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Research Article

On Singular Integrals with Cauchy Kernel on Weight Subspaces: The Basicity Property of Sines and Cosines Systems in Weight Spaces

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A singular operator with Cauchy kernel on the subspaces of weight Lebesgue space is considered. A sufficient condition for a bounded action of this operator from a subspace to another subspace of weight Lebesgue space of functions is found. These conditions are not identical with Muckenhoupt conditions. Moreover, the completeness, minimality, and basicity of sines and cosines systems are considered.

1. Introduction

Consider the following singular operator with Cauchy kernel:

$$[Sf](t) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{f(s)ds}{1-e^{-i(s-t)}}, \tag{1.1}$$

where $f \in L_{p,\rho}(-\pi,\pi)$, $1 < p < +\infty$, is an appropriate density, $\rho(t)$ is a weight function of the form

$$\rho(t) = \prod_{k=1}^{r}|t-t_k|^{p\beta_k}, \tag{1.2}$$

$\{t_k\}_1^r \subset [-\pi,\pi]$ $(t_i \neq t_j$ for $i \neq j)$, $\{\beta_k\}_1^r \subset R$ are real numbers.

Under $L_{p,\rho}$ we understand a Lebesgue weight space with the norm

$$\|f\|_{p,\rho} \equiv \left(\int_{-\pi}^{\pi} |f(t)|^{p}\rho(t)dt\right)^{1/p}. \tag{1.3}$$

Bounded action of the operator S in the spaces $L_{p,\rho}$ plays an important role in many problems of mathematics including the theory of bases. This direction has been well developed and treated in the known monographs. We will need the following.

Statement 1. The operator S is bounded in $L_{p,\rho}$ if and only if the inequalities

$$-\frac{1}{p} < \beta_k < \frac{1}{q}, \quad k = \overline{1,r}, \qquad \frac{1}{p} + \frac{1}{q} = 1, \tag{1.4}$$

are fulfilled.

Concerning this fact a one can see the monograph [1] and papers [2–4]. Inequalities (1.4) are Muckenhoupt condition with respect to the weight function $\rho(t)$ with degrees $p\beta_k$. It is known that the classic system of exponents $\{e^{int}\}_{n\in Z}$ (Z are integers) forms a basis in $L_{p,\rho}$ if and only if inequalities (1.4) hold (see, e.g., [3, 4]).

It turns that if you consider the singular operator acting on the subspace of the weighted Lebesgue space, then inequality (1.4) is not necessary for the bounded action. At different points of degeneration the change interval of the corresponding exponent is expanded. This paper is devoted to studying these issues.

2. Some Necessary Facts

Let $L^0_{p,\omega} \equiv L_{p,\omega}(0,\pi)$, $\omega(t)$—a weight function of the form

$$\omega(t) \equiv \prod_{k=0}^{r} |t-\tau_k|^{p\alpha_k}, \tag{2.1}$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_r = \pi$, $\{\alpha_k\}_0^r \subset R$.

Denote the space of even (odd) functions in $L_{p,\rho}$ by $L^+_{p,\rho}$ ($L^-_{p,\rho}$), that is,

$$L^{\pm}_{p,\rho} \equiv \{f \in L_{p,\rho} : f(-t) = \pm f(t),\ t \in [-\pi,\pi]\}. \tag{2.2}$$

We'll need the following identity:

$$\frac{1}{1-e^{i(\theta-\varphi)}} - \frac{1}{1-e^{i(\theta+\varphi)}} = \frac{i}{2}\frac{\sin\varphi}{\sin((\varphi-\theta)/2)\sin((\theta+\varphi)/2)}. \tag{2.3}$$

Indeed, we have

$$
\begin{aligned}
\frac{1}{1-e^{i(\theta-\varphi)}}-\frac{1}{1-e^{i(\theta+\varphi)}} &= \frac{1-e^{-i(\theta-\varphi)}}{\left(1-e^{i(\theta-\varphi)}\right)\left(1-e^{-i(\theta-\varphi)}\right)}-\frac{1-e^{-i(\theta+\varphi)}}{\left(1-e^{i(\theta+\varphi)}\right)\left(1-e^{-i(\theta+\varphi)}\right)} \\
&= \frac{1-\cos(\theta-\varphi)+i\sin(\theta-\varphi)}{(1-\cos(\theta-\varphi))^2+\sin^2(\theta-\varphi)}-\frac{1-\cos(\theta+\varphi)+i\sin(\theta+\varphi)}{(1-\cos(\theta+\varphi))^2+\sin^2(\theta+\varphi)} \\
&= i\left[\frac{\sin(\theta-\varphi)}{2-2\cos(\theta-\varphi)}-\frac{\sin(\theta+\varphi)}{2-2\cos(\theta+\varphi)}\right] \\
&= i\left[\frac{\cos((\theta-\varphi)/2)}{2\sin((\theta-\varphi)/2)}-\frac{\cos((\theta+\varphi)/2)}{2\sin((\theta+\varphi)/2)}\right] \\
&= \frac{i}{2}\frac{\cos((\theta-\varphi)/2)\sin((\theta+\varphi)/2)-\cos((\theta+\varphi)/2)\sin((\theta-\varphi)/2)}{\sin((\theta-\varphi)/2)\sin((\theta+\varphi)/2)} \\
&= \frac{i}{2}\frac{\sin\varphi}{\sin((\theta-\varphi)/2)\sin((\theta+\varphi)/2)}.
\end{aligned}
\tag{2.4}
$$

