{x}$ solves the variational inequality (3.4), for each $x \in F(\mathcal{S})$. From (3.3), we derive that

$$(\gamma f - A)x_t = -\frac{1}{t}(I - tA)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t \, ds - x_t\right). \tag{3.20}$$

Now, we observe that

$$\begin{aligned}
&\left\langle \frac{1}{\lambda_t}\int_0^{\lambda_t} (I - T(s))x \, ds - \frac{1}{\lambda_t}\int_0^{\lambda_t} (I - T(s))x_t \, ds, J_{\varphi}(x - x_t) \right\rangle \\
&= \langle x - x_t, J_{\varphi}(x - x_t) \rangle - \left\langle \frac{1}{\lambda_t}\int_0^{\lambda_t} (T(s)x - T(s)x_t) ds, J_{\varphi}(x - x_t) \right\rangle \\
&\geq \|x - x_t\| \left\| J_{\varphi}(x - x_t) \right\| - \frac{1}{\lambda_t}\int_0^{\lambda_t} \|T(s)\ x - T(s)x_t\| ds \left\| J_{\varphi}(x - x_t) \right\| \\
&\geq \Phi(\|x - x_t\|) - \|x - x_t\| \left\| J_{\varphi}(x - x_t) \right\| \\
&= \Phi(\|x - x_t\|) - \Phi(\|x - x_t\|) = 0.
\end{aligned} \tag{3.21}$$

It follows from (3.20) that

$$\begin{aligned}\langle(\gamma f-A)x_t,J_\varphi(x-x_t)\rangle&=-\frac{1}{t}\left\langle(I-tA)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_t\,ds-x_t\right),J_\varphi(x-x_t)\right\rangle\\&=-\frac{1}{t}\left\langle(I-tA)\left(\frac{1}{\lambda_t}\int_0^{\lambda_t}T(s)x_t\,ds-\frac{1}{\lambda_t}\int_0^{\lambda_t}x_t\,ds\right),J_\varphi(x-x_t)\right\rangle\\&=-\frac{1}{t}\left\langle\frac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))x\,ds-\frac{1}{\lambda_t}\int_0^{\lambda_t}(I-T(s))x_t\,ds,J_\varphi(x-x_t)\right\rangle\\&\quad+\left\langle A\left(\frac{1}{\lambda_t}\int_0^{\lambda_t}(T(s)-I)x_t\,ds\right),J_\varphi(x-x_t)\right\rangle\\&\le\left\langle A\left(\frac{1}{\lambda_t}\int_0^{\lambda_t}(T(s)-I))x_t\,ds\right),J_\varphi(x-x_t)\right\rangle.\end{aligned}\tag{3.22}$$

Now, replacing t and λ_t with t_{n_j} and λ_{n_j}, respectively, in (3.22), and letting $j\to\infty$, and we notice that $(T(s)-I)x_{n_j}\to(T(s)-I)\tilde{x}=0$ for $\tilde{x}\in F(\mathcal{S})$, we obtain that $\langle(\gamma f-A)\tilde{x},J_\varphi(x-\tilde{x})\rangle\le 0$. That is, $\tilde{x}$ is a solution of the variational inequality (3.4). By uniqueness, as $\tilde{x}=x^*$, we have shown that each cluster point of the net $\{x_t\}$ is equal to x^*. Then, we conclude that $x_t\to x^*$ as $t\to 0$. This proof is complete. □

Theorem 3.3. *Let X be a uniformly convex Banach space which admits a weakly continuous duality mapping J_φ with the gauge function φ such that φ is invariant in $[0,1]$, and let C be a nonempty closed convex subset of X such that $C\pm C\subset C$. Let $\mathcal{S}=\{T(t):t\in\mathbb{R}^+\}$ be a nonexpansive semigroup from C into itself such that $F(\mathcal{S})\neq\emptyset$, let f be a contraction mapping with a coefficient $\alpha\in(0,1)$, and let A be a strongly positive linear bounded operator with a coefficient $\overline{\gamma}>0$ such that $0<\gamma<\overline{\gamma}\varphi(1)/\alpha$. Let $\{\alpha_n\}_{n=0}^\infty,\{\beta_n\}_{n=0}^\infty,\{\gamma_n\}_{n=0}^\infty$ be the sequences in $(0,1)$ and let $\{t_n\}_{n=0}^\infty$ be a positive real divergent sequence. Assume that the following conditions hold:*

