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Research Article

Global Attractor for the Generalized Dissipative KDV Equation with Nonlinearity

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We discuss global attractor for the generalized dissipative KDV equation with nonlinearity under the initial condition $u(x,0) = u_0(x)$. We prove existence of a global attractor in space $H^2(\Omega)$, by using decomposition method with cut-off function and *Kuratowski* α-measure in order to overcome the noncompactness of the classical Sobolev embedding.

1. Introduction

In order to study the longtime behavior of a dissipative evolutionary equation, we generally aim to show that the dynamics of the equation is finite dimensional for long time. In fact, one possible way to express this fact is to prove that dynamical systems describing the evolutional equation comprise the existence of the global attractor [1]. The KDV equation without dissipative and forcing was initially derived as a model for one direction water waves of small amplitude in shallow water, and it was later shown to model a number of other physical stems. In recent years, the KDV equations has been always being an important nonlinear model associated with the science of solids, liquids, and gases from different perspectives both mathematics and physics. As for dissipative KDV equation, existence of a global attractor is a significant feature. In [2], Ghidaglia proved that for the dissipative KDV equation

$$u_t + u_{xxx} + \gamma u = f - uu_x, \quad x \in [0, L], \tag{1.1}$$

with periodic boundary condition $u(x,t) = u(x+L,t)$, there exists a weak global attractor of finite dimension. Later, there are many contributions to the global attractor of the dissipative KDV equation (see [3–10]). In [3], Guo and Wu proved the existence of global attractors for the generalized KDV equation

$$u_t + u_{xxx} - \eta u_{xx} + \gamma u = f - g(u)_x, \quad u(x,0) = u_0. \tag{1.2}$$

However, few efforts are devoted to the existence of global attractor for generalized dissipative four-order KDV equation with nonlinearity in unbounded domains. In this paper, we consider the existence of global attractor for generalized dissipative four-order KDV equation with nonlinearity as follows:

$$u_t - \phi(u)_x + u_{xxxx} + \beta u_{xxx} - \alpha u_{xx} + g(u) = f(x), \tag{1.3}$$

$u(x,0) = u_0$, where $\alpha, \beta > 0$, $(x,t) \in \Omega \times [0,T]$, and Ω is unbounded domain.

As we all know, the solutions to the dissipative equation can be described by a semigroup of solution operators. When the equation is defined in a bounded domain, if the semigroup is asymptotically compact, then the classical theory of semiflow yields the existence of a compact global attractor (see [11–13]). But, when the equation is defined in a unbounded domain, which causes more difficulties when we prove the existence of attractors. Because, in this case, the Sobolev embedding is not compact. Hence, we cannot obtain a compact global attractor using classical theory.

Fortunately, as far as we concerned, there are several methods which can be used to show the existence of attractors in the standard Sobolev spaces even the equations are defined in unbounded domains. One method is to show that the weak asymptotic compactness is equivalent to the strong asymptotic compactness by an energy method (see [9, 10, 14]). A second method is to decompose the solution operator into a compact part and asymptotically small part (see [15–17]). A third method is to prove that the solutions uniformly small for large space and time variables by a cut-off function (see [18, 19]) or by a weight function (see [20]).

Generally speaking, the energy method proposed by Ball depends on the weak continuity of relevant energy functions (see [21, 22]). However, for (1.3) in unbounded domains, it seems that the energy method is not easy to use. Consequently, in this paper, we will show the idea to obtain the existence of global attractor in unbounded domains by showing the solutions are uniformly small for large space by a cut-off function or weight function, and at the same time, we apply decomposition method and *Kuratowski α*-measure to prove our result in order to overcome the noncompactness of the classical Sobolev embedding.

This paper is organized as follows.

In Section 2, firstly, we recall some basic notations; secondly, we make precise assumptions on the nonlinearity $g(u)$ and $\phi(u)$; finally, we state our main result of the global attractor for (1.3).

In Section 3, we show the existence of a absorbing set in $H^2(\Omega)$.

In Section 4, we prove the existence of global attractor.

2. Preliminaries and Main Result

We consider the generalized dissipative four-order KDV equation (1.3), where $\Omega \subset R^n$ is unbounded domain and the initial data $u_0 \in H^3(\Omega)$, $f \in H^1(\Omega)$, $g(u)$ is nonlinearity.

Throughout the paper, we use the notation $H = L^2(\Omega)$, $H^s = H^{s,2}(\Omega)$ with the scalar product and norms given, respectively, by $(\cdot,\cdot)$, $|\cdot|$, and $((\cdot,\cdot))$, $\|\cdot\|$. In the space H^2, we consider the scalar product $((u,v)) = (\int_\Omega \nabla u \cdot \nabla v\, dx)$ and the norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$. While in the space H, we consider the scalar product $(u,v) = (\int_\Omega u \cdot v\, dx)$ and the norm $|u| = (\int_\Omega u^2 dx)^{1/2}$.

Notice

E_i, C, c, c_i denote for different positive constants.

First, we assume that $f \in H$, and $\phi(u), g(u)$ satisfy the following conditions:

(A1): $\phi(u) \in C^3$, $|\phi(u)| \leq A|u|^{5-\sigma}$, $(A > 0, 0 < \sigma \leq 4)$,

(A2): $|\phi''(u)| \leq A|u|^{3-\sigma}$, $|\phi''(u)| \leq A|u|^{2-\sigma}$,

(A3): $g(u) \in C^2$, $g(u) = g_1(u) + g_2(u)$, where $g_1(u) = \gamma u$, $g_2(u) = K|u|^5 (K > 0,\ \gamma > 0)$,

(A4): $g(0) = 0$, $|g'(u)| \leq C$.

Secondly, we can rewrite (1.3) as the following equation with the above assumption:

$$u_t - \phi(u)_x + u_{xxxx} + \beta u_{xxx} - \alpha u_{xx} + \gamma u + g_2(u) = f(x), \tag{2.1}$$

$u(x,0) = u_0$, where $\alpha, \beta > 0$, $(x,t) \in \Omega \times [0,T]$.

Finally, we state our main result is the following theorem.