For compactness of the notation, we assume $e(x)\equiv 1/(1-e^{ix})$. Thus,

$$
e(\theta-\varphi)-e(\theta+\varphi)=\frac{i}{2}\frac{\sin\varphi}{\sin((\theta-\varphi)/2)\sin((\theta+\varphi)/2)}. \tag{2.5}
$$

From this identity, we can easily get the following relations:

$$
\begin{aligned}
e(\theta-\varphi)-e(\theta+\varphi) &= -\frac{i}{2}\frac{\cos(\varphi/2)}{\cos(\theta/2)}\left[\frac{1}{\sin((\varphi-\theta)/2)}+\frac{1}{\sin((\varphi+\theta)/2)}\right], \\
e(\theta-\varphi)-e(\theta+\varphi) &= -\frac{i}{2}\frac{\sin(\varphi/2)}{\sin(\theta/2)}\left[\frac{1}{\sin((\varphi-\theta)/2)}-\frac{1}{\sin((\varphi+\theta)/2)}\right].
\end{aligned}
\tag{2.6}
$$

The authors of the papers [4–7] used these relations earlier while establishing the basicity criterion of the system of sines and cosines with linear phases in L_p. Thus, the following is valid.

Lemma 2.1. *The following identities are true:*

$$
\begin{aligned}
\frac{1}{1-e^{i(\theta-\varphi)}}-\frac{1}{1-e^{i(\theta+\varphi)}} &= -\frac{i}{2}\frac{\cos(\varphi/2)}{\cos(\theta/2)}\left[\frac{1}{\sin((\varphi-\theta)/2)}+\frac{1}{\sin((\varphi+\theta)/2)}\right], \\
\frac{1}{1-e^{i(\theta-\varphi)}}-\frac{1}{1-e^{i(\theta+\varphi)}} &= -\frac{i}{2}\frac{\sin(\varphi/2)}{\sin(\theta/2)}\left[\frac{1}{\sin((\varphi-\theta)/2)}-\frac{1}{\sin((\varphi+\theta)/2)}\right].
\end{aligned}
\tag{2.7}
$$

In the similar way, we obtain

$$\begin{aligned}
&\frac{1}{1-e^{i(\theta-\varphi)}}+\frac{1}{1-e^{i(\theta+\varphi)}}\\
&\quad=\frac{e^{-i((\theta-\varphi)/2)}}{e^{-i((\theta-\varphi)/2)}-e^{i((\theta-\varphi)/2)}}+\frac{e^{-i((\theta+\varphi)/2)}}{e^{-i((\theta+\varphi)/2)}-e^{i((\theta+\varphi)/2)}}\\
&\quad=-\frac{1}{2i}\frac{e^{-i((\theta-\varphi)/2)}}{\sin((\theta-\varphi)/2)}-\frac{1}{2i}\frac{e^{-i((\theta+\varphi)/2)}}{\sin((\theta+\varphi)/2)}\\
&\quad=-\frac{1}{2i}\left[\frac{\cos((\theta-\varphi)/2)-i\sin((\theta-\varphi)/2)}{\sin((\theta-\varphi)/2)}+\frac{\cos((\theta+\varphi)/2)-i\sin((\theta+\varphi)/2)}{\sin((\theta+\varphi)/2)}\right]\\
&\quad=-\frac{1}{2i}\left[\frac{\cos((\theta-\varphi)/2)}{\sin((\theta-\varphi)/2)}+\frac{\cos((\theta+\varphi)/2)}{\sin((\theta+\varphi)/2)}\right]+1\\
&\quad=1-\frac{1}{2i}\frac{\sin\theta}{\sin((\theta-\varphi)/2)\sin((\theta+\varphi)/2)}.
\end{aligned}\tag{2.8}$$

Further, we must take into account the following relation:

$$\begin{aligned}
\frac{1}{\sin((\theta-\varphi)/2)\sin((\theta+\varphi)/2)}&=\frac{1}{2\sin(\theta/2)\cos(\varphi/2)}\left[\frac{1}{\sin((\theta-\varphi)/2)}+\frac{1}{\sin((\theta+\varphi)/2)}\right],\\
\frac{1}{\sin((\theta-\varphi)/2)\sin((\theta+\varphi)/2)}&=\frac{1}{2\cos(\theta/2)\sin(\varphi/2)}\left[\frac{1}{\sin((\theta-\varphi)/2)}-\frac{1}{\sin((\theta+\varphi)/2)}\right].
\end{aligned}\tag{2.9}$$

As a result, we have

$$e(\theta-\varphi)+e(\theta+\varphi)=1-\frac{1}{2i}\frac{\sin(\theta/2)}{\sin(\varphi/2)}\left[\frac{1}{\sin((\theta-\varphi)/2)}-\frac{1}{\sin((\theta+\varphi)/2)}\right],\tag{2.10}$$

$$e(\theta-\varphi)+e(\theta+\varphi)=1-\frac{1}{2i}\frac{\cos(\theta/2)}{\cos(\varphi/2)}\left[\frac{1}{\sin((\theta-\varphi)/2)}+\frac{1}{\sin((\theta+\varphi)/2)}\right].\tag{2.11}$$

