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Research Article

On Common Solutions for Fixed-Point Problems of Two Infinite Families of Strictly Pseudocontractive Mappings and the System of Cocoercive Quasivariational Inclusions Problems in Hilbert Spaces

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This paper is concerned with a common element of the set of common fixed points for two infinite families of strictly pseudocontractive mappings and the set of solutions of a system of cocoercive quasivariational inclusions problems in Hilbert spaces. The strong convergence theorem for the above two sets is obtained by a novel general iterative scheme based on the viscosity approximation method, and applicability of the results has shown difference with the results of many others existing in the current literature.

1. Introduction

Throughout this paper, we always assume that C is a nonempty closed-convex subset of a real Hilbert space H with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, and 2^H denotes the family of all the nonempty subsets of H.

Let $B : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ a set-valued mapping. We consider the following *quasivariational inclusion problem*, which is to find a point $x \in H$ such that

$$\theta \in Bx + Mx, \tag{1.1}$$

where θ is the zero vector in H. The set of solutions of the problem (1.1) is denoted by $\mathrm{VI}(H, B, M)$. As special cases of the problem (1.1), we have the following.

(i) If $M = \partial\phi : H \to 2^H$, where $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function such that $\mathbb{R}$ is the set of real numbers, and $\partial\phi$ is the subdifferential of ϕ, then the quasivariational inclusion problem (1.1) is equivalent to find $x \in H$ such that

$$\langle Bx, v - x\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall v, y \in H, \tag{1.2}$$

which is called the *mixed quasivariational inequality problem* (see [1]).

(ii) If $M = \partial\delta_C$, where $\delta_C : H \to \{0, +\infty\}$ is the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{1.3}$$

then the quasivariational inclusion (1.1) is equivalent to find $x \in C$ such that

$$\langle Bx, v - x\rangle \geq 0, \quad \forall v \in C, \tag{1.4}$$

which is called *Hartman-Stampacchia variational inequality problem* (see [2–4]).

Recall that P_C is the *metric projection* of H onto C, that is, for each $x \in H$, there exists the unique point in $P_C x \in C$ such that

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|. \tag{1.5}$$

A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C, \tag{1.6}$$

and the mapping $f : C \to C$ is called a *contraction* if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.7}$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. If $C \subset H$ is bounded, closed, and convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty (see [5]). Recall that a mapping $A : C \to C$ is said to be

(i) *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C, \tag{1.8}$$

(ii) *k-Lipschitz continuous* if there exists a constant $k > 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in C, \tag{1.9}$$

if $k = 1$, then A is a nonexpansive,

(iii) *pseudocontractive* if

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + \|(I - A)x - (I - A)y\|^2, \quad \forall x, y \in C, \tag{1.10}$$

(iv) *k-strictly pseudocontractive* if there exists a constant $k \in [0, 1)$ such that

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + k\|(I - A)x - (I - A)y\|^2, \quad \forall x, y \in C, \tag{1.11}$$

and it is obvious that A is a nonexpansive if and only if A is a 0-strictly pseudocontractive,

(v) *α-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C, \tag{1.12}$$

(vi) *α-inverse-strongly monotone (or α-cocoercive)* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C, \tag{1.13}$$

if $\alpha = 1$, then A is called that *firmly nonexpansive*; it is obvious that any α-inverse-strongly monotone mapping A is monotone and $(1/\alpha)$-Lipschitz continuous,

(vii) *relaxed α-cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\alpha)\|Ax - Ay\|^2, \quad \forall x, y \in C, \tag{1.14}$$

(viii) *relaxed (α, r)-cocoercive* if there exists two constants $\alpha, r > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\alpha)\|Ax - Ay\|^2 + r\|x - y\|^2, \quad \forall x, y \in C, \tag{1.15}$$

and it is obvious that any r-strongly monotonicity implies to the relaxed (α, r)-cocoercivity.

The existence common fixed points for a finite family of nonexpansive mappings have been considered by many authors (see [6–9] and the references therein).

In this paper, we study the mapping W_n defined by

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \mu_n S_n U_{n,n+1} + (1-\mu_n) I, \\
U_{n,n-1} &= \mu_{n-1} S_{n-1} U_{n,n} + (1-\mu_{n-1}) I, \\
&\vdots \\
U_{n,k} &= \mu_k S_k U_{n,k+1} + (1-\mu_k) I, \\
U_{n,k-1} &= \mu_{k-1} S_{k-1} U_{n,k} + (1-\mu_{k-1}) I, \\
&\vdots \\
U_{n,2} &= \mu_2 S_2 U_{n,3} + (1-\mu_2) I, \\
W_n = U_{n,1} &= \mu_1 S_1 U_{n,2} + (1-\mu_1) I,
\end{aligned} \tag{1.16}$$

where $\{\mu_i\}$ is nonnegative real sequence in $(0,1)$, for all $i \in \mathbb{N}$, $S_1, S_2, \ldots$ from a family of infinitely nonexpansive mappings of C into itself. It is obvious that W_n is a nonexpansive of C into itself, such a mapping W_n is called a *W-mapping* generated by $S_1, S_2, \ldots, S_n$ and $\mu_1, \mu_2, \ldots, \mu_n$.

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping in a real Hilbert space H,

$$\min_{x\in\Omega} \left\{ \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle \right\}, \tag{1.17}$$

where A is a bounded linear operator on H, Ω is the fixed-point set of a nonexpansive mapping S on H, and b is a given point in H. Recall that A is a strongly positive bounded linear operator on H if there exists a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.18}$$

Marino and Xu [10] introduced the following iterative scheme based on the viscosity approximation method introduced by Moudafi [11]:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S x_n, \quad \forall n \in \mathbb{N}, \tag{1.19}$$

where $x_1 \in H$, A is a strongly positive bounded linear operator on H, f is a contraction on H, and S is a nonexpansive on H. They proved that under some appropriateness conditions imposed on the parameters, if $F(S) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.19) converges strongly to the unique solution $z = P_{F(S)}(I - A + \gamma f)z$ of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in F(S), \tag{1.20}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(S)}\left\{\frac{1}{2}\langle Ax, x\rangle - h(x)\right\}, \tag{1.21}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Iiduka and Takahashi [12] introduced an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality (1.4) as in the following theorem.

Theorem IT. *Let C be a nonempty closed-convex subset of a real Hilbert space H. Let B be an α-inverse-strongly monotone mapping of C into H, and let S be a nonexpansive mapping of C into itself such that $F(S) \cap \mathrm{VI}(C, B) \neq \emptyset$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is the sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n B x_n), \tag{1.22}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \subset [a, b]$ such that $0 < a < b < 2\alpha$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

then $\{x_n\}$ converges strongly to $P_{F(S)\cap \mathrm{VI}(C,B)}x$.

Definition 1.1 (see [13]). Let $M : H \to 2^H$ be a multivalued maximal monotone mapping, then the single-valued mapping $J_{M,\lambda} : H \to H$ defined by $J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u)$, for all $u \in H$, is called *the resolvent operator associated with* M, where λ is any positive number, and I is the identity mapping.

Recently, Zhang et al. [13] considered the problem (1.1). To be more precise, they proved the following theorem.

Theorem ZLC. *Let H be a real Hilbert space, let $B : H \to H$ be an α-inverse-strongly monotone mapping, let $M : H \to 2^H$ be a maximal monotone mapping, and let $S : H \to H$ be a nonexpansive mapping. Suppose that the set $F(S) \cap \mathrm{VI}(H, B, M) \neq \emptyset$, where $\mathrm{VI}(H, B, M)$ is the set of solutions of quasivariational inclusion* (1.1). *Suppose that $x_1 = x \in H$ and $\{x_n\}$ is the sequence defined by*

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda B x_n), \\ x_{n+1} &= \alpha_n x + (1 - \alpha_n) S y_n, \end{aligned} \tag{1.23}$$

for all $n \in \mathbb{N}$, where $\lambda \in (0, 2\alpha)$ and $\{\alpha_n\} \subset (0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

then $\{x_n\}$ converges strongly to $P_{F(S)\cap \mathrm{VI}(H,B,M)}x$.

Peng et al. [14] introduced an iterative scheme

$$\begin{aligned}\Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H,\\ y_n = J_{M,\lambda}(u_n - \lambda B u_n),\\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S y_n,\end{aligned} \tag{1.24}$$

for all $n \in \mathbb{N}$, where $x_1 \in H$, B is an α-cocoercive mapping on H, f is a contraction on H, S is a nonexpansive on H, M is a maximal monotone mapping of H into 2^H, and Φ is a bifunction from $H \times H$ into $\mathbb{R}$.

We note that their iteration is well defined if we let $C = H$, and the appropriateness of the control conditions α_n and λ of their iteration should be $\{\alpha_n\} \subset (0,1)$ and $\lambda \in (0, 2\alpha)$ (see Theorem 3.1 in [14]). They proved that under some appropriateness imposed on the other parameters, if $\Omega = F(S) \cap \mathrm{VI}(H,B,M) \cap \mathrm{EP}(\Phi) \neq \emptyset$, then the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ generated by (1.24) converge strongly to $z = P_\Omega f(z)$ of the variational inequality

$$\langle z - f(z), x - z\rangle \geq 0, \quad \forall x \in \Omega, \tag{1.25}$$

where $\mathrm{EP}(\Phi)$ is the set of solutions of *equilibrium problem* defined by

$$\mathrm{EP}(\Phi) = \{x \in H : \Phi(x,y) \geq 0, \ \forall y \in H\}. \tag{1.26}$$

Moreover, Plubtieng and Sripard [15] introduced an iterative scheme

$$\begin{aligned}\Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H,\\ y_n = J_{M,\lambda}(u_n - \lambda B u_n),\\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A) S_n y_n,\end{aligned} \tag{1.27}$$

for all $n \in \mathbb{N}$, where $x_1 \in H$, A is a strongly bounded linear operator on H, B is an α-cocoercive mapping on H, f is a contraction on H, S_n is a nonexpansive on H, M is a maximal monotone mapping of H into 2^H, and Φ is a bifunction from $H \times H$ into $\mathbb{R}$.

We note that the appropriateness of the control conditions α_n and λ of their iteration should be $\{\alpha_n\} \subset (0,1)$ and $\lambda \in (0, 2\alpha)$ (see Theorem 3.2 in [15]). They proved that under some appropriateness imposed on the other parameters, if $\Omega = \bigcap_{n=1}^{\infty} F(S_n) \cap \mathrm{VI}(H,B,M) \cap \mathrm{EP}(\Phi) \neq \emptyset$, then the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ generated by (1.27) converge strongly to $z = P_\Omega(I - A + \gamma f)z$.