Theorem 2.1. *Let the generalized dissipative of four-order KDV equation with nonlinearity given by* (2.1)*. Assume that $\phi(u), g(u)$ satisfy conditions (A1)–(A4) and, moreover, $u_0 \in H^2$, $f \in H^1$, then for $\alpha, \beta, \gamma > 0$, there exists a global attractor $\mathcal{A}$ of the problem* (2.1)*, that is, there is a bounded absorbing set $B \in H^2$ in which sense the trajectories are attract to $\mathcal{A}$, such that*

$$\mathcal{A} = \omega(B) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B}, \tag{2.2}$$

where $S(t)$ is semigroup operator generated by the problem (2.1)*.*

3. Existence of Absorbing Set in Space $H^2(\Omega)$

In this section, we will show the existence of an absorbing set in space $H^2(\Omega)$ by obtaining uniformly in time estimates. In order to do this, we start with the following lemmas.

Lemma 3.1. *Assume that $g(u)$ satisfied (A4), furthermore, $u_0 \in H$, $f \in H$, then for the solution u of the problem* (2.1), *one has the estimates*

$$\|u\|^2 \leq \|u_0\|^2 \exp(-ct) + \frac{\|f\|^2}{C} - \exp(-ct), \tag{3.1}$$

$$\lim_{t\to\infty} \|u\|^2 \leq \frac{\|f\|^2}{C} = E_0, \tag{3.2}$$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \|u_x\|^2 dt \leq \frac{\|f\|^2}{2\alpha C}. \tag{3.3}$$

Proof. Taking the inner product of (2.1) with u, we have

$$\left(u_t - \phi(u)_x + u_{xxxx} + \beta u_{xxx} - \alpha u_{xx} + \gamma u + g_2(u), u\right) = (f, u), \tag{3.4}$$

where

$$\begin{gathered} \left(u, \phi(u)_x\right) = 0, \\ (u, u_{xxx}) = 0, \\ (u, g(u)) \geq C\|u\|^2, \\ \left|(f, u)\right| \leq \frac{\|f\|^2}{C} + \frac{C\|u\|^2}{2}, \end{gathered} \tag{3.5}$$

here, we apply Young's inequality and the condition (A4).

Thus, from (3.4), we get

$$\frac{d}{dt}\|u\|^2 + 2\alpha\|u\| + 2C\|u\|^2 \leq \frac{\|f\|^2}{C}. \tag{3.6}$$

By virtue of Gronwall's inequality and (3.6), one has (3.1) and which implies (3.2) and (3.3). □

Lemma 3.2. *In addition to the conditions of Lemma 3.1, one supposes that*

$$\phi(u) \in C^2, \quad g(u) \in C^1, \quad \left|\phi(u)\right| \leq A|u|^{5-\sigma} \quad (A > 0,\ 0 < \sigma \leq 4), \tag{3.7}$$

then one has the estimate

$$\|u_x\|^2 \leq 4\exp(-2c_8 t)\varphi(u_0) + \frac{2}{c_8}\left(2\frac{\|f\|^2}{\alpha} + c_7\right)\left(1 - \exp(-2c_8 t)\right), \tag{3.8}$$

where

$$\varphi(u) = \|u_x\|^2 - \frac{1}{\beta}\int_\Omega \phi(u)dx. \tag{3.9}$$

Proof. Taking the inner product of (2.1) with u_{xx}, we have

$$\left(u_t - \phi(u)_x + u_{xxxx} + \beta u_{xxx} - \alpha u_{xx} + \gamma u + g_2(u), u_{xx}\right) = \left(f, u_{xx}\right), \tag{3.10}$$

where

$$\left(u_{xx}, \phi(u)_x\right) = \frac{1}{\beta}\left(u_t - \phi(u)_x + u_{xxxx} - \alpha u_{xx} + g(u) - f\right). \tag{3.11}$$

Noticing that

$$\begin{gathered}\frac{1}{\beta}\left(\phi(u), u_t\right) = \frac{1}{\beta}\int_\Omega \phi(u)u_t dx = \frac{1}{\beta}\frac{d}{dt}\int_\Omega \phi(u)dx, \\ \left(\phi(u), \phi(u)_x\right) = 0.\end{gathered} \tag{3.12}$$

Using Nirberg's interpolation inequality and the Sobolev embedding theory (see [11]), we have

$$\begin{gathered}\left|\frac{1}{\beta}\left(\phi(u), u_{xxxx}\right)\right| \le \frac{\alpha}{2}\|u_{xx}\|^2 + c_2, \\ \left|\frac{1}{\beta}\left(\phi(u), -\alpha u_{xx}\right)\right| \le \frac{\alpha}{2}\|u_{xx}\|^2 + c_3.\end{gathered} \tag{3.13}$$

Due to Lemma 3.1 and conditions of Lemma 3.2, we get that

$$\begin{gathered}\left|\frac{1}{\beta}\left(\phi(u), g(u)\right)\right| \le \frac{AK}{\beta}\|u\|_{5-\sigma} \le c_4, \\ \left|\frac{1}{\beta}\left(\phi(u), f\right)\right| \le \frac{A}{\beta}\|f\|\|u\|_{5-\sigma} \le c_5, \\ \left|\left(g(u), u_{xx}\right)\right| \le C\|u_{xx}\|^2, \\ \left|\left(f, u_{xx}\right)\right| \le \frac{\alpha}{2}\|u_{xx}\|^2 + \frac{2}{\alpha}\|f\|^2.\end{gathered} \tag{3.14}$$

From (3.10) and above inequalities, we get

$$\frac{d}{dt}\left(\|u_x\|^2 - \frac{2}{\alpha}\int_\Omega \phi(u)dx\right) + \frac{\alpha}{2}\|u_{xx}\|^2 + (C-\gamma)\|u_x\|^2 \le \frac{2}{\alpha}\|f\|^2 + c_6 = c_7. \tag{3.15}$$

Setting $C - \gamma = c_8$, then we can obtain that

$$\varphi(u) = \frac{1}{2}\int_\Omega \|u_x\|^2 dx - \frac{1}{\beta}\int_\Omega \int_0^u \phi(s)ds\,dx \ge \frac{1}{4}\|u_{xx}\|^2 - c_9. \tag{3.16}$$

Thus, by Gronwall's inequality and (3.15), we get that

$$\|u_x\|^2 \le 4\exp(-2c_8 t)\varphi(u_0) + \frac{2}{c_8}(1-\exp(-2c_8 t))\left(\frac{2\|f\|^2}{\alpha} + c_7\right), \tag{3.17}$$

which implies

$$\begin{gathered}\lim_{t\to\infty}\|u_x\|^2 \le E_1, \\ \lim_{t\to\infty}\frac{1}{t}\int_0^t \|u_{xx}\|^2 d\mathrm{t} \le C.\end{gathered} \tag{3.18}$$