Assume

$$K_\varphi(\theta)=e(\theta-\varphi)+e(\theta+\varphi)-1.\tag{2.12}$$

Thus,

$$\begin{aligned}
K_\varphi(-\theta)=-\frac{1}{2i}\frac{\cos(\theta/2)}{\cos(\varphi/2)}\left[\frac{1}{-\sin((\theta+\varphi)/2)}-\frac{1}{\sin((\theta-\varphi)/2)}\right]&=\frac{1}{2i}\frac{\cos(\theta/2)}{\cos(\varphi/2)}\\
\times\left[\frac{1}{\sin((\theta+\varphi)/2)}+\frac{1}{\sin((\theta-\varphi)/2)}\right]&=-K_\varphi(\theta).
\end{aligned}\tag{2.13}$$

In sequel the following lemma is valid.

Lemma 2.2. *The following identities are true:*

$$\frac{1}{1-e^{i(\theta-\varphi)}}+\frac{1}{1-e^{i(\theta+\varphi)}}=1-\frac{1}{2i}\frac{\sin(\theta/2)}{\sin(\varphi/2)}\left[\frac{1}{\sin((\theta-\varphi)/2)}-\frac{1}{\sin((\theta+\varphi)/2)}\right],$$
$$\frac{1}{1-e^{i(\theta-\varphi)}}+\frac{1}{1-e^{i(\theta+\varphi)}}=1-\frac{1}{2i}\frac{\cos(\theta/2)}{\cos(\varphi/2)}\left[\frac{1}{\sin((\theta-\varphi)/2)}-\frac{1}{\sin((\theta+\varphi)/2)}\right]. \tag{2.14}$$

3. Boundedness of Singular Operators on Subspace of Even Functions

Let $f \in L^{-}_{p,\rho}$. We have

$$\begin{aligned}[Sf](t) &= \frac{1}{2\pi i}\int_{-\pi}^{\pi}\frac{f(s)ds}{1-e^{-i(s-t)}} = \frac{1}{2\pi i}\left[\int_{0}^{\pi}\frac{f(s)ds}{1-e^{i(t-s)}}+\int_{-\pi}^{0}\frac{f(s)ds}{1-e^{i(t-s)}}\right]\\ &= \frac{1}{2\pi i}\int_{0}^{\pi}\left[\frac{1}{1-e^{i(t-s)}}-\frac{1}{1-e^{i(t+s)}}\right]f(s)ds = -\frac{1}{4\pi}\int_{0}^{\pi}K_1(t,s)f(s)ds\\ &= -\frac{1}{4\pi}\int_{0}^{\pi}K_2(t,s)f(s)ds,\end{aligned} \tag{3.1}$$

where the kernels $K_i(t,s)$, $i=1,2$, are determined by the expressions

$$\begin{aligned}K_1(t,s) &= \frac{\cos(s/2)}{\cos(t/2)}\left[\frac{1}{\sin((s-t)/2)}+\frac{1}{\sin((s+t)/2)}\right],\\ K_2(t,s) &= \frac{\sin(s/2)}{\sin(t/2)}\left[\frac{1}{\sin((s-t)/2)}-\frac{1}{\sin((s+t)/2)}\right].\end{aligned} \tag{3.2}$$

Continue the weight $\omega(t)$ to the interval $(-\pi,0)$ by parity and denote by μ:

$$\mu(t)=\begin{cases}\omega(t), & 0\le t\le\pi,\\ \omega(-t), & -\pi\le t<0.\end{cases} \tag{3.3}$$

It is obvious that $\{\pm\tau_k\}_0^r \subset [-\pi,\pi]$ are the degeneration points of $\mu(t)$. Thus, $\mu(t)=\omega(|t|)$, $t\in[-\pi,\pi]$, that is,

$$\mu(t)=\prod_{k=0}^{r}||t|-\tau_k|^{p\alpha_k}. \tag{3.4}$$

Accept the denotation

$$f\sim g \text{ as } t\longrightarrow a, \quad \text{if } \exists\delta>0:\delta\le\left|\frac{f}{g}\right|\le\delta^{-1}, \text{ as } t\longrightarrow a. \tag{3.5}$$

It is easy to see that the following holds

$$||t| - \tau_k| \sim |t - \tau_k||t + \tau_k|, \quad t \in [-\pi, \pi], \text{ for } k > 1. \tag{3.6}$$

As a result, for μ, we get the representation

$$\mu(t) \sim \rho_0(t) \equiv |t|^{p\alpha_0} \prod_{k=1}^{r} |t - \tau_k|^{p\alpha_k} |t + \tau_k|^{p\alpha_k}, \quad t \in [-\pi, \pi]. \tag{3.7}$$