(C1) $\lim_{n\to\infty}\alpha\_n=0$ *and* $\sum_{n=0}^\infty\alpha_n=\infty$,

(C2) $\lim_{n\to\infty}\gamma_n=0$,

(C3) $\beta_n=o(\alpha_n)$,

Then the sequence $\{x_n\}$ defined by

$$\begin{gathered}x_0\in C\ \text{chosen arbitrarily};\\ z_n=\gamma_nx_n+(1-\gamma_n)\frac{1}{t_n}\int_0^{t_n}T(s)x_n\,ds;\\ y_n=\alpha_n\gamma f(z_n)+(I-\alpha_nA)z_n;\\ x_{n+1}=\beta_nx_n+(1-\beta_n)y_n,\quad\forall n\ge 0,\end{gathered}\tag{3.23}$$

converges strongly to the common fixed point x^* that is obtained in Lemma 3.2.

Proof. From the condition $(C1)$, we may assume, with no loss of generality, that $\alpha_n \le \varphi(1)\|A\|^{-1}$ for each $n \ge 0$. From Lemma 2.2, we have $\|I - \alpha_n A\| \le \varphi(1)(1 - \alpha_n \overline{\gamma})$.

Firstly, we show that $\{x_n\}$ is bounded. Let $p \in F(\mathcal{S})$; we get

$$\begin{aligned}
\|z_n - p\| &= \left\| \gamma_n (x_n - p) + (1 - \gamma_n) \frac{1}{t_n} \int_0^{t_n} (T(s)x_n - p)\, ds \right\| \\
&\le \gamma_n \|x_n - p\| + (1 - \gamma_n) \frac{1}{t_n} \int_0^{t_n} \|T(s)x_n - p\|\, ds \\
&\le \gamma_n \|x_n - p\| + (1 - \gamma_n) \|x_n - p\| \\
&= \|x_n - p\|, \\
\|y_n - p\| &= \|\alpha_n (\gamma f(z_n) - Ap) + (I - \alpha_n A)(z_n - p)\| \\
&\le \alpha_n \gamma \|f(z_n) - f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + \|I - \alpha_n A\| \|z_n - p\| \\
&\le (1 - \alpha_n (\varphi(1)\overline{\gamma} - \gamma\alpha)) \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|.
\end{aligned} \tag{3.24}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n (x_n - p) + (1 - \beta_n)(y_n - p)\| \\
&\le \beta_n \|x_n - p\| + (1 - \beta_n) \|y_n - p\| \\
&\le \beta_n \|x_n - p\| + (1 - \beta_n)\left((1 - \alpha_n (\varphi(1)\overline{\gamma} - \gamma\alpha)) \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\|\right) \\
&= (1 - \alpha_n (\varphi(1)\overline{\gamma} - \gamma\alpha)(1 - \beta_n)) \|x_n - p\| + \alpha_n (\varphi(1)\overline{\gamma} - \gamma\alpha)(1 - \beta_n) \frac{\|\gamma f(p) - Ap\|}{\varphi(1)\overline{\gamma} - \gamma\alpha} \\
&\le \max\left\{ \|x_n - p\|, \frac{\|\gamma f(p) - Ap\|}{\varphi(1)\overline{\gamma} - \gamma\alpha} \right\}.
\end{aligned} \tag{3.25}$$

By induction on n, we have

$$\|x_n - p\| \le \max\left\{ \|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\varphi(1)\overline{\gamma} - \gamma\alpha} \right\}, \quad \forall n \ge 0. \tag{3.26}$$

Thus, $\{x_n\}$ is bounded. Since $\{x_n\}$ is bounded, then $\|(1/t_n) \int_0^{t_n} T(s)x_n\, ds - p\| \le \|x_n - p\|$ and $\{A((1/t_n) \int_0^{t_n} T(s)x_n\, ds)\}$ and $\{f(z_n)\}$ are also bounded.