On the other hand, Li and Wu [16] introduced an iterative scheme for finding a common element of the set of fixed points of a k-strictly pseudocontractive mapping with a fixed point and the set of solutions of relaxed cocoercive quasivariational inclusions as follows:

$$\begin{aligned} y_n &= J_{M,\lambda}(x_n - \lambda B x_n), \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n)I - \alpha_n A\big)\big(\mu S_k x_n + (1-\mu) y_n\big), \end{aligned} \tag{1.28}$$

for all $n \in \mathbb{N}$, where $x_1 \in H$, A is a strongly positive bounded linear operator on H, f is a contraction on H, S_k is a mapping on H defined by $S_k x = kx + (1-k)Sx$ for all $x \in H$, such that S is a k-strictly pseudocontractive mapping on H with a fixed point, B is relaxed cocoercive and Lipschitz continuous mappings on H, and M is a maximal monotone mapping of H into 2^H.

They proved that under the missing condition of μ, which should be $0 < \mu < 1$ (see Theorem 2.1 in [16]) and some appropriateness imposed on the other parameters, if $\Omega = F(S) \cap \mathrm{VI}(H, B, M) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.28) converges strongly to $z = P_\Omega(I - A + \gamma f)z$.

Very recently, Tianchai and Wangkeeree [17] introduced an implicit iterative scheme for finding a common element of the set of common fixed points of an infinite family of a k_n-strictly pseudocontractive mapping and the set of solutions of the system of generalized relaxed cocoercive quasivariational inclusions as follows:

$$\begin{aligned} z_n &= J_{M_2,\lambda_2}(x_n - \lambda_2 (B_2 + C_2) x_n), \\ y_n &= J_{M_1,\lambda_1}(z_n - \lambda_1 (B_1 + C_1) z_n), \\ x_{n+1} &= \alpha_n \gamma f(W_n x_n) + \beta_n x_n + \big((1-\beta_n)I - \alpha_n A\big)\big(\gamma_n W_n x_n + (1-\gamma_n) y_n\big), \end{aligned} \tag{1.29}$$

for all $n \in \mathbb{N}$, where $x_1 \in H$, A is a strongly positive bounded linear operator on H, f is a contraction on H, W_n is a W-mapping on H generated by $\{S_n\}$ and $\{\mu_n\}$ such that $S_n x = \delta_n x + (1-\delta_n) T_n x$ for all $x \in H$, T_n is a k_n-strictly pseudocontractive mapping on H with a fixed point, M_i is a maximal monotone mapping of H into 2^H, and B_i, C_i are two mappings of relaxed cocoercive and Lipschitz continuous mappings on H for each $i = 1, 2$.

They proved that under some appropriateness imposed on the parameters, if $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \cap F(D) \neq \emptyset$ such that the mapping $D : H \to H$ defined by

$$Dx = J_{M_1,\lambda_1}((I - \lambda_1 (B_1 + C_1)) J_{M_2,\lambda_2}(I - \lambda_2 (B_2 + C_2)) x), \quad \forall x \in H, \tag{1.30}$$

then the sequence $\{x_n\}$ generated by (1.29) converges strongly to $z = P_\Omega(I - A + \gamma f)z$.

In this paper, we introduce a novel general iterative scheme (1.32) below by the viscosity approximation method to find a common element of the set of common fixed points for two infinite families of strictly pseudocontractive mappings and the set of solutions of a system of cocoercive quasivariational inclusions problems in Hilbert spaces. Firstly, we introduce a mapping W_n, where W_n is a W-mapping generated by $\{R_n\}$ and $\{\mu_n\}$ for solving a common fixed point for two infinite families of strictly pseudocontractive mappings by iteration such that the mapping $R_n : H \to H$ defined by

$$R_n x = \alpha x + (1-\alpha)(\alpha S_n x + (1-\alpha) T_n x), \quad \forall x \in H, \tag{1.31}$$

for all $n \in \mathbb{N}$, where $\{S_n : H \to H\}$ and $\{T_n : H \to H\}$ are two infinite families of k_1 and k_2-strictly pseudocontractive mappings with a fixed point, respectively, and $\{\mu_n\} \subset (0, \mu]$ for some $\mu \in (0,1)$. It follows that a linear general iterative scheme of the mappings W_n and $J_{M_i,\lambda_i}(I - \lambda_i C_i)$ is obtained as follows:

$$\begin{gathered} y_n = \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1-\epsilon_n)I - \beta_n B - \alpha_n A) y_n, \end{gathered} \tag{1.32}$$

for all $n \in \mathbb{N}$, where $x_1 = u \in H$, $M_i : H \to 2^H$ is a maximal monotone mapping, $C_i : H \to H$ is a cocoercive mapping for each $i = 1, 2, \ldots, N$, $f : H \to H$ is a contraction mapping, and $A, B : H \to H$ are two mappings of the strongly positive linear bounded self-adjoint operator mappings.

As special cases of the iterative scheme (1.32), we have the following.

(i) If $\epsilon_n = 0$ for all $n \in \mathbb{N}$, then (1.32) is reduced to the iterative scheme

$$\begin{gathered} y_n = \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + (I - \beta_n B - \alpha_n A) y_n, \quad \forall n \in \mathbb{N}. \end{gathered} \tag{1.33}$$

(ii) If $B \equiv I$, then (1.32) is reduced to the iterative scheme

$$\begin{gathered} y_n = \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\epsilon_n - \beta_n)I - \alpha_n A) y_n, \quad \forall n \in \mathbb{N}. \end{gathered} \tag{1.34}$$

(iii) If $\epsilon_n = 0$ for all $n \in \mathbb{N}$, then (1.34) is reduced to the iterative scheme

$$\begin{gathered} y_n = \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) y_n, \quad \forall n \in \mathbb{N}. \end{gathered} \tag{1.35}$$

(iv) If $\beta_n = 0$ for all $n \in \mathbb{N}$, then (1.34) is reduced to the iterative scheme

$$\begin{gathered} y_n = \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + ((1-\epsilon_n)I - \alpha_n A) y_n, \quad \forall n \in \mathbb{N}. \end{gathered} \tag{1.36}$$

(v) If $\epsilon_n = 0$ for all $n \in \mathbb{N}$, then (1.36) is reduced to the iterative scheme

$$y_n = \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n),$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \quad \forall n \in \mathbb{N}. \tag{1.37}$$

(vi) If $\gamma = 1$ and $A \equiv I$, then (1.37) is reduced to the iterative scheme

$$y_n = \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n),$$
$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, \quad \forall n \in \mathbb{N}. \tag{1.38}$$

(vii) If $M_i \equiv C_i \equiv 0$ for each $i = 1, 2, \ldots, N$ and $\sum_{i=1}^{N}\rho_i = 1$, then (1.32) is reduced to the iterative scheme

$$y_n = \gamma_n W_n x_n + (1-\gamma_n)x_n,$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + \big((1-\epsilon_n)I - \beta_n B - \alpha_n A\big)y_n, \quad \forall n \in \mathbb{N}. \tag{1.39}$$

Furthermore, if $S_n \equiv T_n$ for all $n \in \mathbb{N}$, then the mapping $R_n : H \to H$ in (1.31) is reduced to

$$R_n x = \alpha x + (1-\alpha)T_n x, \quad \forall x \in H, \tag{1.40}$$

for all $n \in \mathbb{N}$. It follows that the iterative scheme (1.32) is reduced to find a common element of the set of common fixed points for an infinite family of strictly pseudocontractive mappings and the set of solutions of a system of cocoercive quasivariational inclusions problems in Hilbert spaces.

It is well known that the class of strictly pseudocontractive mappings contains the class of nonexpansive mappings; it follows that if the mapping R_n is defined as (1.31) and $k_1 = k_2 = 0$, then the iterative scheme (1.32) is reduced to find a common element of the set of common fixed points for two infinite families of nonexpansive mappings and the set of solutions of a system of cocoercive quasivariational inclusions problems in Hilbert spaces, and if the mapping R_n is defined as (1.40) and $k_1 = k_2 = 0$, then the iterative scheme (1.32) is reduced to find a common element of the set of common fixed points for an infinite family of nonexpansive mappings and the set of solutions of a system of cocoercive quasivariational inclusions problems in Hilbert spaces.

We suggest and analyze the iterative scheme (1.32) above under some appropriateness conditions imposed on the parameters, the strong convergence theorem for the above two sets is obtained, and applicability of the results has shown difference with the results of many others existing in the current literature.

2. Preliminaries

We collect the following lemmas which are used in the proof for the main results in the next section.

Lemma 2.1. *Let C be a nonempty closed-convex subset of a Hilbert space H then the following inequalities hold:*

(1) $\langle x - P_C x, P_C x - y\rangle \geq 0, \forall x \in H, y \in C$,

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \forall x, y \in H$.

Lemma 2.2 (see [10]). *Let H be a Hilbert space, let $f : H \to H$ be a contraction with coefficient $0 < \alpha < 1$, and let $A : H \to H$ be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$, then*

(1) *if $0 < \gamma < \overline{\gamma}/\alpha$, then*

$$\langle x - y, (A - \gamma f)x - (A - \gamma f)y\rangle \geq (\overline{\gamma} - \gamma\alpha)\|x - y\|^2, \quad \forall x, y \in H, \tag{2.1}$$

(2) *if $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.3 (see [18]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \eta_n)a_n + \delta_n, \quad n \geq 1, \tag{2.2}$$

where $\{\eta_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\lim_{n\to\infty}\eta_n = 0$ *and* $\sum_{n=1}^{\infty}\eta_n = \infty$,

(2) $\limsup_{n\to\infty}(\delta_n/\eta_n) \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$,

then $\lim_{n\to\infty}a_n = 0$.

Lemma 2.4 (see [9]). *Let C be a nonempty closed-convex subset of a Hilbert space H, define mapping W_n as (1.16), let $S_i : C \to C$ be a family of infinitely nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$, and let $\{\mu_i\}$ be a sequence such that $0 < \mu_i \leq \mu < 1$, for all $i \geq 1$, then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(S_i)$ for each $n \geq 1$,*

(2) *for each $x \in C$ and for each positive integer k, $\lim_{n\to\infty}U_{n,k}x$ exists,*

(3) *the mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C, \tag{2.3}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S_i)$, and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\mu_1, \mu_2, \ldots$.