Therefore, we prove Lemma 3.2. □

Lemma 3.3. *Suppose that $\phi(u)$, $g(u)$ satisfy (A2), (A3) and, moreover, the following conditions hold true:*

(1) $\phi(u) \in C^3, g(u) \in C^2$,

(2) $u_0 \in H^2, f \in H^1$,

then for the solution u of the problem of (2.1), *one has the following estimate*

$$\|u_{xx}\|^2 \le \|u_{xx}(0)\|^2 \exp(-2c_{13}t) + \frac{c_{14}}{c_{13}}(1-\exp(-2c_{13}t)); \tag{3.19}$$

furthermore,

$$\begin{gathered}\lim_{t\to\infty}\|u_{xx}\|^2 \le E_2, \\ \lim_{t\to\infty}\frac{1}{t}\int_0^t \|u_{xxx}\|^2 dt \le C.\end{gathered} \tag{3.20}$$

Proof. Taking the inner product of (2.1) with u_{xxxx}, we have

$$\left(u_t - \phi(u)_x + u_{xxxx} + \beta u_{xxx} - \alpha u_{xx} + \gamma u + g_2(u), u_{xxxx}\right) = \left(f, u_{xxxx}\right), \tag{3.21}$$

where

$$\begin{aligned}\left(u_{xxxx}, \phi(u)_x\right) &= -\left(\phi(u)_{xx}, u_{xxx}\right) = \left(\phi''(u)u_x^2 + \phi'(u)u_{xx}, u_{xxxx}\right)\\ &\le A\|u\|^{3-\sigma}\|u_{xx}\|^2\|u_{xxx}\| + A\|u\|^{4-\sigma}\|u_{xx}\|^2\|u_{xxx}\|.\end{aligned} \tag{3.22}$$

By Young's inequality and Lemmas 3.1 and 3.2, thus from (3.22), we have

$$\left|\left(\phi(u), u_{xxxx}\right)\right| \le c_{10}u_{xx}^2 + \alpha\|u_{xxx}\|^2 + c_{11}. \tag{3.23}$$

Due to Lemmas 3.1 and 3.2 and (A3), we obtain

$$\begin{aligned}\left(g(u), u_{xxxx}\right) &= \left(\gamma u, u_{xxxx}\right) + \left(g_2(u), u_{xxxx}\right)\\ &= \gamma\|u_{xx}\|^2 + \left(g_2(u), u_{xxxx}\right)\\ &= \gamma\|u_{xx}\|^2 + \left(g_2''(u)u_{xx}^2 + g_2'(u)u_{xx}, u_{xx}\right)\\ &\le \gamma\|u_{xx}\|^2 + K\|u\|^2\|u_x\|^2\|u_{xx}\| + K\|u\|^3\|u_{xx}\|^2.\end{aligned} \tag{3.24}$$

Using Young's inequality, we have

$$\begin{aligned}\left|\left(g(u), u_{xxxx}\right)\right| &\le \gamma\|u_{xx}\|^2 + \frac{\alpha}{2}\|u_{xx}\|^2 + c_{12},\\ \left|\left(f, u_{xxxx}\right)\right| = \left|\left(-f_x, u_{xxx}\right)\right| &\le \frac{\alpha}{2}\|u_{xxx}\|^2 + \frac{2}{\alpha}\left\|f_x\right\|^2.\end{aligned} \tag{3.25}$$

By (3.21), (3.23) and (3.25), we get

$$\frac{1}{2}\frac{d}{dt}\|u_{xx}\|^2 + \frac{\alpha}{2}\|u_{xxx}\|^2 + \left(c_{10} - \gamma - \frac{\alpha}{2}\right)\|u_{xx}\|^2 \le \frac{2}{\alpha}\left\|f_x\right\|^2 + c_{11} + c_{12}, \tag{3.26}$$

that is,

$$\frac{d}{dt}\|u_{xx}\|^2 + \alpha\|u_{xxx}\|^2 + c_{13}\|u_{xx}\|^2 \le c_{14}, \tag{3.27}$$

where

$$c_{13} = 2\left(c_{10} - \gamma - \frac{\alpha}{2}\right),$$
$$c_{14} = 2\left(\frac{2}{\alpha}\|f_x\|^2 + c_{11} + c_{12}\right). \tag{3.28}$$

By virtue of Gronwall's inequality, we have

$$\|u_{xx}\|^2 \le \|u_{xx}(0)\|^2 \exp(-2c_{13}t) + \frac{c_{14}}{c_{13}}(1 - \exp(-2c_{13}t)), \tag{3.29}$$

and (3.27) implies

$$\lim_{t\to\infty}\|u_{xx}\|^2 \le E_2,$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \|u_{xxx}\|^2 dt \le C. \tag{3.30}$$

Therefore, we prove Lemma 3.3. □

Lemma 3.4. *Suppose that $\phi(u), g(u)$ satisfy (A2), (A3) and, moreover, the following conditions hold true:*

(1) $\phi(u) \in C^3$, $g(u) \in C^2$,

(2) $u_0 \in H^3$, $f \in H^2$,

then for the solution u of the problem of (2.1), *we have the following estimates:*

$$\|u_{xxx}\|^2 \le \|u_{xxx}(0)\|^2 \exp(-2\gamma t) + \frac{c_{17}}{\gamma}(1 - \exp(-2\gamma t)), \tag{3.31}$$

where

$$c_{17} = 2\left(c_{16} + \frac{2}{\alpha}\|f_{xx}\|^2\right); \tag{3.32}$$

furthermore,

$$\lim_{t\to\infty}\|u_{xxx}\|^2 \le E_3,$$
$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \|u_{xxxx}\|^2 dt \le C. \tag{3.33}$$

