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Research Article

Strong Convergence Theorems for Family of Nonexpansive Mappings and System of Generalized Mixed Equilibrium Problems and Variational Inequality Problems

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We introduce a new iterative scheme by hybrid method for finding a common element of the set of common fixed points of infinite family of nonexpansive mappings, the set of common solutions to a system of generalized mixed equilibrium problems, and the set of solutions to a variational inequality problem in a real Hilbert space. We then prove strong convergence of the scheme to a common element of the three sets. We give some applications of our results. Our results extend important recent results.

1. Introduction

Let K be a nonempty closed and convex subset of a real Hilbert space H. A mapping $A : K \to H$ is called *monotone* if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in K. \tag{1.1}$$

The variational inequality problem is to find an $x^* \in K$ such that

$$\langle y - x^*, Ax^* \rangle \geq 0, \quad \forall y \in K \tag{1.2}$$

(see, e.g., [1]). We will denote the set of solutions to the variational inequality problem (1.2) by $\mathrm{VI}(K, A)$.

A mapping $A : K \to H$ is called *inverse-strongly monotone* (see, e.g., [2, 3]) if there exists a positive real number α such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$, for all $x, y \in K$. For such a case, A is called α-inverse-strongly monotone.

A mapping $T : K \to K$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{1.3}$$

for all $x, y \in K$. A point $x \in K$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is the set $F(T) := \{x \in K : Tx = x\}$.

Let $\varphi : K \to \mathbb{R}$ be a real-valued function and $A : K \to H$ a nonlinear mapping. Suppose that $F : K \times K$ into $\mathbb{R}$ is an equilibrium bifunction. That is, $F(u, u) = 0$, for all $u \in K$. The generalized mixed equilibrium problem is to find $x \in K$ (see, e.g., [4–6]) such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \tag{1.4}$$

for all $y \in K$. We will denote the set of solutions of this generalized mixed equilibrium problem by $\text{GMEP}(F, A, \varphi)$. Thus

$$\text{GMEP}(F, A, \varphi) := \{x^* \in K : F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^*\rangle \geq 0,\ \forall y \in K\}. \tag{1.5}$$

If $\varphi = 0$, $A = 0$, then problem (1.4) reduces to equilibrium problem studied by many authors (see, e.g., [7–14]), which is to find $x^* \in K$ such that

$$F(x^*, y) \geq 0, \tag{1.6}$$

for all $y \in K$. The set of solutions of (1.6) is denoted by $\text{EP}(F)$.

If $\varphi = 0$, then problem (1.4) reduces to generalized equilibrium problem studied by many authors (see, e.g., [15–18]), which is to find $x^* \in K$ such that

$$F(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0, \tag{1.7}$$

for all $y \in K$. The set of solutions of (1.7) is denoted by $\text{GEP}(F, A)$.

If $A = 0$, then problem (1.4) reduces to mixed equilibrium problem considered by many authors (see, e.g., [19–21]), which is to find $x^* \in K$ such that

$$F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \tag{1.8}$$

for all $y \in K$. The set of solutions of (1.8) is denoted by $\text{MEP}(F, \varphi)$.

The generalized mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and equilibrium problems as special cases (see, e.g., [22]). Numerous problems in physics, optimization, and economics reduce to find a solution of problem (1.4). Several methods have been proposed to solve the fixed point problems, variational inequality problems, and equilibrium problems in the literature. See, for example, [23–33].

One of the iterative processes (see Halpern [34]) which is often used to approximate a fixed point of a nonexpansive mapping T is defined as follows. Take an initial guess $x_0 \in K$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. The iteration process (1.9) has been proved to be strongly convergent both in Hilbert spaces [34–36] and uniformly smooth Banach spaces [37, 38] when the sequence $\{\alpha_n\}$ satisfies the conditions

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) either $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}\alpha_n/\alpha_{n+1} = 1$.

Motivated by (1.9), Martinez-Yanes and Xu [39] introduced the following iterative scheme for a single nonexpansive mapping T in a Hilbert space:

$$\begin{gathered} x_0 \in K, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \left\{z \in K : \left\|y_n - z\right\|^2 \leq \|x_n - z\|^2 + \alpha_n\left(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle\right)\right\}, \\ Q_n = \{z \in K : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x_0, \end{gathered} \tag{1.10}$$

where P_K denotes the metric projection of H onto a closed and convex subset K of H. They proved that if $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$, then the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Furthermore, algorithm (1.10) has been modified by many authors for relatively nonexpansive mappings and quasi-ϕ-nonexpansive mappings in Banach spaces (see, e.g., [40–43]).

Recently, Ceng and Yao [44] introduced a new iterative scheme of approximating a common element of the set of solutions to mixed equilibrium problem and set of common fixed points of finite family of nonexpansive mappings in a real Hilbert space H. In the proof process of their results, they imposed the following condition on a nonempty closed and convex subset K of H:

(E) $A : K \to \mathbb{R}$ is η-strongly convex and its derivative A' is sequentially continuous from weak topology to the strong topology.