Lemma 2.5 (see [13]). *The resolvent operator $J_{M,\lambda}$ associated with M is single-valued and nonexpansive for all $\lambda > 0$.*

Lemma 2.6 (see [13]). *$u \in H$ is a solution of quasivariational inclusion* (1.1) *if and only if $u = J_{M,\lambda}(u - \lambda Bu)$, for all $\lambda > 0$, that is,*

$$\mathrm{VI}(H, B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0. \tag{2.4}$$

Lemma 2.7 (see [19]). *Let C be a nonempty closed-convex subset of a strictly convex Banach space X. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n = 1$, then a mapping S on C defined by*

$$Sx = \sum_{n=1}^{\infty} \alpha_n T_n x, \tag{2.5}$$

for $x \in C$, is well defined, nonexpansive, and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.

Lemma 2.8 (see [2]). *Let C be a nonempty closed-convex subset of a Hilbert space H and $S : C \to C$ a nonexpansive mapping, then $I - S$ is demiclosed at zero. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$.*

Lemma 2.9 (see [20]). *Let C be a nonempty closed-convex subset of a real Hilbert space H and $T : C \to C$ a k-strict pseudocontraction. Define $S : C \to C$ by $Sx = \alpha x + (1 - \alpha)Tx$ for each $x \in C$, then, as $\alpha \in [k, 1)$, S is a nonexpansive such that $F(S) = F(T)$.*

3. Main Results

Lemma 3.1. *Let C be a nonempty closed-convex subset of a real Hilbert space H, and let $S, T : C \to C$ be two mappings of k_1 and k_2-strictly pseudocontractive mappings with a fixed point, respectively. Suppose that $F(S) \cap F(T) \neq \emptyset$ and define a mapping $R : C \to C$ by*

$$Rx = \alpha x + (1 - \alpha)(\alpha Sx + (1 - \alpha)Tx), \quad \forall x \in C, \tag{3.1}$$

where $\alpha \in [k, 1) \setminus \{0\}$ such that $k = \max\{k_1, k_2\}$, then R is well defined, nonexpansive, and $F(R) = F(S) \cap F(T)$.

Proof. Define the mappings $S_1, T_1 : C \to C$ as follows:

$$S_1 x = \alpha x + (1 - \alpha)Sx, \qquad T_1 x = \alpha x + (1 - \alpha)Tx, \tag{3.2}$$

for all $x \in C$. By Lemma 2.9, we have S_1 and T_1 as nonexpansive such that $F(S_1) = F(S)$ and $F(T_1) = F(T)$. Therefore, for all $x \in C$, we have

$$\begin{aligned} Rx &= \alpha x + (1 - \alpha)(\alpha Sx + (1 - \alpha)Tx) \\ &= \alpha x + \alpha(1 - \alpha)Sx + (1 - \alpha)^2 Tx \end{aligned}$$

$$\begin{aligned}&= \alpha^2 x + \alpha(1-\alpha)Sx + (1-\alpha)\alpha x + (1-\alpha)^2 Tx \\ &= \alpha(\alpha x + (1-\alpha)Sx) + (1-\alpha)(\alpha x + (1-\alpha)Tx) \\ &= \alpha S_1 x + (1-\alpha)T_1 x.\end{aligned} \tag{3.3}$$

It follows from Lemma 2.7 that R is well defined, nonexpansive, and $F(R) = F(S_1) \cap F(T_1) = F(S) \cap F(T)$. □

Theorem 3.2. *Let H be a real Hilbert space, let $M_i : H \to 2^H$ be a maximal monotone mapping, and let $C_i : H \to H$ be a ξ_i-cocoercive mapping for each $i = 1, 2, \ldots, N$. Let $A, B : H \to H$ be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients $\overline{\delta}, \overline{\beta} \in (0,1]$ such that $\overline{\delta} \le \|A\| \le 1$ and $\|B\| = \overline{\beta}$, respectively, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0,1)$. Let $\{S_n : H \to H\}$ and $\{T_n : H \to H\}$ be two infinite families of k_1 and k_2-strictly pseudocontractive mappings with a fixed point such that $k_1, k_2 \in [0,1)$, respectively. Define a mapping $R_n : H \to H$ by*

$$R_n x = \alpha x + (1-\alpha)(\alpha S_n x + (1-\alpha)T_n x), \quad \forall x \in H, \tag{3.4}$$

for all $n \in \mathbb{N}$, where $\alpha \in [k,1) \setminus \{0\}$ such that $k = \max\{k_1, k_2\}$. Let $W_n : H \to H$ be a W-mapping generated by $\{R_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0,\mu]$, for some $\mu \in (0,1)$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(S_n)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(H, C_i, M_i)) \ne \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ is generated iteratively by

$$\begin{aligned} y_n &= \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N} \rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1-\epsilon_n)I - \beta_n B - \alpha_n A) y_n, \end{aligned} \tag{3.5}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$, $\{\beta_n\}, \{\epsilon_n\} \subset [0,1)$ such that $\epsilon_n \le \alpha_n$, $\rho_i \in (0,1)$, and $\lambda_i \in (0, 2\xi_i]$ for each $i = 1, 2, \ldots, N$ satisfying the following conditions:

- (C1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty}(\epsilon_n/\alpha_n) = 0$,
- (C2) $0 < \lim_{n\to\infty} \gamma_n < 1$ *and* $\limsup_{n\to\infty} \beta_n < 1$,
- (C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{i=1}^{N} \rho_i = 1$,
- (C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, *and* $\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty$,
- (C5) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} \prod_{i=1}^{n} \mu_i < \infty$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $w \in \Omega$ where $w = P_\Omega(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \ge 0, \quad \forall y \in \Omega. \tag{3.6}$$

Proof. From $\|B\| = \overline{\beta} \in (0,1]$, $\epsilon_n \le \alpha_n$ for all $n \in \mathbb{N}$, (C1) and (C2), we have $\alpha_n \to 0$, $\epsilon_n \to 0$ as $n \to \infty$ and $\limsup_{n\to\infty} \beta_n < 1$. Thus, we may assume without loss of generality that

$\alpha_n < (1-\epsilon_n-\beta_n\|B\|)\|A\|^{-1}$ for all $n\in\mathbb{N}$. For any $x,y\in H$ and for each $i=1,2,\ldots,N$, by the ξ_i-cocoercivity of C_i, we have

$$\begin{aligned}\|(I-\lambda_iC_i)x-(I-\lambda_iC_i)y\|^2&=\|(x-y)-\lambda_i(C_ix-C_iy)\|^2\\&=\|x-y\|^2-2\lambda_i\langle x-y,C_ix-C_iy\rangle+\lambda_i^2\|C_ix-C_iy\|^2\\&\le\|x-y\|^2-(2\xi_i-\lambda_i)\lambda_i\|C_ix-C_iy\|^2\\&\le\|x-y\|^2,\end{aligned}\tag{3.7}$$

which implies that $I-\lambda_iC_i$ is a nonexpansive. Since A and B are two mappings of the linear bounded self-adjoint operators, we have

$$\begin{aligned}\|A\|&=\sup\{|\langle Ax,x\rangle|:\ x\in H,\|x\|=1\},\\\|B\|&=\sup\{|\langle Bx,x\rangle|:\ x\in H,\|x\|=1\}.\end{aligned}\tag{3.8}$$

Observe that

$$\begin{aligned}\langle((1-\epsilon_n)I-\beta_nB-\alpha_nA)x,x\rangle&=(1-\epsilon_n)\langle x,x\rangle-\beta_n\langle Bx,x\rangle-\alpha_n\langle Ax,x\rangle\\&\ge1-\epsilon_n-\beta_n\|B\|-\alpha_n\|A\|\\&>0.\end{aligned}\tag{3.9}$$

Therefore, we obtain that $(1-\epsilon_n)I-\beta_nB-\alpha_nA$ is positive. Thus, by the strong positivity of A and B, we get

$$\begin{aligned}\|(1-\epsilon_n)I-\beta_nB-\alpha_nA\|&=\sup\{\langle((1-\epsilon_n)I-\beta_nB-\alpha_nA)x,x\rangle:\ x\in H,\|x\|=1\}\\&=\sup\{(1-\epsilon_n)\langle x,x\rangle-\beta_n\langle Bx,x\rangle-\alpha_n\langle Ax,x\rangle:\ x\in H,\|x\|=1\}\\&\le1-\epsilon_n-\beta_n\overline{\beta}-\alpha_n\overline{\delta}\\&\le1-\beta_n\overline{\beta}-\alpha_n\overline{\delta}.\end{aligned}\tag{3.10}$$

Define the sequences of mappings $\{P_n:H\to H\}$ and $\{Q_n:H\to H\}$ as follows:

$$\begin{aligned}P_nx&=\alpha_n\gamma f(x)+\beta_nBx+((1-\epsilon_n)I-\beta_nB-\alpha_nA)Q_nx,\quad\forall x\in H,\\Q_nx&=\gamma_nW_nx+(1-\gamma_n)\sum_{i=1}^{N}\rho_iJ_{M_i,\lambda_i}(I-\lambda_iC_i)x,\quad\forall x\in H,\end{aligned}\tag{3.11}$$

for all $n \in \mathbb{N}$. Firstly, we prove that P_n has a unique fixed point in H. Note that for all $x, y \in H$, by (3.11), (C3), the nonexpansiveness of W_n, J_{M_i,λ_i}, and $I - \lambda_i C_i$, we have

$$\begin{aligned}
\|Q_n x - Q_n y\| &\leq \gamma_n \|W_n x - W_n y\| \\
&\quad + (1-\gamma_n)\sum_{i=1}^{N}\rho_i \|J_{M_i,\lambda_i}(I-\lambda_i C_i)x - J_{M_i,\lambda_i}(I-\lambda_i C_i)y\| \\
&\leq \gamma_n\|x-y\| + (1-\gamma_n)\sum_{i=1}^{N}\rho_i\|(I-\lambda_i C_i)x - (I-\lambda_i C_i)y\| \\
&\leq \gamma_n\|x-y\| + (1-\gamma_n)\left(\sum_{i=1}^{N}\rho_i\right)\|x-y\| \\
&= \|x-y\|.
\end{aligned} \tag{3.12}$$

Therefore, Q_n is a nonexpansive. It follows from (3.10), (3.11), (3.12), the contraction of f, and the linearity of A and B that

$$\begin{aligned}
\|P_n x - P_n y\| &\leq \alpha_n\gamma\|f(x) - f(y)\| + \beta_n\|B\|\|x-y\| \\
&\quad + \|(1-\epsilon_n)I - \beta_n B - \alpha_n A\|\|Q_n x - Q_n y\| \\
&\leq \alpha_n\gamma\delta\|x-y\| + \beta_n\overline{\beta}\|x-y\| + \left(1-\beta_n\overline{\beta} - \alpha_n\overline{\delta}\right)\|x-y\| \\
&= \left(1-\left(\overline{\delta}-\gamma\delta\right)\alpha_n\right)\|x-y\|.
\end{aligned} \tag{3.13}$$

Hence, P_n is a contraction with coefficient $1-(\overline{\delta}-\gamma\delta)\alpha_n \in (0,1)$. Therefore, Banach contraction principle guarantees that P_n has a unique fixed point in H, and so the iteration (3.5) is well defined.