Proof. Taking the inner product of (2.1) with u_{xxxxxx}, we have

$$(u_t - \phi(u)_x + u_{xxxx} + \beta u_{xxx} - \alpha u_{xx} + \gamma u + g_2(u), u_{xxxxxx}) = (f, u_{xxxxxx}), \tag{3.34}$$

where

$$
\begin{aligned}
(u_t, u_{xxxxxx}) &= -\frac{1}{2}\frac{d}{dt}\|u_{xxx}\|^2,\\
(u_{xxx}, u_{xxxxxx}) &= 0,\\
(\alpha u_{xx}, u_{xxxxxx}) &= \alpha\|u_{xxxx}\|^2,\\
\left|(u_{xxxxxx}, \phi(u)_x)\right| &= (\phi(u)_{xxx}, u_{xxxx}) = (\phi'''(u)u_x^3 + 3\phi''(u)u_x u_{xx} + \phi'(u)u_{xxx}, u_{xxxx})\\
&= (\phi'''(u)u_x^3) + 3(\phi''(u)u_x u_{xx}, u_{xxxx}).
\end{aligned}
\tag{3.35}
$$

$$
\begin{aligned}
\left|(u_{xxxxxx}, \phi(u)_x)\right| &= (\phi(u)_{xxx}, u_{xxxx}) = (\phi'''(u)u_x^3 + 3\phi''(u)u_x u_{xx} + \phi'(u)u_{xxx}, u_{xxxx})\\
&= (\phi'''(u)u_x^3) + 3(\phi''(u)u_x u_{xx}, u_{xxxx}).
\end{aligned}
\tag{3.36}
$$

Using Nirberg's interpolation inequality and Young's inequality, from (3.36) and Lemmas 3.1–3.3, we have

$$
\left|(u_{xxxxxx}, \phi(u)_x)\right| \le \alpha\|u_{xxxx}\|^2 + c_{15}. \tag{3.37}
$$

Due to the condition (4.3), we get

$$
(g(u), u_{xxxxxx}) = (\gamma u, u_{xxxxxx}) + (g_2(u), u_{xxxxxx}). \tag{3.38}
$$

By direct calculations, it is easy to get that

$$
\begin{aligned}
(\gamma u, u_{xxxxxx}) &= -\gamma\|u_{xxx}\|^2,\\
(g_2(u), u_{xxxxxx}) &= (g_2(u)_{xx}, u_{xxxx})\\
&= \left(g_2(u)_x^2 + g_2'(u)_{xx}, u_{xxxx}\right)\\
&\le \frac{\alpha}{2}\|u_{xxxx}\|^2 + c_{16},\\
\left|(f, u_{xxxxxx})\right| &\le \frac{\alpha}{2}\|u_{xxxx}\|^2 + \frac{2}{\alpha}\|f_{xx}\|^2.
\end{aligned}
\tag{3.39}
$$

Due to (3.34)–(3.39), we have

$$
\frac{1}{2}\frac{d}{dt}\|u_{xxx}\|^2 + \frac{\alpha}{4}\|u_{xxxx}\|^2 + \gamma\|u_{xxx}\|^2 \le \frac{2}{\alpha}\|f_{xx}\|^2 + c_{16}, \tag{3.40}
$$

that is,

$$
\frac{d}{dt}\|u_{xxx}\|^2 + \frac{\alpha}{2}\|u_{xxxx}\|^2 + 2\gamma\|u_{xxx}\|^2 \le c_{17}, \tag{3.41}
$$

where

$$c_{17} = 2\left(\frac{2}{\alpha}\|f_{xx}\|^2 + c_{16}\right). \tag{3.42}$$

Using Gronwall's inequality, we deduce that

$$\|u_{xxx}\|^2 \le \|u_{xxx}(0)\|^2 \exp(-2\gamma t) + \frac{c_{17}}{\gamma}(1 - \exp(-2\gamma t)), \tag{3.43}$$

moreover, (3.41) implies

$$\begin{gathered} \lim_{t\to\infty} \|u_{xxx}\|^2 \le E_3, \\ \lim_{t\to\infty} \frac{1}{t}\int_0^t \|u_{xxxx}\|^2 dt \le C; \end{gathered} \tag{3.44}$$

Therefore, we prove Lemma 3.4. □

In a similar way as above, we can get the uniformly estimates of $\|u_{xxxx}\|$, $\|u_t\|$ and we omit them here.

Next, we will show the existence of global solution for the problem (2.1) as follows.

Lemma 3.5. *Suppose that the following conditions hold true:*

(1) $u_0 \in H^{m+1}$, $f \in H^m$,

(2) $\phi(u) \in C^{m+1}$, $\|\phi(u)\| \le A|u|^{5-\sigma}$, $(\sigma, A > 0)$,

(3) $g(u) \in C^m$, $|g_2(u)| \le K|u|^5$,

(4) $g(u)$ *satisfies (A3), (A4) and* $g(u)$ *is Lipschitz continuous, that is,*

$$\left|g(u) - g(v)\right| \le C|u - v|, \tag{3.45}$$

then there exists a unique global solution u for the problem (2.1) *such that* $u \in L^\infty(0, T; H^m(\Omega))$, *and furthermore,the semigroup operator* $S(t)$ *associated with the problem of* (2.1) *is continuous and there exists an absorbing set* $B \subset H^2(\Omega)$, *where*

$$B \in \left\{u \mid u \in H^2(\Omega), \|u\| \le E_0, \|u_x\| \le E_1, \|u_{xx}\| \le E_2\right\}. \tag{3.46}$$

Proof. Similar to the proof of Lemmas 3.1–3.4, we have

$$\|u\|_{H^m} \le C. \tag{3.47}$$

At the same time, we use the Galerkin method (see [11]) and Lemmas 3.1–3.4 to prove the existence of global solution for the problem (2.1). So, we omit them here.

Next, we will prove the uniqueness of the global solution.

Assume that u, v are two solutions of the problem (2.1) and $w = u - v$, then we have

$$w_t - \left(\phi'(u) - \phi'(v)\right)w_x + w_{xxxx} + \beta w_{xxx} - \alpha w_{xx} + g(u) - g(v) = 0. \tag{3.48}$$

Taking the inner product in H of (2.1) with w, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \left(\left(\phi'(u) - \phi'(v)\right)w_x, w\right) + \|w_{xx}\|^2 + \beta(w_{xxx}, w) + \alpha\|w_x\|^2 + \left(g(u) - g(v), w\right) = 0, \tag{3.49}$$

where

$$\begin{gathered}(w, w_x) = 0, \\ (w_{xxx}, w) = 0.\end{gathered} \tag{3.50}$$

Due to the condition

$$\left|g(u) - g(v)\right| \leq C|u - v|, \tag{3.51}$$

and from (3.49), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \|w_{xx}\|^2 + \alpha\|w_x\|^2 + C\|w\|^2 \leq 0, \tag{3.52}$$

that is,

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + C\|w\|^2 \leq 0. \tag{3.53}$$

By application of Gronwall's inequality, we get $w = 0$.