It is obvious that the singular integral S boundedly acts in $L_{p,\mu}(-\pi,\pi)$ if and only if it boundedly acts in $L_{p,\rho_0}(-\pi,\pi)$. Statement 1 is valid also in the case if the Cauchy kernel is replaced by the Hilbert kernel $(1/\sin((\varphi - t)/2))$. So, assume that the inequalities $-1/p < \alpha_k < 1/q$, $k = \overline{0,r}$, are fulfilled. Then, from Statement 1, we directly get that S boundedly acts from L^-_{p,ρ_0} to L_{p,ρ_0} and so from $L^-_{p,\mu}$ to $L_{p,\mu}$. Assume

$$k^{\pm}(s,t) = \frac{\sin(s/2)}{\sin(t/2)} \frac{1}{\sin((s \pm t)/2)}, \quad (s,t) \in [-\pi,\pi] \times [-\pi,\pi], \tag{3.8}$$

and consider the integral operator $I^{\pm}$:

$$I^{\pm} f = \int_{-\pi}^{\pi} k^{\pm}(s,t) f(s) ds, \tag{3.9}$$

with the kernel $k^{\pm}(s,t)$. We have $(f \in L^-_{p,\mu}(-\pi,\pi))$:

$$\|Sf\|_{p,\mu} = \frac{1}{4\pi} \left\| I^- \tilde{f} - I^+ \tilde{f} \right\|_{p,\mu} \leq \frac{1}{4\pi} \left(\left\| I^- \tilde{f} \right\|_{p,\mu} + \left\| I^+ \tilde{f} \right\|_{p,\mu} \right), \tag{3.10}$$

where

$$\tilde{f}(t) = \begin{cases} f(t), & t \in [0,\pi], \\ 0, & t \in [-\pi, 0). \end{cases} \tag{3.11}$$

Consequently,

$$\begin{aligned} \left\| I^- \tilde{f} \right\|^p_{p,\mu} &= \int_{-\pi}^{\pi} \left| \left(I^- \tilde{f} \right)(t) \right|^p \mu(t) dt \sim \int_{-\pi}^{\pi} \left| I^- \tilde{f} \right|^p \rho_0(t) dt \\ &= \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \frac{\tilde{f}(s) \sin(s/2) ds}{\sin((s-t)/2)} \right|^p \rho_0(t) \left| \sin \frac{t}{2} \right|^{-p} dt. \end{aligned} \tag{3.12}$$

It is obvious that

$$\rho_0(t) \left| \sin \frac{t}{2} \right|^{-p} \sim \theta(t), \tag{3.13}$$

where $\theta(t) = |s|^{p(\alpha_0-1)}\prod_{k=1}^{r}|t-\tau_k|^{p\alpha_k}|t+\tau_k|^{p\alpha_k}$. So, if the inequalities

$$-\frac{1}{p} < \alpha_0 - 1 < \frac{1}{q}, \qquad -\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad k = \overline{1,r}, \tag{3.14}$$

hold, then from Statement 1 we obtain

$$\begin{aligned}\left\|I^{-}\widetilde{f}\right\|_{L_{p,\mu}}^{p} &\le c\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi}\frac{\widetilde{f}(s)\sin(s/2)}{\sin((s-t)/2)}ds\right|^{p}\theta(t)dt \le c\left\|\widetilde{f}(s)\sin\frac{s\theta(s)}{2}\right\|_{L_{p,\theta}}^{p}\\ &\le c\int_{0}^{\pi}\left|f(s)s\right|^{p}\theta(s)ds = c\left\|f\right\|_{L_{p,\omega}^{0}}^{p},\end{aligned} \tag{3.15}$$

where c is a constant independent from f (different in different places). As a result, we get that if the inequalities (3.14) hold, the operator I^{-} boundedly acts from $L_{p,\omega}^{0}$ to $L_{p,\mu}$. The same conclusion is true for the operator I^{+} as well. As a result, we get that while fulfilling the conditions

$$\alpha_0 \in \left(-\frac{1}{p},\frac{1}{q}\right)\bigcup\left(\frac{1}{q},1+\frac{1}{q}\right), \qquad \alpha_k \in \left(-\frac{1}{p},\frac{1}{q}\right), \quad k = \overline{1,r}, \tag{3.16}$$

the operator S boundedly acts from $L_{p,\mu}^{-}$ to $L_{p,\mu}$. On the other hand, it is easily seen that $[Sf]\,(-t) = [Sf](t)$. As a result, we get that the operator S boundedly acts from $L_{p,\mu}^{-}$ to $L_{p,\mu}^{+}$.

Now, consider the case when $\alpha_0 = 1/q$ and α_k, $k = \overline{1,r}$, satisfy conditions (3.16). Let $p_{\pm\varepsilon} = p \pm \varepsilon$ and $q_{\pm\varepsilon}$ be a number conjugated to $p_{\pm\varepsilon}$. It is obvious that the relations

$$\alpha_0 \in \left(-\frac{1}{p_{\pm\varepsilon}},\frac{1}{q_{\pm\varepsilon}}\right)\bigcup\left(\frac{1}{q_{\pm\varepsilon}},1+\frac{1}{q_{\pm\varepsilon}}\right), \qquad \alpha_k \in \left(-\frac{1}{p_{\pm\varepsilon}},\frac{1}{q_{\pm\varepsilon}}\right), \quad k = \overline{1,r}, \tag{3.17}$$

are fulfilled for sufficiently small $\varepsilon > 0$. Then, from the previous reasonings we get that the operator S boundedly acts from $L_{p_{\pm\varepsilon},\mu}^{-}$ to $L_{p_{\pm\varepsilon},\mu}^{+}$. As a result, it follows from the Riesz-Torin theorem (see, e.g., [7, page 144]) that the operator S boundedly acts from $L_{p,\mu}^{-}$ to $L_{p,\mu}^{+}$. We get the following.