We remark here that this condition (E) has been used by many authors for approximation of solution to mixed equilibrium problem in a real Hilbert space (see, e.g., [45, 46]). However, it is observed that condition (E) does not include the case $A(x) = \|x\|^2/2$ and $\eta(x,y) = x - y$. Furthermore, Peng and Yao [19],Wangkeeree and Wangkeeree [47], and other authors replaced condition (E) with these conditions:

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq K$ and $y_x \in K$ such that, for any $z \in K \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0, \tag{1.11}$$

or

(B2) K is a bounded set.

Consequently, conditions (B1) and (B2) have been used by many authors in approximating solution to generalized mixed equilibrium (mixed equilibrium) problems in a real Hilbert space (see, e.g., [19, 47]).

In [48], Takahashi et al. proved the following convergence theorem using hybrid method.

Theorem 1.1 (Takahashi et al. [48]). *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let T be a nonexpansive mapping of K into itself such that $F(T) \neq \emptyset$. For $C_1 = K$, $x_1 = P_{C_1}x_0$, define sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ of K as follows:*

$$\begin{gathered} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 1, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \quad n \geq 1, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1. \end{gathered} \tag{1.12}$$

Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$ satisfies $0 \leq \alpha_n < \alpha < 1$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{F(T)}x_0$.

Motivated by the results of Takahashi et al. [48], Kumam [49] studied the problem of approximating a common element of set of solutions to an equilibrium problem, set of solutions to variational inequality problem, and set of fixed points of a nonexpansive mapping in a real Hilbert space. In particular, he proved the following theorem.

Theorem 1.2 (Kumam [49]). *Let K be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $K \times K$ satisfying (A1)–(A4), and let B be a β-inverse-strongly monotone mapping of K into H. Let T be a nonexpansive mapping of K into H such that $F(T) \cap \mathrm{EP}(F) \cap \mathrm{VI}(K,B) \neq \emptyset$. For $C_1 = K$, $x_1 = P_{C_1}x_0$, define sequences $\{x_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ of K as follows:*

$$\begin{gathered} F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in K, \\ y_n = \alpha_n x_n + (1-\alpha_n)TP_K(z_n - \lambda_n Bz_n), \quad n \geq 1, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \quad n \geq 1, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1. \end{gathered} \tag{1.13}$$

Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset [0,1)$, $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$, and $\{\lambda_n\}_{n=1}^{\infty} \subset [0,2\beta]$ satisfy

$$\liminf_{n\to\infty} r_n > 0, \qquad 0 < c \leq \lambda_n \leq f < 2\beta, \qquad \lim_{n\to\infty}\alpha_n = 0. \tag{1.14}$$

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{F(T)\cap\mathrm{EP}(F)\cap\mathrm{VI}(K,B)}x_0$.

Quite recently, Chantarangsi et al. [50] proved the following convergence theorem for approximation of fixed point of a nonexpansive mapping which is also a common solution to a system of generalized mixed equilibrium problems and variational inequality problem in a real Hilbert space.

Theorem 1.3 (Chantarangsi et al. [50]). *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let θ_1, θ_2 be bifunctions from $K \times K$ satisfying (A1)–(A4), Ψ_1 an ξ-inverse-strongly monotone mapping of K into H, Ψ_2 a β-inverse-strongly monotone mapping of K into H with assumption (B1) or (B2), and $T : K \to K$ a nonexpansive mapping. Let B be an ω-Lipschitz continuous and relaxed (υ, ν) co-coercive mapping of K into H, $f : K \to K$ a contraction mapping with coefficient $\eta \in (0,1)$, and A a strongly positive linear bounded selfadjoint operator with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\eta$. Suppose that $F := F(T) \cap \mathrm{GMEP}(\theta_1, \varphi, \Psi_1) \cap \mathrm{GMEP}(\theta_2, \varphi, \Psi_2) \cap \mathrm{VI}(K, B) \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{x_n\}_{n=1}^{\infty}$ be generated by*

$$\begin{aligned} u_n &= T_{r_n}^{(\theta_1,\varphi)}(x_n - r_n \Psi_1 x_n), \\ v_n &= T_{s_n}^{(\theta_2,\varphi)}(u_n - s_n \Psi_1 u_n), \\ z_n &= P_K(v_n - \alpha_n B S v_n), \\ y_n &= \varepsilon_n \gamma f(x_n) + \beta_n x_n + \big((1-\beta_n) I - \varepsilon_n A\big) z_n, \\ x_{n+1} &= \gamma_n x_n + (1-\gamma_n) y_n, \quad \forall n \geq 1, \end{aligned} \tag{1.15}$$

where $\{r_n\} \subset [a,b] \subset [0, 2\xi]$, $\{s_n\} \subset [c,d] \subset [0, 2\beta]$, $\{\gamma_n\} \subset [h,j] \subset (0,1)$, $\{\gamma_n\}$, $\{\varepsilon_n\}$, $\{\beta_n\}$ are three sequences in (0,1) satisfying the following conditions:

(C1) $\lim_{n\to\infty} \varepsilon_n = 0$ *and* $\sum_{n=1}^{\infty} \varepsilon_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C3) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\xi$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(C4) $0 < \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n < 2\beta$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$,

(C5) $\{\alpha_n\} \subset [e,g] \subset (0, (2(\nu - \upsilon\omega^2))/\omega^2)$, $\nu > \upsilon\omega^2$ *and* $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$.

Then, $\{x_n\}$ converges strongly to $z = P_F(\gamma f + (I - A))(z)$.

Motivated by the ongoing research and the above-mentioned results, we modify algorithm (1.10) and introduce *a new iterative scheme for finding a common element of the set of fixed points of an infinite family of nonexpansive mappings, the set of common solutions to a system of generalized mixed equilibrium problems, and the set of solutions to a variational inequality problem in a real Hilbert space*. Furthermore, we show that our new iterative scheme converges strongly to a common element of the three sets. In the proof process of our results, we use conditions (B1) and (B2) mentioned above. Our result extends many important recent results. Finally, we give some applications of our results.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let K be a nonempty closed and convex subset of H. The strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written $x_n \to x$ as $n \to \infty$.

For any point $u \in H$, there exists a unique point $P_K u \in K$ such that

$$\|u - P_K u\| \leq \|u - y\|, \quad \forall y \in K. \tag{2.1}$$

P_K is called the *metric projection* of H onto K. We know that P_K is a nonexpansive mapping of H onto K. It is also known that P_K satisfies

$$\langle x - y, P_K x - P_K y \rangle \geq \|P_K x - P_K y\|^2, \tag{2.2}$$

for all $x, y \in H$. Furthermore, $P_K x$ is characterized by the properties $P_K x \in K$ and

$$\langle x - P_K x, P_K x - y \rangle \geq 0, \tag{2.3}$$

for all $y \in K$ and

$$\|x - P_K x\|^2 \leq \|x - y\|^2 - \|y - P_K x\|^2, \quad \forall x \in H, \ y \in K. \tag{2.4}$$

In the context of the variational inequality problem, (2.3) implies that

$$x^* \in \mathrm{VI}(K, A) \Longleftrightarrow x^* = P_K(x^* - \lambda A x^*), \quad \forall \lambda \geq 0. \tag{2.5}$$

If A is an α-inverse-strongly monotone mapping of K into H, then it is obvious that A is a $(1/\alpha)$-Lipschitz continuous. We also have that, for all $x, y \in K$ and $r > 0$,

$$\begin{aligned} \|(I - rA)x - (I - rA)y\|^2 &= \|x - y - r(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2r\langle Ax - Ay, x - y \rangle + r^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2. \end{aligned} \tag{2.6}$$

So, if $r \leq 2\alpha$, then $I - rA$ is a nonexpansive mapping of K into H.

For solving the generalized mixed equilibrium problem for a bifunction $F : K \times K \to \mathbb{R}$, let us assume that F, φ, and K satisfy the following conditions:

(A1) $F(x, x) = 0$ for all $x \in K$,

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y, \in K$,

(A3) for each $x, y, z \in K$, $\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$,

(A4) for each $x \in K$, $y \mapsto F(x, y)$ is convex and lower semicontinuous,

(B1) for each $x \in H$ and $r > 0$ there exist a bounded subset $D_x \subseteq K$ and $y_x \in K$ such that for any $z \in K \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x \rangle < 0, \tag{2.7}$$

(B2) K is a bounded set.

Then, we have the following lemma.

Lemma 2.1 (Wangkeeree and Wangkeeree [47]). *Assume that $F : K \times K \to \mathbb{R}$ satisfies (A1)–(A4), and let $\varphi : K \to \mathbb{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(F,\varphi)} : H \to K$ as follows:*

$$T_r^{(F,\varphi)}(x) = \left\{ z \in K : F(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in K \right\}, \tag{2.8}$$

for all $z \in H$. Then, the following hold:

(1) *for each $x \in H$, $T_r^{(F,\varphi)} \neq \emptyset$,*

(2) $T_r^{(F,\varphi)}$ *is single-valued,*

(3) $T_r^{(F,\varphi)}$ *is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y \right\|^2 \leq \left\langle T_r^{(F,\varphi)}x - T_r^{(F,\varphi)}y, x - y \right\rangle, \tag{2.9}$$

(4) $F(T_r^{(F,\varphi)}) = \mathrm{GMEP}(F)$,

(5) $\mathrm{GMEP}(F)$ *is closed and convex.*

We will also use the following lemma in our results.