Finally, we recall some basic results in [11, 23] and by Lemmas 3.1–3.3, it is easy to prove that there exists an absorbing set

$$B \in \left\{u \mid u \in H^2(\Omega), \|u\| \leq E_0, \|u_x\| \leq E_1, \|u_{xx}\| \leq E_2\right\} \tag{3.54}$$

in $H^2(\Omega)$. But as for the continuity of semigroup $S(t)$, we can apply the following Lemmas 3.6, and 3.7 to prove the result. □

Lemma 3.6. *Suppose that $u_0 \in H^2$, $f \in H^1$, and $\phi(u), g(u)$ satisfy (A1)–(A4), there exists constant $C > 0$, such that*

$$\begin{gathered}\left\|u_\eta\right\|^2, \left\|u_{\eta x}\right\|^2, \left\|u_{\eta xx}\right\|^2, \left\|u_{\eta xxx}\right\|^2, \left\|u_{\eta xxxx}\right\|^2 \leq C, \quad \forall \eta \in (0,1),\ t \geq 0, \\ \left\|u_\eta\right\|^2, \left\|u_{\eta x}\right\|^2, \left\|u_{\eta xx}\right\|^2 \leq C\eta, \quad \forall \eta \in (0,1),\ t \geq t_0 > 0.\end{gathered} \tag{3.55}$$

Now, we use the decomposition method to prove the continuity of $S(t)$ for sake of overcoming the difficult of noncompactness.

Set $f \in H^1(\Omega)$, $\lambda_L(x) \in C_0^\infty(\Omega)$, $0 \le \lambda_L \le 1$, satisfies

$$\lambda_L(x) = \begin{cases} 1 & |x| \le L, \\ 0 & |x| \le L+1, \end{cases} \tag{3.56}$$

then, for all $\eta \in (0,1)$, there exists $L_\eta > 0$, such that

$$\|f - f_\eta\|_{1,2}^2 \le \eta, \quad f_\eta = f \times \lambda_L(\eta). \tag{3.57}$$

Assume that u_η is solution of the following equation:

$$\begin{gathered} u_{\eta t} + \phi(u_\eta)_x + u_{\eta xxxx} + \beta u_{\eta xxx} - \alpha u_{\eta xx} + g(u_\eta) = f - f_\eta, \\ u_\eta(x,0) = 0. \end{gathered} \tag{3.58}$$

Setting

$$\begin{gathered} S_{1\eta}(t)u_0 = u_\eta, \\ w_\eta = S_{2\eta}(t)u_0 = S(t)u_0 - S_{1\eta}(t)u_0 \end{gathered} \tag{3.59}$$

is a solution of the equation as follows:

$$\begin{gathered} w_{\eta t} + \phi(w_\eta)_x + w_{\eta xxxx} + \beta w_{\eta xxx} - \alpha w_{\eta xx} + g(w_\eta) = f_\eta, \\ w_\eta(x,0) = 0. \end{gathered} \tag{3.60}$$

Now, we prove the Lemma 3.6.

Proof. We take the scalar product in space H of (3.58) with u_η, we get

$$\frac{1}{2}\frac{d}{dt}\|u_\eta\|^2 + \lambda\|u_{\eta xx}\|^2 + \alpha\|u_{\eta x}\|^2 + \gamma\|u_\eta\|^2 + (g_2(u_\eta), u_\eta) \le \|f - f_\eta\|\|u_\eta\|. \tag{3.61}$$

Due to (A3) and Young's inequality, we get

$$\begin{gathered} (g_2(u_\eta), u_\eta) \le \frac{\gamma}{2}\|u_\eta\|^2, \\ \|f - f_\eta\|\|u_\eta\| \le \frac{\gamma}{2}\|u_\eta\|^2 + \frac{\eta^2}{\gamma}. \end{gathered} \tag{3.62}$$

From (3.61), we obtain the following inequality:

$$\frac{d}{dt}\|u_\eta\|^2 + 2\gamma\|u_\eta\|^2 \leq \frac{\eta^2}{\gamma}. \tag{3.63}$$

By Gronwall's inequality, one has

$$\|u_\eta\|^2 \leq \|u_0\|^2 \exp(-2\gamma t) + \frac{\eta^2}{2\gamma^2}(1 - \exp(-2\gamma t)). \tag{3.64}$$

Hence, there exists $C > 0$, such that

$$\|u_\eta\|^2 \leq C, \tag{3.65}$$

and implies

$$\|u_\eta\|^2 \leq C\eta, \quad \forall t \geq t_0. \tag{3.66}$$

We take the scalar product in space H of (3.58) with $u_{\eta xx}$ and similar to the proof of Lemma 3.2, we have

$$\frac{d}{dt}\|u_{\eta x}\|^2 + \|u_{\eta xxx}\|^2 + 2\gamma\|u_{\eta x}\|^2 \leq C\eta. \tag{3.67}$$

By application of Gronwall's inequality, we deduce that

$$\|u_{\eta x}\|^2 \leq \|u_x(0)\|^2 \exp(-2\gamma t) + \frac{C\eta^2}{2\gamma}(1 - \exp(-2\gamma t)). \tag{3.68}$$

So, there exists $C > 0$, such that

$$\|u_{\eta x}\|^2 \leq C, \tag{3.69}$$

and implies

$$\|u_{\eta x}\|^2 \leq C\eta, \quad \forall t \geq t_0. \tag{3.70}$$

We take the scalar product in space H of (3.58) with $u_{\eta xxxx}$ and similar to the proof of Lemma 3.3, we have

$$\frac{1}{2}\frac{d}{dt}\|u_{\eta xx}\|^2 + \frac{1}{2}\|u_{\eta xxxx}\|^2 + \gamma\|u_{\eta xx}\|^2 \leq C\eta. \tag{3.71}$$

It is easy to prove that

$$\|u_{\eta xx}\|^2 \le C, \quad \|u_{\eta xx}\|^2 \le C\eta, \quad \forall t \ge t_0. \tag{3.72}$$