Next, we prove that $\{x_n\}$ is bounded. Pick $p \in \Omega$. Therefore, by Lemma 2.6, we have

$$p = J_{M_i,\lambda_i}(I-\lambda_i C_i)p, \tag{3.14}$$

for each $i = 1, 2, \ldots, N$. By (3.14), the nonexpansiveness of J_{M_i,λ_i}, and $I - \lambda_i C_i$, we have

$$\begin{aligned}
\|J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n) - p\| &= \|J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n) - J_{M_i,\lambda_i}(p - \lambda_i C_i p)\| \\
&\leq \|(x_n - \lambda_i C_i x_n) - (p - \lambda_i C_i p)\| \\
&\leq \|x_n - p\|.
\end{aligned} \tag{3.15}$$

Let $t_n = \sum_{i=1}^{N} \rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n)$. By (3.14), (C3), the nonexpansiveness of J_{M_i,λ_i}, and $I - \lambda_i C_i$, we have

$$\begin{aligned}
\|t_n - p\| &= \left\| \sum_{i=1}^{N} \rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n) - \sum_{i=1}^{N} \rho_i p \right\| \\
&= \left\| \sum_{i=1}^{N} \rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n) - \sum_{i=1}^{N} \rho_i J_{M_i,\lambda_i}(p - \lambda_i C_i p) \right\| \\
&\leq \sum_{i=1}^{N} \rho_i \| J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n) - J_{M_i,\lambda_i}(p - \lambda_i C_i p) \| \\
&\leq \sum_{i=1}^{N} \rho_i \| (x_n - \lambda_i C_i x_n) - (p - \lambda_i C_i p) \| \\
&\leq \left(\sum_{i=1}^{N} \rho_i \right) \|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{3.16}$$

Since $R_n x = \alpha x + (1-\alpha)(\alpha S_n x + (1-\alpha) T_n x)$, where $\alpha \in [k,1) \setminus \{0\}$, $\{S_n\}$ and $\{T_n\}$ are two infinite families of k_1 and k_2-strict pseudocontractions with a fixed point, respectively, such that $k = \max\{k_1, k_2\}$; therefore, by Lemma 3.1, we have that R_n is a nonexpansive and $F(R_n) = F(S_n) \cap F(T_n)$ for all $n \in \mathbb{N}$. It follows from Lemma 2.4(1) that we get $F(W_n) = \bigcap_{i=1}^{n} F(R_i) = (\bigcap_{i=1}^{n} F(S_i)) \cap (\bigcap_{i=1}^{n} F(T_i))$, which implies that $W_n p = p$. Hence, by (3.16) and the nonexpansiveness of W_n, we have

$$\begin{aligned}
\|y_n - p\| &= \|\gamma_n W_n x_n + (1-\gamma_n) t_n - p\| \\
&= \|\gamma_n (W_n x_n - p) + (1-\gamma_n)(t_n - p)\| \\
&\leq \gamma_n \|W_n x_n - W_n p\| + (1-\gamma_n)\|t_n - p\| \\
&\leq \gamma_n \|x_n - p\| + (1-\gamma_n)\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{3.17}$$

By (3.10), (3.17), the contraction of f, and the linearity of A and B, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \gamma f(x_n) + \beta_n B x_n + ((1-\epsilon_n)I - \beta_n B - \alpha_n A) y_n - p\| \\
&= \|\alpha_n (\gamma f(x_n) - Ap) + \beta_n B(x_n - p) \\
&\quad + ((1-\epsilon_n)I - \beta_n B - \alpha_n A)(y_n - p) - \epsilon_n p\| \\
&\leq \alpha_n \|\gamma f(x_n) - Ap\| + \beta_n \|B\| \|x_n - p\| \\
&\quad + \|(1-\epsilon_n)I - \beta_n B - \alpha_n A\| \|y_n - p\| + \epsilon_n \|p\|
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \gamma \left\| f(x_n) - f(p) \right\| + \alpha_n \left\| \gamma f(p) - Ap \right\| + \beta_n \overline{\beta} \left\| x_n - p \right\| \\
&\quad + \left(1 - \beta_n \overline{\beta} - \alpha_n \overline{\delta}\right) \left\| x_n - p \right\| + \alpha_n \|p\| \\
&\leq \left(1 - \left(\overline{\delta} - \gamma\delta\right)\alpha_n\right) \left\| x_n - p \right\| + \alpha_n (\left\| \gamma f(p) - Ap \right\| + \|p\|) \\
&\leq \max\left\{ \left\| x_n - p \right\|, \frac{\left\| \gamma f(p) - Ap \right\| + \|p\|}{\overline{\delta} - \gamma\delta} \right\}.
\end{aligned} \tag{3.18}$$

It follows from induction that

$$\left\| x_{n+1} - p \right\| \leq \max\left\{ \left\| x_1 - p \right\|, \frac{\left\| \gamma f(p) - Ap \right\| + \|p\|}{\overline{\delta} - \gamma\delta} \right\}, \tag{3.19}$$

for all $n \in \mathbb{N}$. Hence, $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{W_n x_n\}$, $\{t_n\}$, $\{f(x_n)\}$, $\{Ay_n\}$, $\{Bx_n\}$, and $\{By_n\}$.

Next, we prove that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. By (C3), the nonexpansiveness of J_{M_i,λ_i}, and $I - \lambda_i C_i$, we have

$$\begin{aligned}
\|t_{n+1} - t_n\| &= \left\| \sum_{i=1}^{N} \rho_i J_{M_i,\lambda_i}(x_{n+1} - \lambda_i C_i x_{n+1}) - \sum_{i=1}^{N} \rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n) \right\| \\
&\leq \sum_{i=1}^{N} \rho_i \| J_{M_i,\lambda_i}(x_{n+1} - \lambda_i C_i x_{n+1}) - J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n) \| \\
&\leq \sum_{i=1}^{N} \rho_i \| (x_{n+1} - \lambda_i C_i x_{n+1}) - (x_n - \lambda_i C_i x_n) \| \\
&\leq \left(\sum_{i=1}^{N} \rho_i \right) \|x_{n+1} - x_n\| \\
&= \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.20}$$

By the nonexpansiveness of R_i and $U_{n,i}$, we have

$$\begin{aligned}
\|W_{n+1} x_n - W_n x_n\| &= \|U_{n+1,1} x_n - U_{n,1} x_n\| \\
&= \left\| \mu_1 R_1 U_{n+1,2} x_n + (1 - \mu_1) x_n - (\mu_1 R_1 U_{n,2} x_n + (1 - \mu_1) x_n) \right\| \\
&\leq \mu_1 \|U_{n+1,2} x_n - U_{n,2} x_n\| \\
&= \mu_1 \left\| \mu_2 R_2 U_{n+1,3} x_n + (1 - \mu_2) x_n - (\mu_2 R_2 U_{n,3} x_n + (1 - \mu_2) x_n) \right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \mu_1\mu_2\|U_{n+1,3}x_n - U_{n,3}x_n\| \\
&\vdots \\
&\leq \left(\prod_{i=1}^{n}\mu_i\right)\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
&\leq M\prod_{i=1}^{n}\mu_i,
\end{aligned}
\tag{3.21}
$$

for some constant M such that $M \geq \|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \geq 0$. Therefore, from (3.21), by the nonexpansiveness of W_{n+1}, we have

$$
\begin{aligned}
\|W_{n+1}x_{n+1} - W_nx_n\| &\leq \|W_{n+1}x_{n+1} - W_{n+1}x_n\| + \|W_{n+1}x_n - W_nx_n\| \\
&\leq \|x_{n+1} - x_n\| + M\prod_{i=1}^{n}\mu_i.
\end{aligned}
\tag{3.22}
$$

Since

$$
\begin{aligned}
y_{n+1} - y_n &= \left(\gamma_{n+1}W_{n+1}x_{n+1} + (1-\gamma_{n+1})t_{n+1}\right) - \left(\gamma_nW_nx_n + (1-\gamma_n)t_n\right) \\
&= \gamma_{n+1}(W_{n+1}x_{n+1} - W_nx_n) + (\gamma_{n+1} - \gamma_n)(W_nx_n - t_n) \\
&\quad + (1-\gamma_{n+1})(t_{n+1} - t_n),
\end{aligned}
\tag{3.23}
$$

combining (3.20), (3.22), and (3.23), we have

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\leq \gamma_{n+1}\|W_{n+1}x_{n+1} - W_nx_n\| + |\gamma_{n+1} - \gamma_n|\|W_nx_n - t_n\| \\
&\quad + (1-\gamma_{n+1})\|t_{n+1} - t_n\| \\
&\leq \gamma_{n+1}\left(\|x_{n+1} - x_n\| + M\prod_{i=1}^{n}\mu_i\right) + |\gamma_{n+1} - \gamma_n|\|W_nx_n - t_n\| \\
&\quad + (1-\gamma_{n+1})\|x_{n+1} - x_n\| \\
&\leq \|x_{n+1} - x_n\| + M\prod_{i=1}^{n}\mu_i + |\gamma_{n+1} - \gamma_n|\|W_nx_n - t_n\|.
\end{aligned}
\tag{3.24}
$$

By the linearity of A and B, we have

$$
\begin{aligned}
x_{n+2} - x_{n+1} &= \left(\alpha_{n+1}\gamma f(x_{n+1}) + \beta_{n+1}Bx_{n+1} + \left((1-\epsilon_{n+1})I - \beta_{n+1}B - \alpha_{n+1}A\right)y_{n+1}\right) \\
&\quad - \left(\alpha_n\gamma f(x_n) + \beta_nBx_n + \left((1-\epsilon_n)I - \beta_nB - \alpha_nA\right)y_n\right) \\
&= \left((1-\epsilon_{n+1})I - \beta_{n+1}B - \alpha_{n+1}A\right)(y_{n+1} - y_n) + (\beta_n - \beta_{n+1})By_n
\end{aligned}
$$

$$
\begin{aligned}
&+ (\alpha_n - \alpha_{n+1}) A y_n + (\epsilon_n - \epsilon_{n+1}) y_n + \alpha_{n+1}\gamma\big(f(x_{n+1}) - f(x_n)\big) \\
&+ \gamma(\alpha_{n+1} - \alpha_n) f(x_n) + \beta_{n+1} B(x_{n+1} - x_n) \\
&+ (\beta_{n+1} - \beta_n) B x_n.
\end{aligned}
\tag{3.25}
$$