Statement 2. Let the inequalities

$$-\frac{1}{p} < \alpha_0 < 2 - \frac{1}{p}, \qquad -\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad k = \overline{1,r}, \tag{3.18}$$

be fulfilled. Then, the operator S boundedly acts from $L_{p,\mu}^{-}$ to $L_{p,\mu}^{+}$.

Now, consider the representation of the operator S by the kernel $K_1(t,s)$. Having paid attention to the expression

$$\frac{\cos(s/2)}{\cos(t/2)} = \frac{\sin((\pi-s)/2)}{\sin((\pi-t)/2)}, \tag{3.19}$$

similar to the previous case we establish that the boundedness of the operator S holds also in the case when the change interval of the exponent α_r extends by $(-1/p, 2-1/p)$. In the conclusion we get that the following main theorem is valid.

Theorem 3.1. *Let the weight function ω be defined by the expression* (2.1) *and assume that the inequalities*

$$-\frac{1}{p} < \alpha_0 < 2 - \frac{1}{p}, \qquad -\frac{1}{p} < \alpha_m < 2 - \frac{1}{p}, \qquad -\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad k = \overline{1, r-1}, \tag{3.20}$$

are fulfilled. Then the singular operator S:

$$Sf = \int_{-\pi}^{\pi} k(t,s) f(s) ds, \tag{3.21}$$

with Cauchy kernel $k(t,s) = 1/(1 - e^{-i(s-t)})$, boundedly acts from $L^{-}_{p,\mu}$ to $L^{+}_{p,\mu}$, where $\mu(t) = \omega(|t|)$, $t \in (-\pi, \pi)$.

4. Boundedness of Singular Operators on Subspace of Odd Functions

Let the weight function $\omega(t)$ be defined by expression (2.1) and assume $\mu(t) \equiv \omega(|t|)$, $t \in (-\pi, \pi)$. Denote by $K(s;t)$ the following Cauchy-type kernel

$$K(s;t) = \frac{1}{1 - e^{i(t-s)}} - \frac{1}{2}, \quad (s;t) \in (-\pi, \pi) \times (-\pi, \pi). \tag{4.1}$$

Appropriate integral operator denote by $\mathcal{K}$:

$$[\mathcal{K} f](t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(s;t) f(s) ds. \tag{4.2}$$

Let the following inequalities

$$-\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad k = \overline{0, r}, \tag{4.3}$$

be fulfilled. We have

$$\int_{-\pi}^{\pi} |f(t)| dt = \int_{-\pi}^{\pi} |f(t)| \mu^{1/p} \mu^{-1/p} dt \le \|f\|_{p,\mu} \left(\int_{-\pi}^{\pi} \mu^{-q/p} dt \right)^{1/q}. \tag{4.4}$$

From (4.3) it follows that $-q\alpha_k > -1$, $k = \overline{0, r}$, and as a result $\mu^{-q/p} \in L_1(-\pi, \pi)$. As a result, from Statement 1 we obtain that the integral operator $\mathcal{K}$ boundedly acts in $L_{p,\mu}$, if the inequalities (4.3) hold. In particular, it follows that the operator $\mathcal{K}$ boundedly acts from $L^+_{p,\mu}$ to $L_{p,\mu}$, if the inequalities (4.3) are fulfilled, that is,

$$\|\mathcal{K}f\|_{L_{p,\mu}} \le c\|f\|_{L^+_{p,\mu}}, \quad \forall f \in L^+_{p,\mu}, \tag{4.5}$$

where c is a constant independent from f. On the other hand for $f \in L^+_{p,\mu}$ we have

$$[\mathcal{K}f](t) = \frac{1}{2\pi}\int_0^{\pi} f(s)[K(s;t) + K(-s;t) - 1]ds. \tag{4.6}$$

Pay an attention to the relation (2.13), we obtain that $[\mathcal{K}f](-t) = -[\mathcal{K}f](t)$, $t \in (-\pi,\pi)$. Then from (4.5) yields

$$\|\mathcal{K}f\|_{L^0_{p,\omega}} \le c\|f\|_{L^0_{p,\omega}}, \quad \forall f \in L^0_{p,\omega}. \tag{4.7}$$

Now, let

$$-1-\frac{1}{p} < \alpha_0 < -\frac{1}{p}, \qquad -\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad k = \overline{1,r} \tag{4.8}$$

be fulfilled. Assume

$$K^{\pm}(s;t) = -\frac{1}{2i}\frac{\sin(t/2)}{\sin(s/2)\sin((t \mp s)/2)}, \quad (s;t) \in (0,\pi)\times(0,\pi). \tag{4.9}$$

Denote the integral operator with kernel $K^{\pm}(s;t)$ by $S^{\pm}$, that is,

$$[S^{\pm}f](t) = \frac{1}{2\pi}\int_0^{\pi} K^{\pm}(s;t)f(s)ds, \quad t \in (0,\pi). \tag{4.10}$$