Next, we show that $\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0$, for all $h \geq 0$. From (3.23), we note that

$$\left\|x_{n+1} - \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right\| \leq \|x_{n+1} - y_n\| + \|y_n - z_n\| + \left\|z_n - \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right\|$$
$$\leq \beta_n\|x_n - y_n\| + \alpha_n\|\gamma f(z_n) - Az_n\| + \gamma_n\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right\|. \tag{3.27}$$

By the conditions $(C1)$–$(C3)$, then (3.27), we obtain

$$\lim_{n\to\infty}\left\|x_{n+1} - \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right\| = 0. \tag{3.28}$$

Moreover, we note that

$$\|x_{n+1} - T(h)x_{n+1}\| \leq \left\|x_{n+1} - \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right\| + \left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right)\right\|$$
$$+ \left\|T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right) - T(h)x_{n+1}\right\|$$
$$\leq 2\left\|x_{n+1} - \frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right\| + \left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right)\right\|. \tag{3.29}$$

Define the set $K = \{z \in C : \|z - p\| \leq \|x_0 - p\| + \|\gamma f(p) - Ap\|/(\varphi(1)\overline{\gamma} - \gamma\alpha)\}$. Then K is a nonempty bounded closed convex subset of C, which is $T(s)$—invariant for each $s \geq 0$ and contains $\{x_n\}$; it follows from Lemma 2.3 that

$$\lim_{n\to\infty}\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds - T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right)\right\| = 0, \quad \forall h \geq 0. \tag{3.30}$$

Then, for all $h \geq 0$, from (3.28) and (3.30), into (3.29), we obtain $\lim_{n\to\infty}\|x_{n+1} - T(h)x_{n+1}\| = 0$, and hence

$$\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0, \quad \forall h \geq 0. \tag{3.31}$$

Next, we show that $\limsup_{n\to\infty}\langle\gamma f(x^*) - Ax^*, J_\varphi(x_n - x^*)\rangle \leq 0$. We can take subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\lim_{j\to\infty}\left\langle\gamma f(x^*) - Ax^*, J_\varphi\left(x_{n_j} - x^*\right)\right\rangle = \limsup_{n\to\infty}\langle\gamma f(x^*) - Ax^*, J_\varphi(x_n - x^*)\rangle. \tag{3.32}$$

By the assumption that X is uniformly convex, hence it is reflexive and $\{x_n\}$ is bounded; then there exists a subsequence $\{x_{n_j}\}$ which converges weakly to some $x \in C$ as $j \to \infty$. Since J_φ is weakly continuous, from Lemma 2.1, we have

$$\limsup_{j\to\infty} \Phi\left(\left\|x_{n_j} - z\right\|\right) = \limsup_{j\to\infty} \Phi\left(\left\|x_{n_j} - x\right\|\right) + \Phi(\|z - x\|), \quad \forall z \in C. \tag{3.33}$$

Let $H(z) = \limsup_{j\to\infty}\Phi(\|x_{n_j} - z\|)$, for all $z \in C$.

It follows that $H(z) = H(x) + \Phi(\|z - x\|)$, for all $z \in C$.

From (3.31), we have

$$\begin{aligned} H(T(h)x) &= \limsup_{j\to\infty} \Phi\left(\left\|x_{n_j} - T(h)x\right\|\right) \\ &= \limsup_{j\to\infty} \Phi\left(\left\|T(h)x_{n_j} - T(h)x\right\|\right) \\ &\le \limsup_{j\to\infty} \Phi\left(\left\|x_{n_j} - x\right\|\right) = H(x). \end{aligned} \tag{3.34}$$

On the other hand, we note that

$$\begin{aligned} H(T(h)x) &= \limsup_{j\to\infty} \Phi\left(\left\|x_{n_j} - x\right\|\right) + \Phi(\|T(h)x - x\|) \\ &= H(x) + \Phi(\|T(h)x - x\|). \end{aligned} \tag{3.35}$$

Combining (3.34) with (3.35), we obtain $\Phi(\|T(h)x - x\|) \le 0$.