Therefore, by (3.10), (3.24), (3.25), and the contraction of f, we have

$$
\begin{aligned}
\|x_{n+2} - x_{n+1}\| \le{}& \big\|(1-\epsilon_{n+1})I - \beta_{n+1}B - \alpha_{n+1}A\big\| \|y_{n+1} - y_n\| + |\beta_n - \beta_{n+1}| \|By_n\| \\
&+ |\alpha_n - \alpha_{n+1}| \|Ay_n\| + |\epsilon_n - \epsilon_{n+1}| \|y_n\| + \alpha_{n+1}\gamma \|f(x_{n+1}) - f(x_n)\| \\
&+ \gamma|\alpha_{n+1} - \alpha_n| \|f(x_n)\| + \beta_{n+1}\|B\|\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|Bx_n\| \\
\le{}& \left(1 - \beta_{n+1}\overline{\beta} - \alpha_{n+1}\overline{\delta}\right) \|y_{n+1} - y_n\| + |\beta_n - \beta_{n+1}| \|By_n\| \\
&+ |\alpha_n - \alpha_{n+1}| \|Ay_n\| + |\epsilon_n - \epsilon_{n+1}| \|y_n\| + \alpha_{n+1}\gamma\delta \|x_{n+1} - x_n\| \\
&+ \gamma|\alpha_{n+1} - \alpha_n| \|f(x_n)\| + \beta_{n+1}\overline{\beta}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n| \|Bx_n\| \\
\le{}& (1 - \eta_n)\|x_{n+1} - x_n\| + \delta_n,
\end{aligned}
\tag{3.26}
$$

where $\eta_n := (\overline{\delta} - \gamma\delta)\alpha_{n+1} \in (0,1)$ and

$$
\delta_n := M\prod_{i=1}^{n}\mu_i + N\big(|\gamma_n - \gamma_{n+1}| + |\epsilon_n - \epsilon_{n+1}| + |\beta_n - \beta_{n+1}| + |\alpha_n - \alpha_{n+1}|\big),
\tag{3.27}
$$

such that

$$
N = \max\left\{\sup_{n\ge1}\|W_n x_n - t_n\|,\ \sup_{n\ge1}\big(\|By_n\| + \|Bx_n\|\big),\ \sup_{n\ge1}\|y_n\|,\ \sup_{n\ge1}\big(\|Ay_n\| + \gamma\|f(x_n)\|\big)\right\}.
\tag{3.28}
$$

By (C1), (C3), (C4), and (C5), we can find that $\lim_{n\to\infty}\eta_n = 0$, $\sum_{n=1}^{\infty}\eta_n = \infty$, and $\sum_{n=1}^{\infty}\delta_n < \infty$; therefore, by (3.26) and Lemma 2.3, we obtain

$$
\|x_{n+1} - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\tag{3.29}
$$

Next, we prove that $\|x_n - y_n\| \to 0$ as $n \to \infty$. By the linearity of B, we have

$$
\begin{aligned}
\|x_{n+1} - y_n\| &= \|\alpha_n\gamma f(x_n) + \beta_n Bx_n + ((1-\epsilon_n)I - \beta_n B - \alpha_n A)y_n - y_n\| \\
&= \|\alpha_n(\gamma f(x_n) - Ay_n) + \beta_n B(x_n - x_{n+1}) + \beta_n B(x_{n+1} - y_n) - \epsilon_n y_n\| \\
&\le \alpha_n\|\gamma f(x_n) - Ay_n\| + \beta_n\|B\|\|x_n - x_{n+1}\| + \beta_n\|B\|\|x_{n+1} - y_n\| + \epsilon_n\|y_n\| \\
&\le \alpha_n\big(\|\gamma f(x_n) - Ay_n\| + \|y_n\|\big) + \beta_n\overline{\beta}\|x_n - x_{n+1}\| + \beta_n\overline{\beta}\|x_{n+1} - y_n\|.
\end{aligned}
\tag{3.30}
$$

It follows that

$$\left(1-\beta_n\overline{\beta}\right)\|x_{n+1}-y_n\| \leq \alpha_n(\|\gamma f(x_n)-Ay_n\|+\|y_n\|)+\beta_n\overline{\beta}\|x_n-x_{n+1}\|. \tag{3.31}$$

Hence, by (C1), (C2), (3.29), and (3.31), we have

$$\|x_{n+1}-y_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.32}$$

Since

$$\|x_n-y_n\| \leq \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|, \tag{3.33}$$

therefore, by (3.29) and (3.32), we obtain

$$\|x_n-y_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.34}$$

For all $x, y \in H$, by Lemma 2.2(2), the nonexpansiveness of P_Ω, the contraction of f, and the linearity of A, we have

$$\begin{aligned}\|P_\Omega(I-A+\gamma f)x-P_\Omega(I-A+\gamma f)y\| &\leq \|(I-A+\gamma f)x-(I-A+\gamma f)y\| \\ &\leq \gamma\|f(x)-f(y)\|+\|I-A\|\|x-y\| \\ &\leq \gamma\delta\|x-y\|+\left(1-\overline{\delta}\right)\|x-y\| \\ &= \left(1-\left(\overline{\delta}-\gamma\delta\right)\right)\|x-y\|.\end{aligned} \tag{3.35}$$

Therefore, $P_\Omega(I-A+\gamma f)$ is a contraction with coefficient $1-(\overline{\delta}-\gamma\delta) \in (0,1)$; Banach contraction principle guarantees that $P_\Omega(I-A+\gamma f)$ has a unique fixed point, say $w \in H$, that is, $w = P_\Omega(I-A+\gamma f)w$. Hence, by Lemma 2.1(1), we obtain

$$\langle (A-\gamma f)w, y-w\rangle \geq 0, \quad \forall y \in \Omega. \tag{3.36}$$

Next, we claim that

$$\limsup_{n\to\infty}\langle \gamma f(w)-Aw, x_n-w\rangle \leq 0. \tag{3.37}$$

To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(w)-Aw, x_n-w\rangle = \lim_{i\to\infty}\langle \gamma f(w)-Aw, x_{n_i}-w\rangle. \tag{3.38}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $\overline{w}$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup \overline{w}$ as $i \to \infty$.

Next, we prove that $\overline{w} \in \Omega$. Define the sequence of mappings $\{Q_n : H \to H\}$ and the mapping $Q : H \to H$ by

$$Q_n x = \gamma_n W_n x + (1-\gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(I-\lambda_i C_i)x, \quad \forall x \in H,$$
$$Qx = \lim_{n\to\infty} Q_n x, \tag{3.39}$$

for all $n \in \mathbb{N}$. Therefore, by (C2) and Lemma 2.4(3), we have

$$Qx = aWx + (1-a)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(I-\lambda_i C_i)x, \quad \forall x \in H, \tag{3.40}$$

where $0 < a = \lim_{n\to\infty}\gamma_n < 1$. From (C3), Lemma 2.4(3), we have that W and $\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(I-\lambda_i C_i)$ are nonexpansive. Therefore, by (C3), Lemmas 2.4(3), 2.6, 2.7, and 3.1, we have

$$\begin{aligned} F(Q) &= F(W)\cap F\left(\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(I-\lambda_i C_i)\right)\\ &= \left(\bigcap_{i=1}^{\infty}F(R_i)\right)\cap\left(\bigcap_{i=1}^{N}F(J_{M_i,\lambda_i}(I-\lambda_i C_i))\right)\\ &= \left(\bigcap_{i=1}^{\infty}F(S_i)\right)\cap\left(\bigcap_{i=1}^{\infty}F(T_i)\right)\cap\left(\bigcap_{i=1}^{N}\mathrm{VI}(H,C_i,M_i)\right), \end{aligned} \tag{3.41}$$

that is, $F(Q) = \Omega$. From (3.34), we have $\|y_{n_i} - x_{n_i}\| \to 0$ as $i \to \infty$. Thus, from (3.5) and (3.39), we get $\|Qx_{n_i} - x_{n_i}\| \to 0$ as $i \to \infty$. It follows from $x_{n_i} \rightharpoonup \overline{w}$ and by Lemma 2.8 that $\overline{w} \in F(Q)$, that is, $\overline{w} \in \Omega$. Therefore, from (3.36) and (3.38), we obtain

$$\begin{aligned} \limsup_{n\to\infty}\langle \gamma f(w) - Aw, x_n - w\rangle &= \lim_{i\to\infty}\langle \gamma f(w) - Aw, x_{n_i} - w\rangle\\ &= \langle (\gamma f - A)w, \overline{w} - w\rangle \le 0. \end{aligned} \tag{3.42}$$

Next, we prove that $x_n \to w$ as $n \to \infty$. Since $w \in \Omega$, the same as in (3.17), we have

$$\|y_n - w\| \le \|x_n - w\|. \tag{3.43}$$

Therefore, by (3.10), (3.43), Lemma 2.1(2), the contraction of f, and the linearity of A and B, we have

$$\begin{aligned} \|x_{n+1} - w\|^2 &= \left\|\alpha_n\gamma f(x_n) + \beta_n B x_n + ((1-\epsilon_n)I - \beta_n B - \alpha_n A)y_n - w\right\|^2\\ &= \big\|\alpha_n(\gamma f(x_n) - Aw) + \beta_n B(x_n - w)\\ &\quad + ((1-\epsilon_n)I - \beta_n B - \alpha_n A)(y_n - w) - \epsilon_n w\big\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \left\|\beta_n B(x_n - w) + \left((1-\epsilon_n)I - \beta_n B - \alpha_n A\right)(y_n - w)\right\|^2 \\
&\quad + 2\left\langle \alpha_n(\gamma f(x_n) - Aw) - \epsilon_n w, x_{n+1} - w\right\rangle \\
&\le \left(\beta_n\|B\|\|x_n - w\| + \left\|(1-\epsilon_n)I - \beta_n B - \alpha_n A\right\|\left\|y_n - w\right\|\right)^2 \\
&\quad + 2\alpha_n\gamma\left\langle f(x_n) - f(w), x_{n+1} - w\right\rangle \\
&\quad + 2\alpha_n\left\langle \gamma f(w) - Aw, x_{n+1} - w\right\rangle - 2\epsilon_n\langle w, x_{n+1} - w\rangle \\
&\le \left(\beta_n\overline{\beta}\|x_n - w\| + \left(1 - \beta_n\overline{\beta} - \alpha_n\overline{\delta}\right)\|x_n - w\|\right)^2 \\
&\quad + 2\alpha_n\gamma\delta\|x_n - w\|\|x_{n+1} - w\| \\
&\quad + 2\alpha_n\left\langle \gamma f(w) - Aw, x_{n+1} - w\right\rangle - 2\epsilon_n\langle w, x_{n+1} - w\rangle \\
&\le \left(1 - \alpha_n\overline{\delta}\right)^2\|x_n - w\|^2 + \alpha_n\gamma\delta\left(\|x_n - w\|^2 + \|x_{n+1} - w\|^2\right) \\
&\quad + 2\alpha_n\left\langle \gamma f(w) - Aw, x_{n+1} - w\right\rangle - 2\epsilon_n\langle w, x_{n+1} - w\rangle.
\end{aligned} \tag{3.44}
$$