Taking into account the relation (2.10), for $f \in L^+_{p,\mu}$ we have

$$\begin{aligned}[\mathcal{K}f](t) &= \frac{1}{2\pi}\int_0^{\pi} f(s)[K(s;t) + K(-s;t) - 1]ds \\ &= \frac{1}{2\pi}\int_0^{\pi}\left[K^+(s;t) - K^-(s;t)\right]f(s)ds = \left[S^+ - S^-\right]f,\end{aligned} \tag{4.11}$$

that is,

$$[\mathcal{K}f](t) = \left[\left(S^+ - S^-\right)f\right](t), \quad t \in (-\pi,\pi). \tag{4.12}$$

Take into account the nonparity $[\mathcal{K}f](t)$ on $(-\pi,\pi)$ we obtain

$$2\|\mathcal{K}f\|^p_{L^0_{p,\omega}} = \|\mathcal{K}f\|^p_{L_{p,\mu}} \le \left(\|S^+f\|_{L_{p,\mu}} + \|S^-f\|_{L_{p,\mu}}\right)^p. \tag{4.13}$$

Let

$$\tilde{f}(t) = \begin{cases} f(t), & t \in (0,\pi), \\ 0, & t \in (-\pi, 0). \end{cases} \tag{4.14}$$

Consider the operator S^{+}. We have

$$\left[S^{+} f\right](t) = \frac{\sin(t/2)}{2\pi} \int_{0}^{\pi} \frac{f(s)\sin^{-1}(s/2)}{\sin((t-s)/2)} ds = \frac{\sin(t/2)}{2\pi} \int_{-\pi}^{\pi} \frac{\tilde{f}(s)\sin^{-1}(s/2)}{\sin((t-s)/2)} ds. \tag{4.15}$$

Thus

$$\left\|S^{+} f\right\|_{L_{p,\mu}}^{p} = c \int_{-\pi}^{\pi} \left|\left[S^{+} f\right](t)\right|^{p} \mu(t) dt = c \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \frac{\tilde{f}(s)\sin^{-1}(s/2)}{\sin((t-s)/2)} ds \right|^{p} \left| \sin \frac{t}{2} \right|^{p} \mu(t) dt. \tag{4.16}$$

Further, we must take into account the expression $\sin(t/2) \sim t$, $t \in (-\pi, \pi)$. As a result, from the previous relation we have

$$\left\|S^{+} f\right\|_{L_{p,\mu}}^{p} \leq c \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} \frac{\tilde{f}(s)\sin^{-1}(s/2)}{\sin((t-s)/2)} ds \right|^{p} \tilde{\mu}(t) dt, \tag{4.17}$$

where

$$\tilde{\mu}(t) = |t|^{p(\alpha_0+1)} \prod_{k=1}^{r} \left||t| - t_k\right|^{p\alpha_k}. \tag{4.18}$$

It is clear that for weight function $\tilde{\mu}(t)$ Muckenhoupt condition is fulfilled and applying Statement 1 to the expression (4.17) we obtain

$$\begin{aligned} \left\|S^{+} f\right\|_{L_{p,\mu}}^{p} &\leq c \left\| \tilde{f}(s) \sin^{-1} \frac{s}{2} \right\|_{L_{p,\tilde{\mu}}}^{p} = c \int_{-\pi}^{\pi} \left|\tilde{f}(s)\right|^{p} \left| \sin^{-1} \frac{s}{2} \right|^{p} \tilde{\mu}(s) ds \\ &\leq c \int_{0}^{\pi} \left|f(s)\right|^{p} |s|^{-p} \tilde{\mu}(s) ds = c \int_{0}^{\pi} \left|f(s)\right|^{p} \omega(s) ds = c \|f\|_{L_{p,\mu}^{+}}^{p}. \end{aligned} \tag{4.19}$$

In the similar way we establish the validity of the inequality

$$\left\|S^{-} f\right\|_{L_{p,\mu}} \leq c \|f\|_{L_{p,\mu}^{+}}, \quad \forall f \in L_{p,\mu}^{+}. \tag{4.20}$$

If the inequalities (4.8) hold, as a result, we have

$$\left\|\mathcal{K} f\right\|_{L_{p,\mu}} \leq c \|f\|_{L_{p,\mu}^{+}}, \quad \forall f \in L_{p,\mu}^{+}. \tag{4.21}$$

Consider the case

$$\alpha_0 = -\frac{1}{p}, \qquad -\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad k = \overline{1,r}. \tag{4.22}$$

Take sufficiently small $\varepsilon > 0$ and determine $p_\varepsilon^\pm = p \pm \varepsilon$. Acting similarly to the case $L_{p,\mu}^-$ (par.3) and accept the Riesz-Torin theorem we obtain boundedly acting of the operator $\mathcal{K}$ from $L_{p,\mu}^+$ to $L_{p,\mu}^-$ (since $[\mathcal{K}f](t) \in L_{p,\mu}^-$). Thus, if the following inequalities

$$-1 - \frac{1}{p} < \alpha_0 < \frac{1}{q}, \qquad -\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad k = \overline{1,r}, \tag{4.23}$$

are fulfilled, then the operator $\mathcal{K}$ boundedly acts from $L_{p,\mu}^+$ to $L_{p,\mu}^-$.

Using the identity (2.11) in the similar way we establish that the same conclusion with respect to the operator $\mathcal{K}$ is true in the case when the change interval of the exponent α_r is expanded on $(-1 - 1/p, 1/q)$. As a result, we obtain the validity of the following theorem.