We take the scalar product in space H of (3.58) with $u_{\eta xxxxxx}$ and similar to the proof of Lemma 3.4, we have

$$\frac{1}{2}\frac{d}{dt}\|u_{\eta xxx}\|^2 + \frac{1}{2}\|u_{\eta xxxxx}\|^2 + \gamma\|u_{\eta xxx}\|^2 \le \frac{C\eta}{2}\|f - f_\eta\|^2 \tag{3.73}$$

that is,

$$\frac{d}{dt}\|u_{\eta xxx}\|^2 + 2\gamma\|u_{\eta xxx}\|^2 \le C\eta. \tag{3.74}$$

Hence, by Gronwall's inequality, we get

$$\|u_{\eta xxx}\|^2 \le C. \tag{3.75}$$

At the same time, we have

$$\|u_{\eta xxxx}\|^2 \le C \tag{3.76}$$

and we omit them here. □

Lemma 3.7. *Under the conditions of Lemma 3.6, one has the following estimates*

$$\begin{aligned} \|xw_\eta\|^2 &\le C_1(\eta), \\ \|xw_{\eta x}\|^2 &\le C_2(\eta), \\ \|xw_{\eta xx}\|^2 &\le C_3(\eta), \end{aligned} \tag{3.77}$$

where $C_i(\eta) > 0$, $(i = 1, 2, 3)$.

Proof. We take the scalar product in space H of (3.60) with $x^2 w_\eta$ and noticing that

$$\left(w_{\eta xxxx}, x^2 w_\eta\right) = \|xw_{\eta xx}\|^2 + 4(w_{\eta xx}, xw_{\eta x}) - 2\|w_{\eta x}\|^2, \tag{3.78}$$

it is easy to get that

$$\frac{d}{dt}\|xw_\eta\|^2 + \|xw_{\eta xx}\|^2 + \gamma\|xw_\eta\|^2 \le C(\eta). \tag{3.79}$$

By Gronwall's inequality, we have

$$\|xw_\eta\|^2 \leq C_1(\eta). \tag{3.80}$$

From (3.60), we obtain

$$w_{\eta txx} + \phi(w_\eta)_{xxx} + w_{\eta xxxxxxxx} + \beta w_{\eta xxxxxxx} - \alpha w_{\eta xxxx} + g(w_\eta)_{xx} = f_{\eta xx} \tag{3.81}$$

We take the scalar product in space H of (3.61) with $x^2 w_{\eta xx}$ and noticing that

$$\begin{aligned}
\left(w_{\eta xxxxxxx}, x^2 w_{\eta xxx}\right) &= \|xw_{\eta xxxx}\|^2 + 4(w_{\eta xx}, xw_{\eta xxx}) - 2\|w_{\eta xxx}\|^2, \\
\left(w_{\eta xxxxxx}, x^2 w_{\eta xx}\right) &= -4(w_{\eta xx}, xw_{\eta xxxx}), \\
\left(f_{\eta xx}, x^2 w_{\eta xx}\right) &= -4\left(f_\eta, x^2 w_{\eta xxxx}\right), \\
\left(\phi(w_\eta)_{xxx}, x^2 w_{\eta xx}\right) &= \left(\phi(w_\eta)_{xx}, \left(x^2 w_{\eta xx}\right)_{xx}\right) \\
&= \left(\phi'(w_\eta) w_{\eta x}, 2w_{\eta xx} + 4xw_{\eta xxx} + x^2 w_{\eta xxxx}\right) \\
&= 4(\phi'(w_\eta) w_{\eta x}, xw_{\eta xxx}), \\
\left(g(w_\eta)_{xx}, x^2 w_{\eta xx}\right) &= -\left(\left(g(w_\eta)_{xx}\right)_x, \left(x^2 w_{\eta xx}\right)_x\right), \\
&= -\left(g(w_\eta)_{xx}\right)_x, 2xw_{\eta xx} + x^2 w_{\eta xxx} \\
&= \left(-g'(w_\eta) w_{\eta x}, x^2 w_{\eta xxx}\right).
\end{aligned} \tag{3.82}$$

By Young's inequality and the Sobolev embedding theory (see [11]) and (3.81)-(3.82), we deduce that

$$\frac{d}{dt}\|xw_{\eta xx}\|^2 + 2\gamma \|xw_{\eta xx}\|^2 \leq C(\eta). \tag{3.83}$$

Using Gronwall's inequality, we obtain

$$\begin{aligned}
&\|xw_{\eta xx}\|^2 \leq C_3(\eta), \\
\|xw_{\eta x}\|^2 &= \int_\Omega x^2 w_{\eta x}^2 dx - 2\int_\Omega xw_{\eta x} dx - \int_\Omega x^2 w_{\eta xx} w_\eta dx \\
&\leq 2\|xw_\eta\| \|w_{\eta x}\| + \|xw_{\eta xx}\| \|xw_\eta\| \\
&\leq C_2(\eta).
\end{aligned} \tag{3.84}$$

The proof of Lemma 3.7 is completed. □

Using Lemmas 3.6 and 3.7, we can prove that $S(t)$ is continuous.

4. Existence of Global Attractor in Space $H^2(\Omega)$

In this section, we prove that the semigroup operator $S(t)$ associated with the problem (2.1) possesses a global attractor in space $H^2(\Omega)$.

In order to prove our result, we need the following results.

Lemma 4.1 (see [23]). *Assume that $s > s_1$, $(s, s_1 \in N)$, then the following embedding $H^s(R^n) \cap H^{s_1}(R^n, (1+|x|^2 dx))$ into $H^{s_1}(R^n)$ is compact.*

Proof. Let $B \subset H^s(R^n) \cap H^{s_1}(R^n, (1+|x|^2 dx))$ be a bounded set. It suffices to prove that B has a finite ε-net for any $\varepsilon > 0$. First, since

$$\int_{R^n} |x|^2 \sum_{l>s_1} \left|D^l u\right|^2 dx \leq C, \quad \text{for } u \in B, \tag{4.1}$$

there exists an integer $A > 0$, such that

$$\int_{|x|<A} \sum_{l>s_1} \left|D^l u\right|^2 dx \leq \frac{1}{A^2} \int_{|x|<A} |x|^2 \sum_{l>s_1} \left|D^l u\right|^2 dx \leq \frac{C}{A^2} \leq \frac{\varepsilon^2}{2}. \tag{4.2}$$

Let $\Omega = \{x \| x| < A|\}$, then the imbedding $H^s(\Omega) \hookrightarrow H^{s_1}(\Omega)$ is compact. Thus,

$$B|_\Omega = \{u \mid u = v|_\Omega, v \in B\} \subset H^s(\Omega) \tag{4.3}$$

is relatively compact in $H^{s_1}(\Omega)$ and has a finite $(\varepsilon/\sqrt{2})$-net $B(\widetilde{u_k}, (\varepsilon/\sqrt{2}))$, $k = 1, 2, \ldots, m$ with $\widetilde{u_k} \in B|_\Omega, \widetilde{u_k} = u_k|_{\overline{\Omega}}$ and $u_k \in B$. We claim that $\{B(\widetilde{u_k}, (\varepsilon/\sqrt{2}))\}$ is an ε-net of B in $H^{s_1}(\Omega)$.