If follows that

$$
\begin{aligned}
\|x_{n+1} - w\|^2 &\le \frac{1 - 2\alpha_n\overline{\delta} + \alpha_n\gamma\delta}{1 - \alpha_n\gamma\delta}\|x_n - w\|^2 + \delta_n' \\
&= \left(1 - \frac{2\left(\overline{\delta} - \gamma\delta\right)\alpha_n}{1 - \alpha_n\gamma\delta}\right)\|x_n - w\|^2 + \delta_n' \\
&\le (1 - \eta_n')\|x_n - w\|^2 + \delta_n',
\end{aligned} \tag{3.45}
$$

where $\eta_n' := (\overline{\delta} - \gamma\delta)\alpha_n/(1 - \alpha_n\gamma\delta) \in (0,1)$ and

$$
\delta_n' := \frac{1}{1 - \alpha_n\gamma\delta}\left(\alpha_n^2\overline{\delta}^2\|x_n - w\|^2 + 2\alpha_n\left\langle \gamma f(w) - Aw, x_{n+1} - w\right\rangle - 2\epsilon_n\langle w, x_{n+1} - w\rangle\right). \tag{3.46}
$$

By (3.29), (3.42), (C1), and (C3), we can found that $\lim_{n\to\infty}\eta_n' = 0$, $\sum_{n=1}^{\infty}\eta_n' = \infty$, and $\limsup_{n\to\infty}(\delta_n'/\eta_n') \le 0$. Therefore, by Lemma 2.3, we obtain that $\{x_n\}$ converges strongly to w, and so is $\{y_n\}$. This completes the proof. □

Remark 3.3. The iteration (3.5) is the difference with many others as follows.

(1) Two mappings A and B of the strongly positive linear bounded self-adjoint operator mappings are used in the iteration of $\{x_n\}$, which used only one mapping A by many others.

(2) Three parameters α_n, β_n, and ϵ_n are used in the iteration of $\{x_n\}$, which used only two parameters α_n and β_n by many others.

(3) The parameter β_n can be chosen to be $\beta_n = 0$ for all $n \in \mathbb{N}$, because the condition $\liminf_{n\to\infty}\beta_n > 0$ of Suzuki's Lemma (see [21]) is ignored in the control conditions of the iteration, which is used by many others.

(4) A solving of a common fixed point for two infinite families of strictly pseudocontractive mappings by iteration is obtained by the mapping W_n, where W_n is a W-mapping generated by $\{R_n\}$ and $\{\mu_n\}$ such that R_n is defined as in Theorem 3.2.

4. Applications

Theorem 4.1. *Let H be a real Hilbert space, let $M_i : H \to 2^H$ be a maximal monotone mapping, and let $C_i : H \to H$ be a ξ_i-cocoercive mapping for each $i = 1, 2, \ldots, N$. Let $A, B : H \to H$ be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients $\overline{\delta}, \overline{\beta} \in (0, 1]$ such that $\overline{\delta} \leq \|A\| \leq 1$ and $\|B\| = \overline{\beta}$, respectively, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\{S_n : H \to H\}$ and $\{T_n : H \to H\}$ be two infinite families of k_1 and k_2-strictly pseudocontractive mappings with a fixed point such that $k_1, k_2 \in [0, 1)$, respectively. Define a mapping $R_n : H \to H$ by*

$$R_n x = \alpha x + (1 - \alpha)(\alpha S_n x + (1 - \alpha) T_n x), \quad \forall x \in H, \tag{4.1}$$

for all $n \in \mathbb{N}$, where $\alpha \in [k, 1) \setminus \{0\}$ such that $k = \max\{k_1, k_2\}$. Let $W_n : H \to H$ be a W-mapping generated by $\{R_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0, \mu]$, for some $\mu \in (0, 1)$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(S_n)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(H, C_i, M_i)) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ is generated iteratively by

$$\begin{aligned} y_n &= \gamma_n W_n x_n + (1 - \gamma_n) \sum_{i=1}^{N} \rho_i J_{M_i, \lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n B x_n + (I - \beta_n B - \alpha_n A) y_n, \end{aligned} \tag{4.2}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\gamma_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$, $\rho_i \in (0, 1)$, and $\lambda_i \in (0, 2\xi_i]$ for each $i = 1, 2, \ldots, N$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \lim_{n\to\infty}\gamma_n < 1$ *and* $\limsup_{n\to\infty}\beta_n < 1$,

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{i=1}^{N} \rho_i = 1$,

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(C5) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} \prod_{i=1}^{n} \mu_i < \infty$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $w \in \Omega$ where $w = P_{\Omega}(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in \Omega. \tag{4.3}$$

Proof. It is concluded from Theorem 3.2 immediately, by putting $\epsilon_n = 0$ for all $n \in \mathbb{N}$. □

Theorem 4.2. *Let H be a real Hilbert space, let $M_i : H \to 2^H$ be a maximal monotone mapping, and let $C_i : H \to H$ be a ξ_i-cocoercive mapping for each $i = 1, 2, \dots, N$. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0, 1]$ such that $\overline{\delta} \le \|A\| \le 1$, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\{S_n : H \to H\}$ and $\{T_n : H \to H\}$ be two infinite families of k_1 and k_2-strictly pseudocontractive mappings with a fixed point such that $k_1, k_2 \in [0, 1)$, respectively. Define a mapping $R_n : H \to H$ by*

$$R_n x = \alpha x + (1-\alpha)(\alpha S_n x + (1-\alpha) T_n x), \quad \forall x \in H, \tag{4.4}$$

for all $n \in \mathbb{N}$, where $\alpha \in [k, 1) \setminus \{0\}$ such that $k = \max\{k_1, k_2\}$. Let $W_n : H \to H$ be a W-mapping generated by $\{R_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0, \mu]$, for some $\mu \in (0, 1)$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(S_n)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(H, C_i, M_i)) \ne \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ is generated iteratively by

$$\begin{aligned} y_n &= \gamma_n W_n x_n + (1-\gamma_n) \sum_{i=1}^{N} \rho_i J_{M_i, \lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \epsilon_n - \beta_n) I - \alpha_n A) y_n, \end{aligned} \tag{4.5}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\gamma_n\} \subset (0, 1)$, $\{\beta_n\}, \{\epsilon_n\} \subset [0, 1)$ such that $\epsilon_n \le \alpha_n$, $\rho_i \in (0, 1)$, and $\lambda_i \in (0, 2\xi_i]$ for each $i = 1, 2, \dots, N$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty}(\epsilon_n/\alpha_n) = 0$,

(C2) $0 < \lim_{n\to\infty} \gamma_n < 1$ *and* $\limsup_{n\to\infty} \beta_n < 1$,

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{i=1}^{N} \rho_i = 1$,

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, *and* $\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty$,

(C5) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} \prod_{i=1}^{n} \mu_i < \infty$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $w \in \Omega$ where $w = P_{\Omega}(I - A + \gamma f) w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f) w, y - w \rangle \ge 0, \quad \forall y \in \Omega. \tag{4.6}$$

Proof. It is concluded from Theorem 3.2 immediately, by putting $B \equiv I$. □

Theorem 4.3. *Let H be a real Hilbert space, let $M_i : H \to 2^H$ be a maximal monotone mapping, and let $C_i : H \to H$ be a ξ_i-cocoercive mapping for each $i = 1, 2, \dots, N$. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0, 1]$ such that $\overline{\delta} \le \|A\| \le 1$, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0, 1)$. Let $\{S_n : H \to H\}$ and $\{T_n : H \to H\}$ be two infinite families of k_1 and k_2-strictly pseudocontractive mappings with a fixed point such that $k_1, k_2 \in [0, 1)$, respectively. Define a mapping $R_n : H \to H$ by*

$$R_n x = \alpha x + (1-\alpha)(\alpha S_n x + (1-\alpha) T_n x), \quad \forall x \in H, \tag{4.7}$$

for all $n \in \mathbb{N}$, *where* $\alpha \in [k,1) \setminus \{0\}$ *such that* $k = \max\{k_1, k_2\}$. *Let* $W_n : H \to H$ *be a W-mapping generated by* $\{R_n\}$ *and* $\{\mu_n\}$ *such that* $\{\mu_n\} \subset (0,\mu]$, *for some* $\mu \in (0,1)$. *Assume that* $\Omega := (\bigcap_{n=1}^{\infty} F(S_n)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(H, C_i, M_i)) \neq \emptyset$ *and* $0 < \gamma < \overline{\delta}/\delta$. *For* $x_1 = u \in H$, *suppose that* $\{x_n\}$ *is generated iteratively by*

$$\begin{aligned} y_n &= \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N} \rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) y_n, \end{aligned} \tag{4.8}$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$, $\rho_i \in (0,1)$, *and* $\lambda_i \in (0, 2\xi_i]$ *for each* $i = 1,2,\dots,N$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $0 < \lim_{n\to\infty} \gamma_n < 1$ *and* $\limsup_{n\to\infty} \beta_n < 1$,

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{i=1}^{N} \rho_i = 1$,

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(C5) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} \prod_{i=1}^{n} \mu_i < \infty$,

then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $w \in \Omega$ *where* $w = P_\Omega(I - A + \gamma f)w$ *is a unique solution of the variational inequality*

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in \Omega. \tag{4.9}$$

Proof. It is concluded from Theorem 4.2 immediately, by putting $\epsilon_n = 0$ for all $n \in \mathbb{N}$. □