Theorem 4.1. *Let the weight function ω be defined by the expression* (2.1) *and $\mu(t) \equiv \omega(|t|)$, $t \in (-\pi, \pi)$. Assume that the inequalities*

$$-1 - \frac{1}{p} < \alpha_0 < \frac{1}{q}, \qquad -1 - \frac{1}{p} < \alpha_r < \frac{1}{q}, \qquad -\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad k = \overline{1,r-1}, \tag{4.24}$$

are fulfilled. Then the singular operator $\mathcal{K}$:

$$\mathcal{K}f = \int_{-\pi}^{\pi} K(s;t) f(s) ds, \tag{4.25}$$

with Cauchy-type kernel $K(s;t) = (1/(1 - e^{-i(s-t)})) - 1/2$, boundedly acts from $L_{p,\mu}^+$ to $L_{p,\mu}^-$.

5. Completeness, Minimality, and Basicity of the System of Sines in Weight Space

Consider the system of sines $\{\sin nt\}_{n\in N}$. Let conditions (3.20) be fulfilled. It is easy to see that then the system $\{\sin nt\}_{n\in N}$ is minimal in $L_{p,\omega}^0$. The system $\{(2/\pi)\omega^{-1}(t)\sin nt\}_{n\in N}$, $1/p + 1/q = 1$, is a biorthogonal system to it. Indeed, it is obvious that $L_{q,\omega}^0$ is a space conjugated to $L_{p,\omega}^0$, and an arbitrary continuous functional l_g on $L_{p,\omega}^0$, generated by $g \in L_{q,\omega}^0$, realized by the formula

$$l_g(f) = \int_0^{\pi} f\overline{g}\omega dt, \quad \forall f \in L_{p,\omega}^0, \tag{5.1}$$

where $(\bar{\cdot})$ is a complex conjugation.

Take $g(t) = \omega^{-1}(t)\sin nt$, $n \in N$. We have

$$\left\|\omega^{-1}(t)\sin nt\right\|^{q}_{L^0_{q,\omega}} = \int_0^{\pi}\left|\omega^{-1}(t)\sin nt\right|^{q}\omega(t)dt = \int_0^{\pi}\omega^{1-q}(t)|\sin nt|^{q}dt. \tag{5.2}$$

Since $\sin nt \sim t$ as $t \to 0$ and $\sin nt \sim \pi - t$ as $t \to \pi$ for every fixed $n \in N$, then from relation (5.2) follows that $\{\omega^{-1}\sin nt\}_{n\in N} \subset L^0_{q,\omega}$, if the inequalities

$$\alpha_0 < 2 - \frac{1}{p}, \qquad \alpha_m < 2 - \frac{1}{p}, \qquad \alpha_k < \frac{1}{q}, \quad k = \overline{1, r-1}, \tag{5.3}$$

are fulfilled.

Take $g(t) = (2/\pi)\omega^{-1}(t)\sin nt$ and denote by ϑ_n generated by its functional, that is,

$$\vartheta_n(f) = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt, \quad \forall f \in L^0_{p,\omega}. \tag{5.4}$$

It is clear that $\vartheta_n(\sin kt) = \delta_{nk}$, $\forall n, k \in N$, where δ_{nk} is a Kronecker's symbol. Consider

$$\|\sin nt\|^{p}_{L^0_{p,\omega}} = \int_0^{\pi}|\sin nt|^{p}\omega(t)dt. \tag{5.5}$$

Thus, $\{\sin nt\}_{n\in N} \subset L_{p,\omega}$, if $\alpha_0 > -1/p - 1$, $\alpha_r > -1/p - 1$, $\alpha_k > -1/p$, $k = \overline{0, r-1}$. As a result, we obtain that if the inequalities

$$-1 - \frac{1}{p} < \alpha_0 < 2 - \frac{1}{p}, \qquad -1 - \frac{1}{p} < \alpha_r < 2 - \frac{1}{p}, \qquad -\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad k = \overline{0, r-1}, \tag{5.6}$$

hold, then the system $\{\sin nt\}_{n\in N}$ is minimal in $L^0_{p,\omega}$.

Now consider the completeness of the system $\{\sin nt\}_{n\in N}$ in $L^0_{p,\omega}$. Suppose that for some $g \in L^0_{q,\omega}$,

$$\int_0^{\pi}\sin nt\,\overline{g}\omega\,dt = 0, \quad \forall n \in N, \tag{5.7}$$

holds. We have

$$\int_0^{\pi}|\overline{g}\omega|dt = \int_0^{\pi}|\overline{g}|\omega^{1/q}\omega^{1/p}dt \le \|g\|_{L^0_{q,\omega}}\left(\int_0^{\pi}\omega\,dt\right)^{1/p}. \tag{5.8}$$

It is easy to see that if the inequalities

$$\alpha_k > -\frac{1}{p}, \quad k = \overline{0, r}, \tag{5.9}$$

are fulfilled, then $\omega \in L_1$. Then from the previous relation we get $\overline{g}\omega \in L_1$. Since, the system $\{\sin nt\}_{n\in N}$ is complete in space of continuous on $[0,\pi]$ functions with sup-norm, which vanishes at the ends of the segment $[0,\pi]$, then from (5.7) it follows that $g(t) = 0$ a.e. on $(0,\pi)$. Consequently, the system $\{\sin nt\}_{n\in N}$ is complete in $L^0_{p,\omega}$. So, the following statement is true.