Indeed, for any $u \in B$, $\tilde{u} = u|_{\overline{\Omega}}$, then there exists a $\widetilde{u_k}$ such that

$$\|\widetilde{u_k} - \tilde{u}\|_{H^{s_1}(\Omega)} < \frac{\varepsilon}{\sqrt{2}}. \tag{4.4}$$

Hence,

$$\begin{aligned} \|\tilde{u} - \widetilde{u_k}\|^2_{H^{s_1}(R^n)} &= \|\tilde{u} - \widetilde{u_k}\|^2_{H^{s_1}\Omega} + \|\tilde{u} - \widetilde{u_k}\|^2_{H^{R^n \setminus \Omega}} \\ &\leq \frac{\varepsilon^2}{2} + \int_{|x|<A} |x|^2 \sum_{l \leq s_1} \left|D^l u\right|^2 dx \\ &< \varepsilon^2. \end{aligned} \tag{4.5}$$

This completes the lemma. □

Lemma 4.2 (see [7, 11]). *Let E be Banach space and $\{S(t), t \geq 0\}$ a set of semigroup operators, that is, $S(t) : E \to E$ satisfy*

$$S(t)S(\tau) = S(t+\tau), \qquad S(0) = I, \tag{4.6}$$

where I is the identity operator and E is space $H^2(\Omega)$. We also assume that

(1) *$S(t)$ is bounded, that is, for each $R > 0$, there exists a constant $C > 0$ such that $\|u\|_E \le C$ implies $\|S(t)u\|_E \le R$, $(t \ge 0)$,*

(2) *there is an bounded absorbing set $B_0 \subset E$, that is, for any bounded set $B \subset E$, there exists a constant T, such that $S(t)B \subset B_0$, for $t \ge T$,*

(3) *$S(t)$ is a continuous operator for $t > 0$, then $S(t)$ has a compact global attractor*

$$\mathcal{A} = \omega(B_0) = \bigcap_{s\ge 0} \overline{\bigcup_{t\ge s} S(t)B_0} \tag{4.7}$$

in the space E_0, such that

(1) *$S(t)\mathcal{A} = \mathcal{A}$, $t \ge 0$,*

(2) $\mathrm{dist}(S(t)B, \mathcal{A})_E \to 0$ *as* $t \to +\infty$, *and* $\mathrm{dist}(S(t)B, \mathcal{A})_E$ *denotes the Hausdorff semidistance defined as*

$$\mathrm{dist}(S(t)B, \mathcal{A}) = \sup_{x\in S(t)B} \inf_{y\in\mathcal{A}} d(x, y), \tag{4.8}$$

for any bounded set $B \subset H^2(\Omega)$ in which sense the trajectories are attracted to $\mathcal{A}$ (see[9, 24]), using Kuratowski α-measure in order to overcome the non-compactness of the classical Sobolev embedding.

Firstly, we need the following definitions.

Definition 4.3 (see [11, 25]). Let $\{S(t)\}_{t\ge 0}$ be a semigroup in complete metric space E. For any subset $B \subset E$, the set $\omega(B)$ defined by $\omega(B) = \bigcap_{s\ge 0} \overline{\bigcup_{t\ge s} S(t)B}$ is called the ω-limit set of B.

Remark 4.4. (1) It is easy to see that $\varphi \in \omega(B)$ if and only if there exists a sequence of element $\varphi_n \in B$ and a sequence $t_n \to \infty$, such that

$$S(t_n)\varphi_n \to \varphi, \quad as\ n \longrightarrow \infty. \tag{4.9}$$

(2) If $\omega(B)$ is ω-limit compact set, then, for every bounded subset B of E and for any $\varepsilon > 0$, there exists a $t_0 > 0$, such that $\alpha(\bigcup_{t\ge t_0} S(t)B) \le \varepsilon$.

Definition 4.5 (see [11, 26]). Let $\{S(t)\}_{t\ge 0}$ be a semigroup in complete metric space E. A subset B_0 of E is called an absorbing set in E if, for any bounded subset B of E, there exists some $t_0 \ge 0$, such that $S(t)B \subset B_0$, for all $t \ge t_0$.

Definition 4.6 (see [11, 26]). Let $\{S(t)\}_{t\ge 0}$ be a semigroup in complete metric space E. A subset $\mathcal{A}$ of E is called global attractor for the semigroup if $\mathcal{A}$ is compact and enjoys the following properties:

(1) $\mathcal{A}$ is a invariant set, that is, $S(t)\mathcal{A} = \mathcal{A}$, for any $t \ge t_0$,

(2) $\mathcal{A}$ attract all bounded set of E, that is, for any bounded subset B of E, $\mathrm{dist}(S(t)B, \mathcal{A}) \to 0$, as $t \to \infty$, where $\mathrm{dist}(B, A)$ is Hausdorff semidistance of two set B and A in space E: $\mathrm{dist}(B, A) = \sup_{x\in B} \inf_{y\in A} d(x, y)$.

Definition 4.7 (see [12, 27]). *Kuratowski* α-measure of set B is defined by the formula

$$\alpha(B) = \inf\{\delta \mid B \quad \text{has} \quad \text{a} \quad \text{finite} \quad \text{cover of} \quad \text{diameter} < \delta\}, \tag{4.10}$$

for every bounded set B of a Banach space X.

Secondly, due to Definition 4.6, it is easy to see that *Kuratowski* α-measure of set B has the following properties.