Theorem 4.4. *Let* H *be a real Hilbert space, let* $M_i : H \to 2^H$ *be a maximal monotone mapping, and let* $C_i : H \to H$ *be a* ξ_i*-cocoercive mapping for each* $i = 1,2,\dots,N$. *Let* $A : H \to H$ *be a strongly positive linear bounded self-adjoint operator mapping with coefficient* $\overline{\delta} \in (0,1]$ *such that* $\overline{\delta} \leq \|A\| \leq 1$, *and let* $f : H \to H$ *be a contraction mapping with coefficient* $\delta \in (0,1)$. *Let* $\{S_n : H \to H\}$ *and* $\{T_n : H \to H\}$ *be two infinite families of* k_1 *and* k_2*-strictly pseudocontractive mappings with a fixed point such that* $k_1, k_2 \in [0,1)$, *respectively. Define a mapping* $R_n : H \to H$ *by*

$$R_n x = \alpha x + (1-\alpha)(\alpha S_n x + (1-\alpha) T_n x), \quad \forall x \in H, \tag{4.10}$$

for all $n \in \mathbb{N}$, *where* $\alpha \in [k,1) \setminus \{0\}$ *such that* $k = \max\{k_1, k_2\}$. *Let* $W_n : H \to H$ *be a W-mapping generated by* $\{R_n\}$ *and* $\{\mu_n\}$ *such that* $\{\mu_n\} \subset (0,\mu]$, *for some* $\mu \in (0,1)$. *Assume that* $\Omega := (\bigcap_{n=1}^{\infty} F(S_n)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(H, C_i, M_i)) \neq \emptyset$ *and* $0 < \gamma < \overline{\delta}/\delta$. *For* $x_1 = u \in H$, *suppose that* $\{x_n\}$ *is generated iteratively by*

$$\begin{aligned} y_n &= \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N} \rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} &= \alpha_n \gamma f(x_n) + ((1-\epsilon_n)I - \alpha_n A) y_n, \end{aligned} \tag{4.11}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$, $\{\epsilon_n\} \subset [0,1)$ such that $\epsilon_n \leq \alpha_n$, $\rho_i \in (0,1)$, and $\lambda_i \in (0, 2\xi_i]$ for each $i = 1, 2, \ldots, N$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}(\epsilon_n/\alpha_n) = 0$,

(C2) $0 < \lim_{n\to\infty}\gamma_n < 1$,

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{i=1}^{N} \rho_i = 1$,

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty$,

(C5) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} \prod_{i=1}^{n}\mu_i < \infty$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $w \in \Omega$ where $w = P_\Omega(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in \Omega. \tag{4.12}$$

Proof. It is concluded from Theorem 4.2 immediately, by putting $\beta_n = 0$ for all $n \in \mathbb{N}$. □

Theorem 4.5. *Let H be a real Hilbert space, let $M_i : H \to 2^H$ be a maximal monotone mapping, and let $C_i : H \to H$ be a ξ_i-cocoercive mapping for each $i = 1, 2, \ldots, N$. Let $A : H \to H$ be a strongly positive linear bounded self-adjoint operator mapping with coefficient $\overline{\delta} \in (0,1]$ such that $\overline{\delta} \leq \|A\| \leq 1$, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0,1)$. Let $\{S_n : H \to H\}$ and $\{T_n : H \to H\}$ be two infinite families of k_1 and k_2-strictly pseudocontractive mappings with a fixed point such that $k_1, k_2 \in [0,1)$, respectively. Define a mapping $R_n : H \to H$ by*

$$R_n x = \alpha x + (1-\alpha)(\alpha S_n x + (1-\alpha)T_n x), \quad \forall x \in H, \tag{4.13}$$

for all $n \in \mathbb{N}$, where $\alpha \in [k,1) \setminus \{0\}$ such that $k = \max\{k_1, k_2\}$. Let $W_n : H \to H$ be a W-mapping generated by $\{R_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0,\mu]$, for some $\mu \in (0,1)$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(S_n)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(H, C_i, M_i)) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ is generated iteratively by

$$\begin{gathered} y_n = \gamma_n W_n x_n + (1-\gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \end{gathered} \tag{4.14}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$, $\rho_i \in (0,1)$, and $\lambda_i \in (0, 2\xi_i]$ for each $i = 1, 2, \ldots, N$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \lim_{n\to\infty}\gamma_n < 1$,

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{i=1}^{N} \rho_i = 1$,

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C5) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} \prod_{i=1}^{n}\mu_i < \infty$, *then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $w \in \Omega$ *where* $w = P_\Omega(I - A + \gamma f)w$ *is a unique solution of the variational inequality*

$$\langle (A - \gamma f)w, y - w \rangle \geq 0, \quad \forall y \in \Omega. \tag{4.15}$$

Proof. It is concluded from Theorem 4.4 immediately, by putting $\epsilon_n = 0$ for all $n \in \mathbb{N}$. □

Theorem 4.6. *Let* H *be a real Hilbert space, let* $M_i : H \to 2^H$ *be a maximal monotone mapping, and let* $C_i : H \to H$ *be a* ξ_i*-cocoercive mapping for each* $i = 1, 2, \ldots, N$*. Let* $f : H \to H$ *be a contraction mapping with coefficient* $\delta \in (0,1)$*, and let* $\{S_n : H \to H\}$ *and* $\{T_n : H \to H\}$ *be two infinite families of* k_1 *and* k_2*-strictly pseudocontractive mappings with a fixed point such that* $k_1, k_2 \in [0,1)$*, respectively. Define a mapping* $R_n : H \to H$ *by*

$$R_n x = \alpha x + (1 - \alpha)(\alpha S_n x + (1 - \alpha) T_n x), \quad \forall x \in H, \tag{4.16}$$

for all $n \in \mathbb{N}$*, where* $\alpha \in [k, 1) \setminus \{0\}$ *such that* $k = \max\{k_1, k_2\}$*. Let* $W_n : H \to H$ *be a* W*-mapping generated by* $\{R_n\}$ *and* $\{\mu_n\}$ *such that* $\{\mu_n\} \subset (0, \mu]$*, for some* $\mu \in (0,1)$*. Assume that* $\Omega := (\bigcap_{n=1}^{\infty} F(S_n)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(H, C_i, M_i)) \neq \emptyset$*. For* $x_1 = u \in H$*, suppose that* $\{x_n\}$ *is generated iteratively by*

$$\begin{gathered} y_n = \gamma_n W_n x_n + (1 - \gamma_n) \sum_{i=1}^{N} \rho_i J_{M_i, \lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) y_n, \end{gathered} \tag{4.17}$$

for all $n \in \mathbb{N}$*, where* $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$*,* $\rho_i \in (0,1)$*, and* $\lambda_i \in (0, 2\xi_i]$ *for each* $i = 1, 2, \ldots, N$ *satisfying the following conditions:*

(C1) $\lim_{n \to \infty} \alpha_n = 0$,

(C2) $0 < \lim_{n \to \infty} \gamma_n < 1$,

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{i=1}^{N} \rho_i = 1$,

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(C5) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} \prod_{i=1}^{n} \mu_i < \infty$,

then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $w \in \Omega$ *where* $w = P_\Omega f(w)$ *is a unique solution of the variational inequality*

$$\langle (I - f)w, y - w \rangle \geq 0, \quad \forall y \in \Omega. \tag{4.18}$$

Proof. It is concluded from Theorem 4.5 immediately, by putting $\gamma = \overline{\delta} = 1$ and $A \equiv I$. □

Theorem 4.7. *Let* H *be a real Hilbert space. Let* $A, B : H \to H$ *be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients* $\overline{\delta}, \overline{\beta} \in (0,1]$ *such that* $\overline{\delta} \leq \|A\| \leq 1$ *and* $\|B\| = \overline{\beta}$*, respectively, and let* $f : H \to H$ *be a contraction mapping with coefficient* $\delta \in (0,1)$*. Let* $\{S_n : H \to H\}$ *and* $\{T_n : H \to H\}$ *be two infinite families of* k_1 *and* k_2*-strictly* *pseudocontractive mappings with a fixed point such that $k_1, k_2 \in [0,1)$, respectively. Define a mapping $R_n : H \to H$ by*

$$R_n x = \alpha x + (1-\alpha)(\alpha S_n x + (1-\alpha) T_n x), \quad \forall x \in H, \tag{4.19}$$

for all $n \in \mathbb{N}$, where $\alpha \in [k,1) \setminus \{0\}$ such that $k = \max\{k_1, k_2\}$. Let $W_n : H \to H$ be a W-mapping generated by $\{R_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0,\mu]$, for some $\mu \in (0,1)$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(S_n)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ is generated iteratively by

$$\begin{gathered} y_n = \gamma_n W_n x_n + (1-\gamma_n) x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1-\epsilon_n) I - \beta_n B - \alpha_n A) y_n, \end{gathered} \tag{4.20}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$ and $\{\beta_n\}, \{\epsilon_n\} \subset [0,1)$ such that $\epsilon_n \leq \alpha_n$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} (\epsilon_n/\alpha_n) = 0$,

(C2) $0 < \lim_{n\to\infty} \gamma_n < 1$ *and* $\limsup_{n\to\infty} \beta_n < 1$,

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, *and* $\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty$,

(C5) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} \prod_{i=1}^{n} \mu_i < \infty$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $w \in \Omega$ where $w = P_\Omega(I - A + \gamma f) w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f) w, y - w \rangle \geq 0, \quad \forall y \in \Omega. \tag{4.21}$$

Proof. It is concluded from Theorem 3.2 immediately, by putting $M_i \equiv C_i \equiv 0$ for each $i = 1, 2, \ldots, N$. □

Theorem 4.8. *Let H be a real Hilbert space, let $M_i : H \to 2^H$ be a maximal monotone mapping, and let $C_i : H \to H$ be a ξ_i-cocoercive mapping for each $i = 1, 2, \ldots, N$. Let $A, B : H \to H$ be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients $\overline{\delta}, \overline{\beta} \in (0,1]$ such that $\overline{\delta} \leq \|A\| \leq 1$ and $\|B\| = \overline{\beta}$, respectively, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0,1)$. Let $\{T_n : H \to H\}$ be an infinite family of k-strictly pseudocontractive mappings with a fixed point such that $k \in [0,1)$. Define a mapping $R_n : H \to H$ by*

$$R_n x = \alpha x + (1-\alpha) T_n x, \quad \forall x \in H, \tag{4.22}$$

for all $n \in \mathbb{N}$, *where* $\alpha \in [k, 1)$. *Let* $W_n : H \to H$ *be a* W*-mapping generated by* $\{R_n\}$ *and* $\{\mu_n\}$ *such that* $\{\mu_n\} \subset (0, \mu]$, *for some* $\mu \in (0, 1)$. *Assume that* $\Omega := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(H, C_i, M_i)) \neq \emptyset$ *and* $0 < \gamma < \overline{\delta}/\delta$. *For* $x_1 = u \in H$, *suppose that* $\{x_n\}$ *is generated iteratively by*

$$y_n = \gamma_n W_n x_n + (1 - \gamma_n) \sum_{i=1}^{N} \rho_i J_{M_i, \lambda_i}(x_n - \lambda_i C_i x_n),$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + \big((1 - \epsilon_n) I - \beta_n B - \alpha_n A\big) y_n, \tag{4.23}$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}, \{\gamma_n\} \subset (0, 1)$ *and* $\{\beta_n\}, \{\epsilon_n\} \subset [0, 1)$ *such that* $\epsilon_n \le \alpha_n$, $\rho_i \in (0, 1)$, *and* $\lambda_i \in (0, 2\xi_i]$ *for each* $i = 1, 2, \ldots, N$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} (\epsilon_n / \alpha_n) = 0$,