This implies that $T(h)x = x$; that is, $x \in F(\mathcal{S})$.

Since the duality map J_φ is single-valued and weakly continuous, we get that

$$\begin{aligned} \limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J_\varphi(x_n - x^*)\rangle &= \lim_{j\to\infty}\left\langle \gamma f(x^*) - Ax^*, J_\varphi\left(x_{n_j} - x^*\right)\right\rangle \\ &= \langle \gamma f(x^*) - Ax^*, J_\varphi(x - x^*)\rangle \le 0, \end{aligned} \tag{3.36}$$

as required. Hence,

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J_\varphi(x_{n+1} - x^*)\rangle \le 0. \tag{3.37}$$

Since $\|x_{n+1} - y_n\| = \beta_n\|x_n - y_n\|$, by condition $(C3)$, we obtain that $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$. It follows from (3.37), that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J_\varphi(y_n - x^*)\rangle \le 0. \tag{3.38}$$

Finally, we show that $x_n \to x^*$ as $n \to \infty$. Now, from Lemma 2.1, we have

$$\begin{aligned}
\Phi(\|y_n - x^*\|) &= \Phi(\|\alpha_n(\gamma f(z_n) - Ax^*) + (I - \alpha_n A)(z_n - x^*)\|) \\
&= \Phi(\|\alpha_n(\gamma f(z_n) - \gamma f(x^*)) + \alpha_n(\gamma f(x^*) - Ax^*) + (I - \alpha_n A)(z_n - x^*)\|) \\
&\le \Phi(\|\alpha_n(\gamma f(z_n) - \gamma f(x^*)) + (I - \alpha_n A)(z_n - x^*)\|) \\
&\quad + \alpha_n\langle \gamma f(x^*) - Ax^*, J_\varphi(y_n - x^*)\rangle \\
&\le (1 - \alpha_n(\varphi(1)\overline{\gamma} - \gamma\alpha))\Phi(\|x_n - x^*\|) + \alpha_n\langle \gamma f(x^*) - Ax^*, J_\varphi(y_n - x^*)\rangle.
\end{aligned} \tag{3.39}$$

On the other hand, we note that

$$\begin{aligned}
\Phi(\|x_{n+1} - x^*\|) &= \Phi(\|\beta_n(x_n - x^*) + (1 - \beta_n)(y_n - x^*)\|) \\
&\le (1 - \beta_n)\Phi(\|y_n - x^*\|) + \beta_n\langle x_n - x^*, J_\varphi(x_{n+1} - x^*)\rangle.
\end{aligned} \tag{3.40}$$

It follows from (3.40) that

$$\begin{aligned}
\Phi(\|x_{n+1} - x^*\|) &\le (1 - \alpha_n(\varphi(1)\overline{\gamma} - \gamma\alpha))\Phi(\|x_n - x^*\|) + \alpha_n\langle \gamma f(x^*) - Ax^*, J_\varphi(y_n - x^*)\rangle \\
&\quad + \beta_n\langle x_n - x^*, J_\varphi(x_{n+1} - x^*)\rangle \\
&\le (1-\alpha_n(\varphi(1)\overline{\gamma}-\gamma\alpha))\Phi(\|x_n-x^*\|)+\alpha_n\left[\langle \gamma f(x^*)-Ax^*, J_\varphi(y_n-x^*)\rangle+\frac{\beta_n}{\alpha_n}M\right],
\end{aligned} \tag{3.41}$$

where $M = \sup_{n\ge 0}\{\|x_n - x^*\|\varphi(\|x_{n+1} - x^*\|)\}$.