Statement 3. Let the weight function $\omega(t)$ be defined by expression (2.1). The system of sines $\{\sin nt\}_{n\in N}$ is minimal in $L^0_{p,\omega}$, if the inequalities (5.6) are fulfilled. It is complete in $L^0_{p,\omega}$, if the inequalities (5.9) are fulfilled. Moreover, it forms a basis in $L^0_{p,\omega}$, if the inequalities

$$-\frac{1}{p} < \alpha_k < \frac{1}{q}, \quad k = \overline{0,m} \tag{5.10}$$

hold.

The basicity of system of sines in $L^0_{p,\omega}$, when the inequalities (5.10) hold, follows from the basicity of system of exponent $\{e^{int}\}_{n\in Z}$ in $L_{p,\mu}$, where $\mu(t) = \omega(|t|)$, $t \in (-\pi,\pi)$. The basicity of these systems earlier considered in papers [3, 4, 8, 9].

In the similar way we prove the following statement.

Statement 4. Let the weight function $\omega(t)$ defined by expression (2.1). The system of cosines $1 \cup \{\cos nt\}_{n\in N}$ is minimal (forms a basis) in $L^0_{p,\omega}$, if the inequalities (5.10) are fulfilled. It is complete in $L^0_{p,\omega}$, if the inequalities (5.9) holds.

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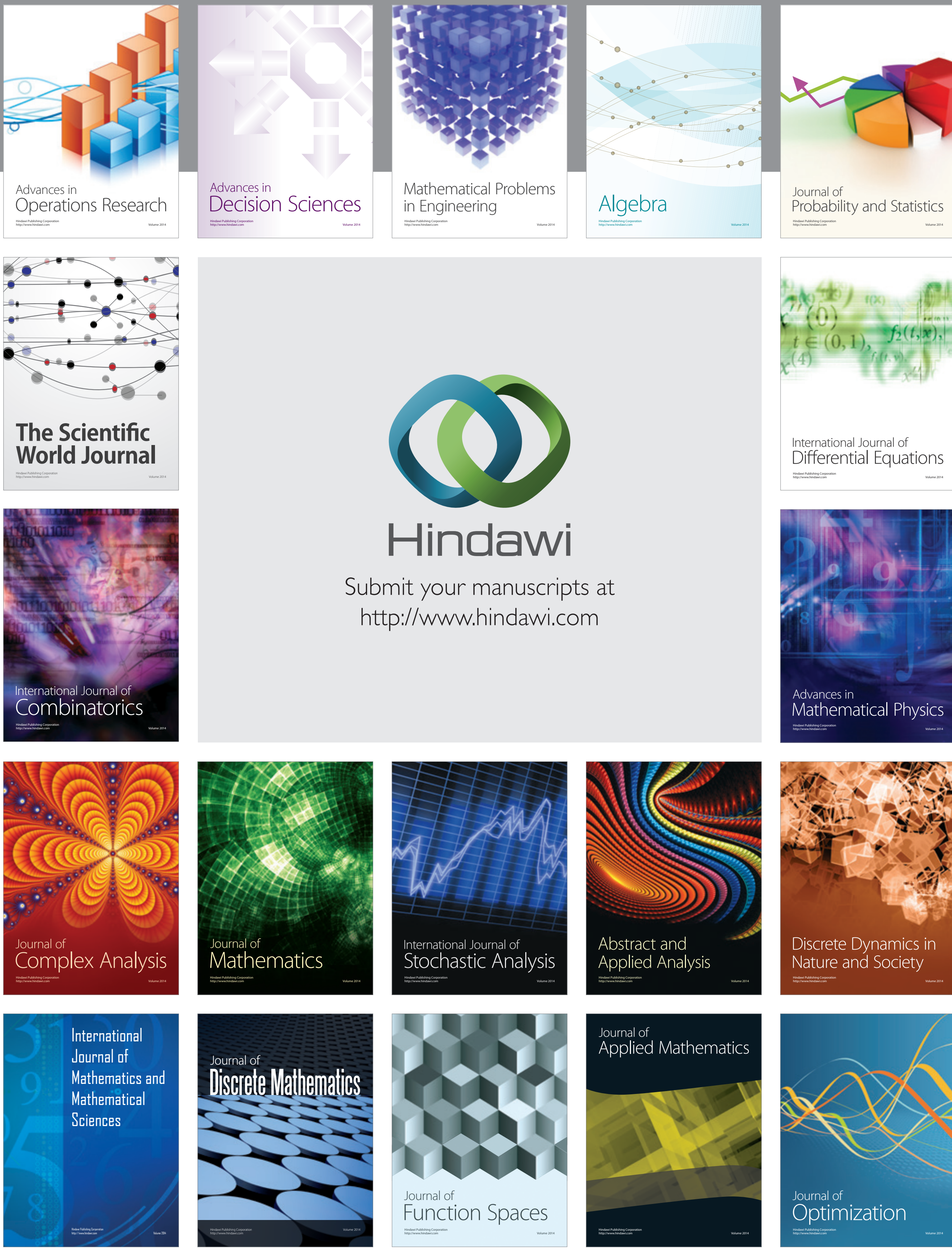
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