(C2) $0 < \lim_{n\to\infty} \gamma_n < 1$ *and* $\limsup_{n\to\infty} \beta_n < 1$,

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{i=1}^{N} \rho_i = 1$,

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, *and* $\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty$,

(C5) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty} \prod_{i=1}^{n} \mu_i < \infty$,

then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $w \in \Omega$ *where* $w = P_{\Omega}(I - A + \gamma f) w$ *is a unique solution of the variational inequality*

$$\big\langle (A - \gamma f) w, y - w \big\rangle \ge 0, \quad \forall y \in \Omega. \tag{4.24}$$

Proof. It is concluded from Theorem 3.2 immediately, by putting $S_n \equiv T_n$ for all $n \in \mathbb{N}$, and note that $\alpha \in [k, 1)$ by Lemma 2.9. □

Theorem 4.9. *Let* H *be a real Hilbert space, let* $M_i : H \to 2^H$ *be a maximal monotone mapping, and let* $C_i : H \to H$ *be a* ξ_i*-cocoercive mapping for each* $i = 1, 2, \ldots, N$. *Let* $A, B : H \to H$ *be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients* $\overline{\delta}, \overline{\beta} \in (0, 1]$ *such that* $\overline{\delta} \le \|A\| \le 1$ *and* $\|B\| = \overline{\beta}$, *respectively, and let* $f : H \to H$ *be a contraction mapping with coefficient* $\delta \in (0, 1)$. *Let* $\{S_n : H \to H\}$ *and* $\{T_n : H \to H\}$ *be two infinite families of nonexpansive mappings. Define a mapping* $R_n : H \to H$ *by*

$$R_n x = \alpha x + (1 - \alpha)(\alpha S_n x + (1 - \alpha) T_n x), \quad \forall x \in H, \tag{4.25}$$

for all $n \in \mathbb{N}$, *where* $\alpha \in (0, 1)$. *Let* $W_n : H \to H$ *be a* W*-mapping generated by* $\{R_n\}$ *and* $\{\mu_n\}$ *such that* $\{\mu_n\} \subset (0, \mu]$, *for some* $\mu \in (0, 1)$. *Assume that* $\Omega := (\bigcap_{n=1}^{\infty} F(S_n)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(H, C_i, M_i)) \neq \emptyset$ *and* $0 < \gamma < \overline{\delta}/\delta$. *For* $x_1 = u \in H$, *suppose that* $\{x_n\}$ *is generated iteratively by*

$$y_n = \gamma_n W_n x_n + (1 - \gamma_n) \sum_{i=1}^{N} \rho_i J_{M_i, \lambda_i}(x_n - \lambda_i C_i x_n),$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + \big((1 - \epsilon_n) I - \beta_n B - \alpha_n A\big) y_n, \tag{4.26}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$ and $\{\beta_n\}, \{\epsilon_n\} \subset [0,1)$ such that $\epsilon_n \leq \alpha_n$, $\rho_i \in (0,1)$, and $\lambda_i \in (0, 2\xi_i]$ for each $i = 1, 2, \ldots, N$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}(\epsilon_n/\alpha_n) = 0$,

(C2) $0 < \lim_{n\to\infty}\gamma_n < 1$ *and* $\limsup_{n\to\infty}\beta_n < 1$,

(C3) $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\sum_{i=1}^{N}\rho_i = 1$,

(C4) $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$, *and* $\sum_{n=1}^{\infty}|\epsilon_{n+1} - \epsilon_n| < \infty$,

(C5) $\sum_{n=1}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty}\prod_{i=1}^{n}\mu_i < \infty$,

then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $w \in \Omega$ where $w = P_\Omega(I - A + \gamma f)w$ is a unique solution of the variational inequality

$$\langle (A - \gamma f)w, y - w\rangle \geq 0, \quad \forall y \in \Omega. \tag{4.27}$$

Proof. It is concluded from Theorem 3.2 immediately, by putting $k_1 = k_2 = 0$. □

Theorem 4.10. *Let H be a real Hilbert space, let $M_i : H \to 2^H$ be a maximal monotone mapping, and let $C_i : H \to H$ be a ξ_i-cocoercive mapping for each $i = 1, 2, \ldots, N$. Let $A, B : H \to H$ be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients $\overline{\delta}, \overline{\beta} \in (0,1]$ such that $\overline{\delta} \leq \|A\| \leq 1$ and $\|B\| = \overline{\beta}$, respectively, and let $f : H \to H$ be a contraction mapping with coefficient $\delta \in (0,1)$. Let $\{T_n : H \to H\}$ be an infinite family of nonexpansive mappings. Define a mapping $R_n : H \to H$ by*

$$R_n x = \alpha x + (1 - \alpha)T_n x, \quad \forall x \in H, \tag{4.28}$$

for all $n \in \mathbb{N}$, where $\alpha \in [0,1)$. Let $W_n : H \to H$ be a W-mapping generated by $\{R_n\}$ and $\{\mu_n\}$ such that $\{\mu_n\} \subset (0, \mu]$, for some $\mu \in (0,1)$. Assume that $\Omega := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{i=1}^{N} \mathrm{VI}(H, C_i, M_i)) \neq \emptyset$ and $0 < \gamma < \overline{\delta}/\delta$. For $x_1 = u \in H$, suppose that $\{x_n\}$ is generated iteratively by

$$\begin{aligned} y_n &= \gamma_n W_n x_n + (1 - \gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n), \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n B x_n + ((1 - \epsilon_n)I - \beta_n B - \alpha_n A)y_n, \end{aligned} \tag{4.29}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$ and $\{\beta_n\}, \{\epsilon_n\} \subset [0,1)$ such that $\epsilon_n \leq \alpha_n$, $\rho_i \in (0,1)$, and $\lambda_i \in (0, 2\xi_i]$ for each $i = 1, 2, \ldots, N$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}(\epsilon_n/\alpha_n) = 0$,

(C2) $0 < \lim_{n\to\infty}\gamma_n < 1$ *and* $\limsup_{n\to\infty}\beta_n < 1$,

(C3) $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\sum_{i=1}^{N}\rho_i = 1$,

(C4) $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$, *and* $\sum_{n=1}^{\infty}|\epsilon_{n+1} - \epsilon_n| < \infty$,

(C5) $\sum_{n=1}^{\infty}|\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=1}^{\infty}\prod_{i=1}^{n}\mu_i < \infty$, *then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $w \in \Omega$ *where* $w = P_\Omega(I - A + \gamma f)w$ *is a unique solution of the variational inequality*

$$\langle (A - \gamma f)w, y - w\rangle \geq 0, \quad \forall y \in \Omega. \tag{4.30}$$

Proof. It is concluded from Theorem 4.8 immediately, by putting $k = 0$. □

Theorem 4.11. *Let* H *be a real Hilbert space. Let* $A, B : H \to H$ *be two mappings of the strongly positive linear bounded self-adjoint operator mappings with coefficients* $\overline{\delta}, \overline{\beta} \in (0,1]$ *such that* $\overline{\delta} \leq \|A\| \leq 1$ *and* $\|B\| = \overline{\beta}$, *respectively, and let* $f : H \to H$ *be a contraction mapping with coefficient* $\delta \in (0,1)$. *Let* $T : H \to H$ *be a nonexpansive mapping. Assume that* $F(T) \neq \emptyset$ *and* $0 < \gamma < \overline{\delta}/\delta$. *For* $x_1 = u \in H$, *suppose that* $\{x_n\}$ *is generated iteratively by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n B x_n + \big((1 - \epsilon_n)I - \beta_n B - \alpha_n A\big)(\sigma_n T x_n + (1 - \sigma_n)x_n), \tag{4.31}$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}, \{\sigma_n\} \subset (0,1)$ *and* $\{\beta_n\}, \{\epsilon_n\} \subset [0,1)$ *such that* $\epsilon_n \leq \alpha_n$ *satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}(\epsilon_n/\alpha_n) = 0$,

(C2) $0 < \lim_{n\to\infty}\sigma_n < 1$ *and* $\limsup_{n\to\infty}\beta_n < 1$,

(C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$,

(C5) $\sum_{n=1}^{\infty} |\epsilon_{n+1} - \epsilon_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n| < \infty$,

then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $w \in F(T)$ *where* $w = P_{F(T)}(I - A + \gamma f)w$ *is a unique solution of the variational inequality*

$$\langle (A - \gamma f)w, y - w\rangle \geq 0, \quad \forall y \in F(T). \tag{4.32}$$

Proof. From Theorem 4.10, putting $\alpha = 0$ and $M_i \equiv C_i \equiv 0$ for all $i = 1, 2, \dots, N$. Setting $T_1 \equiv T$, $T_n \equiv I$ for all $n = 2, 3, \dots$, and let $\mu_n \subset (0, \mu]$ for some $\mu \in (0,1)$ such that $\sum_{n=1}^{\infty}\prod_{i=1}^{n}\mu_i < \infty$. Therefore, from the definition of R_n in Theorem 4.10, we have $R_1 = T_1 = T$ and $R_n = I$ for all $n = 2, 3, \dots$. Since W_n is a W-mapping generated by $\{R_n\}$ and $\{\mu_n\}$, therefore by the definition of $U_{n,i}$ and W_n in (1.16), we have $U_{n,i} = I$ for all $i = 2, 3, \dots$ and $W_n = U_{n,1} = \mu_1 R_1 U_{n,2} + (1 - \mu_1)I = \mu_1 T + (1 - \mu_1)I$. Hence, by Theorem 4.10, we obtain

$$\begin{aligned} y_n &= \gamma_n W_n x_n + (1 - \gamma_n)\sum_{i=1}^{N}\rho_i J_{M_i,\lambda_i}(x_n - \lambda_i C_i x_n) \\ &= \gamma_n(\mu_1 T x_n + (1 - \mu_1)x_n) + (1 - \gamma_n)\left(\sum_{i=1}^{N}\rho_i\right)x_n \\ &= \gamma_n(\mu_1 T x_n + (1 - \mu_1)x_n) + (1 - \gamma_n)x_n \\ &= \sigma_n T x_n + (1 - \sigma_n)x_n, \end{aligned} \tag{4.33}$$

where $\sigma_n := \gamma_n \mu_1$. This completes the proof. □

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