Put $\mu_n := \alpha_n(\varphi(1)\overline{\gamma} - \gamma\alpha)$ and $\delta_n := \alpha_n[\langle \gamma f(x^*) - Ax^*, J_\varphi(y_n - x^*)\rangle + (\beta_n/\alpha_n)M]$. Then (3.41) reduces to formula $\Phi(\|x_{n+1} - x^*\|) \le (1 - \mu_n)\Phi(\|x_n - x^*\|) + \delta_n$. By conditions $(C1)$ and $(C3)$ and noting (3.38), it is easy to see that $\sum_{n=0}^{\infty}\mu_n = \infty$ and $\limsup_{n\to\infty}(\delta_n/\mu_n) = \limsup_{n\to\infty}(1/\varphi(1)\overline{\gamma}-\gamma\alpha)[\langle \gamma f(x^*)-Ax^*, J_\varphi(y_n-x^*)\rangle+(\beta_n/\alpha_n)M] \le 0$. Applying Lemma 2.4, we obtain $\Phi(\|x_n - x^*\|) \to 0$ as $n \to \infty$ this implies that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Taking $\gamma_n = 0$ in (3.23), we can get the following corollary easily.

Corollary 3.4. *Let X be a uniformly convex Banach space which admits a weakly continuous duality mapping J_φ with the gauge function φ such that φ invariant in $[0,1]$, C be a nonempty closed convex subset of X such that $C \pm C \subset C$. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup from C into itself such that $F(\mathcal{S}) \ne \emptyset$, f be a contraction mapping with a coefficient $\alpha \in (0,1)$ and A be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$ such that $0 < \gamma < \overline{\gamma}\varphi(1)/\alpha$. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ be the sequences in $(0,1)$ and $\{t_n\}_{n=0}^{\infty}$ be a positive real divergent sequence. Assume the following conditions are hold:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(C2) $\beta_n = o(\alpha_n)$.

Then the sequence $\{x_n\}$ *defined by*

$$x_0 \in C \text{ chosen arbitrarily},$$
$$y_n = \alpha_n \gamma f\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds, \tag{3.42}$$
$$x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \ge 0,$$

converges strongly to the common fixed point x^*, *in which* $x^* \in F(\mathcal{S})$ *is the unique solution of the variational inequality:*

$$\langle \gamma f(x^*) - Ax^*, J_\varphi(x - x^*)\rangle \le 0, \quad \forall x \in F(\mathcal{S}). \tag{3.43}$$

A strong mean convergence theorem for nonexpansive mapping was first established by Baillon [20] and it was generalized to that for nonlinear semigroups by Reich et al. [21–23]. It is clear that Theorem 3.3 are valid for nonexpansive mappings. Thus, we have the following mean ergodic theorem of viscosity iteration process for nonexpansive mappings in Hilbert spaces.

Corollary 3.5. *Let* H *be a real Hilbert space, and let*C *be a nonempty closed convex subset of* H *such that* $C \pm C \subset C$. *Let* T *be a nonexpansive mapping from* C *into itself such that* $F(T) \ne \emptyset$, f *be a contraction mapping with a coefficient* $\alpha \in (0,1)$, *and let* A *be a strongly positive linear bounded operator with a coefficient* $\overline{\gamma} > 0$ *such that* $0 < \gamma < \overline{\gamma}\varphi(1)/\alpha$. *Let* $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$, *and*$\{\gamma_n\}_{n=0}^\infty$ *be the sequences in* $(0,1)$ *and let* $\{t_n\}_{n=0}^\infty$ *be a positive real divergent sequence. Assume that the following conditions are hold:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$;

(C2) $\lim_{n\to\infty}\gamma_n = 0$;

(C3) $\beta_n = o(\alpha_n)$.