Remark 4.8. (1) If $\mathcal{A}$ is compact set, then $\alpha(\mathcal{A}) = 0$;

(2) $\alpha(\mathcal{A} + \mathcal{B}) \leq \alpha(\mathcal{A}) + \alpha(\mathcal{B})$,

(3) $\alpha(\mathcal{A} \cup \mathcal{B}) \leq \mathrm{Max}\{\alpha(\mathcal{A}), \alpha(\mathcal{B})\}$,

(4) if $\mathcal{A} \leq \mathcal{B}$, $\alpha(\mathcal{A}) \leq \alpha(\mathcal{B})$,

(5) $\alpha(\overline{\mathcal{B}}) \leq \alpha(\mathcal{B})$.

Thirdly, we prove Theorem 2.1.

Proof. Using the result of [11], we have $S(t)$ is ω-limit compact and B is bounded, for any $\varepsilon > 0$, there exists $t \geq 0$ such that

$$\alpha\left(\bigcup_{t \geq t_0} S(t)B\right) \leq \varepsilon. \tag{4.11}$$

Taking $\varepsilon = 1/n, (n = 1, 2, \ldots)$, we can find a sequence $\{t_n\}$, $t_1 < t_2 < \cdots t_n < \cdots$, such that

$$\alpha\left(\bigcup_{t \geq t_n} S(t)B\right) \leq \frac{1}{n}. \tag{4.12}$$

By $\alpha(\overline{\mathcal{B}}) \leq \alpha(\mathcal{B})$, we get

$$\begin{gathered}\alpha\left(\overline{\bigcup_{t \geq t_n} S(t)B}\right) \leq \frac{1}{n}, \\ \mathcal{A} = \omega(B) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)B} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{t \geq t_n} S(t)B}.\end{gathered} \tag{4.13}$$

First, we prove that $\mathcal{A} = \omega(B)$ is variant. As a matter of fact, if $\psi \in S(t)\omega(B)$, then $\psi = S(t)\phi$, for some $\phi \in \omega(B)$. So, there exists a sequence $\phi_n \in B$ and $t_n \to \infty$ such that $S(t)\phi_n \to \phi$, that is,

$$S(t)S(t_n)\phi_n = S(t + t_n)\phi_n \longrightarrow S(t)\phi = \psi, \tag{4.14}$$

which implies that $\psi \in \omega(B)$ and $S(t)\omega(B) \subset \omega(B)$.

Conversely, if $\psi \in \omega(B)$, by (4.9), we can find two sequences $\phi_n \in B$ and $t_n \to \infty$ such that $S(t_n)\phi_n \to \phi$. We need to prove that $\{S(t_n - t)\phi_n\}$ has a subsequence which converges in E. For any $\varepsilon > 0$, there exists a t_ε such that

$$\alpha\left(\bigcup_{t' \geq t_\varepsilon} S(t')B\right) \leq \varepsilon, \tag{4.15}$$

which implies that

$$\alpha\left(\bigcup_{t' \geq t + t_\varepsilon} S(t' - t)B\right) \leq \varepsilon. \tag{4.16}$$

Hence, there exists an integer N, such that

$$\bigcup_{n \geq N} S(t_n - t)\phi_n \subset \bigcup_{t' \geq t + t_\varepsilon} S(t' - t)B. \tag{4.17}$$

Then, it follows that

$$\alpha\left(\bigcup_{n \geq N} S(t_n - t)\phi_n\right) \leq \varepsilon. \tag{4.18}$$

Notice that $\alpha(\bigcup_{n \geq N} S(t_n - t)\phi_n) \leq \varepsilon$ contains only a finite number of elements, where N_0 is fixed such that $t_n - t \geq 0$, as $n \geq N_0$.

By properties (1)–(4) in Remark 4.8, we have

$$\alpha\left(\bigcup_{n \geq N_0} S(t_n - t)\phi_n\right) = \alpha\left(\bigcup_{n \geq N} S(t_n - t)\phi_n\right) \leq \varepsilon. \tag{4.19}$$

Let $\varepsilon \to 0$, then we get that

$$\alpha\left(\bigcup_{n \geq N_0} S(t_n - t)\phi_n\right) = 0. \tag{4.20}$$

This implies that $\{S(t_n - t)\phi_n\}$ is relatively compact. So, there exists a subsequence $t_{n_j} \to \infty$ and $\psi \in E$, such that

$$S\left(t_{n_j} - t\right)\phi_{n_j} \longrightarrow \infty, \quad \text{as } t_{n_j} \longrightarrow \infty. \tag{4.21}$$

It is easy to see that $\psi \in \omega(B)$ and

$$\phi = \lim_{j \to \infty} S\left(t_{n_j} - t\right)\phi_{n_j} = \lim_{j \to \infty} S(t)S\left(t_{n_j} - t\right)\phi_{n_j} = S(t)\psi \tag{4.22}$$

furthermore, $\phi \in S(t)\omega(B)$.

Next, by virtue of Lemma 4.2 and the result of [11, 12], we prove that $\mathcal{A} = \omega(B)$ is an global attractor in E and attracts all bounded subsets of E.

Otherwise, then there exists a bounded subset B_0 of E such that $\mathrm{dist}(S(t)B_0, \mathcal{A})$ does not tend to 0 as $t \to \infty$. Thus, there exists a $\delta > 0$ and a sequence $t_n \to \infty$ such that

$$\mathrm{dist}(S(t_n)B_0, \mathcal{A}) \geq \delta > 0, \quad \forall n \in N. \tag{4.23}$$

For each n, there exist $b_n \in B_0$, $(n = 1, 2, \ldots)$ satisfying

$$\mathrm{dist}(S(t_n)b_n, \mathcal{A}) \geq \frac{\delta}{2} > 0. \tag{4.24}$$

Whereas B is an absorbing set, $S(t_n)B_0$ and $S(t_n)b_n$ belong to B, for n sufficiently large. As in the discussion above, we obtain that $S(t_n)b_n$ is relatively compact admits at least one cluster point γ,

$$\gamma = \lim_{n_i \to \infty} S(t_{n_i})b_{n_i} = \lim_{n_i \to \infty} S(t_{n_i} - t_1)S(t_1)b_{n_i}, \tag{4.25}$$

where t_1 follows $S(t_1)B_0 \subset B$. So, $\gamma \in \mathcal{A} = \omega(B)$ and this contradicts (4.24). The proof is complete. □

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