Then the sequence $\{x_n\}$ *defined by*

$$x_0 \in C \text{ chosen arbitrarily},$$
$$z_n = \gamma_n x_n + (1-\gamma_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n,$$
$$y_n = \alpha_n \gamma f(z_n) + (I - \alpha_n A)z_n, \tag{3.44}$$
$$x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \ge 0,$$

converges strongly to the common fixed point x^*, *in which* $x^* \in F(T)$ *is the unique solution of the variational inequality:*

$$\langle \gamma f(x^*) - Ax^*, x - x^*\rangle \le 0, \quad \forall x \in F(T). \tag{3.45}$$

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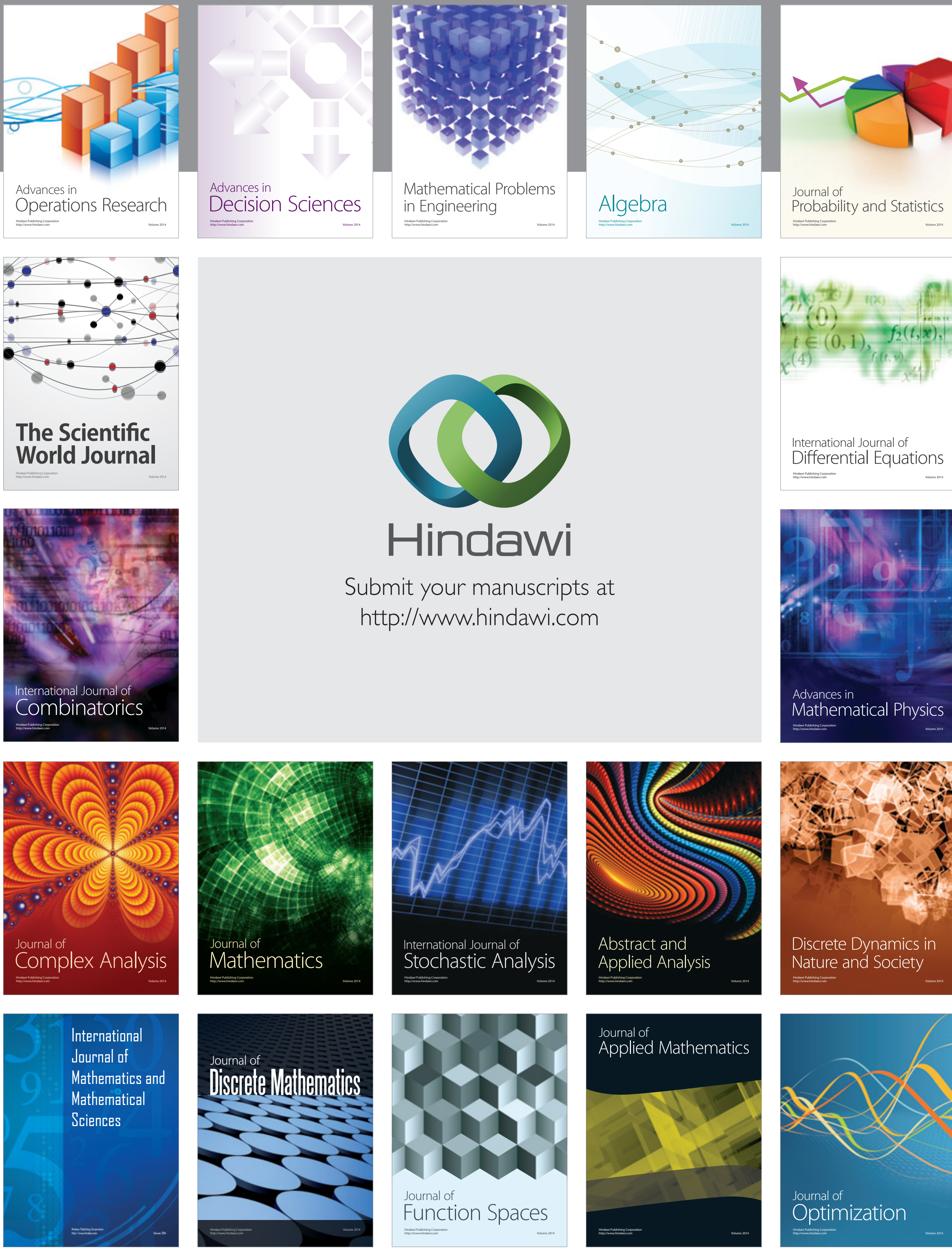
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