Lemma 2.2 (Baillon and Haddad [51]). *Let E be a Banach space, let f be a continuously Fréchet differentiable convex functional on E, and let ∇f be the gradient of f. If ∇f is $(1/\alpha)$-Lipschitz continuous, then ∇f is α-inverse-strongly monotone.*

3. Main Results

Theorem 3.1. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $m = 1, 2$, let F_m be a bifunction from $K \times K$ satisfying (A1)–(A4), $\varphi_m : K \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function with assumption (B1) or (B2), A be an α-inverse-strongly monotone mapping of K into H, and B a β-inverse-strongly monotone mapping of K into H, and, for each $i = 1, 2, \ldots$, let $T_i : K \to K$ be a nonexpansive mapping such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let D be a γ-inverse-strongly monotone mapping of K into H. Suppose that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, A, \varphi_1) \cap \mathrm{GMEP}(F_2, B, \varphi_2) \cap \mathrm{VI}(K, D) \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \ldots)$, and $\{x_n\}_{n=0}^{\infty}$ be generated by $x_0 \in K$, $C_{1,i} = K$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1}x_0$,*

$$z_n = T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n),$$

$$u_n = T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n),$$

$$w_n = P_K(u_n - s_n D u_n),$$

$$y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) T_i w_n,$$

$$C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \le \| x_n - z \|^2 + \alpha_{n,i} \left(\| x_0 \|^2 + 2 \langle x_n - x_0, z \rangle \right) \right\},$$
$$C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i},$$
$$x_{n+1} = P_{C_{n+1}} x_0, \quad n \ge 1. \tag{3.1}$$

Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0,1)$ *(*$i = 1,2,\ldots$*),* $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$*, and* $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 2\beta]$ *satisfy*

(i) $0 < a \le r_n \le b < 2\alpha$,

(ii) $0 < c \le \lambda_n \le f < 2\beta$,

(iii) $\lim_{n\to\infty} \alpha_{n,i} = 0$,

(iv) $0 < h \le s_n \le j < 2\gamma$.

Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_F x_0$.

Proof. Let $x^* \in F$. Then,

$$\begin{aligned} \| w_n - x^* \|^2 &= \| P_K(u_n - s_n D u_n) - P_K(x^* - s_n D x^*) \|^2 \\ &\le \| (u_n - s_n D u_n) - (x^* - s_n D x^*) \|^2 \\ &\le \| u_n - x^* \|^2 + s_n (s_n - 2\gamma) \| D u_n - D x^* \|^2 \\ &\le \| u_n - x^* \|^2. \end{aligned} \tag{3.2}$$

Since both $I - r_n A$ and $I - \lambda_n B$ are nonexpansive for each $n \ge 1$ and $x^* = T_{r_n}^{(F_1,\varphi_1)}(x^* - r_n A x^*)$, $x^* = T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_n B x^*)$, from (2.6), we have that

$$\begin{aligned} \| u_n - x^* \|^2 &= \left\| T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) - x^* \right\|^2 \\ &= \left\| T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) - T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_n B x^*) \right\|^2 \\ &\le \| (I - \lambda_n B) z_n - (I - \lambda_n B) x^* \|^2 \\ &\le \| z_n - x^* \|^2 + \lambda_n (\lambda_n - 2\beta) \| B z_n - B x^* \|^2 \\ &\le \| z_n - x^* \|^2 \quad (\text{since } \lambda_n < 2\beta, \ \forall n \ge 1), \end{aligned}$$

$$\begin{aligned}\|z_n - x^*\|^2 &= \left\|T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n) - x^*\right\|^2 \\ &= \left\|T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n) - T_{r_n}^{(F_1,\varphi_1)}(x^* - r_n A x^*)\right\|^2 \\ &\le \|(I - r_n A)x_n - (I - r_n A)x^*\|^2 \\ &\le \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ax^*\|^2 \\ &\le \|x_n - x^*\|^2.\end{aligned} \tag{3.3}$$

Therefore,

$$\|u_n - x^*\| \le \|x_n - x^*\|. \tag{3.4}$$

Let $n = 1$, then $C_{1,i} = K$ is closed convex for each $i = 1, 2, \ldots$. Now assume that $C_{n,i}$ is closed convex for some $n > 1$. Then, from definition of $C_{n+1,i}$, we know that $C_{n+1,i}$ is closed convex for the same $n > 1$. Hence, $C_{n,i}$ is closed convex for $n \ge 1$ and for each $i = 1, 2, \ldots$. This implies that C_n is closed convex for $n \ge 1$. Furthermore, we show that $F \subset C_n$. For $n = 1$, $F \subset K = C_{1,i}$. For $n \ge 2$, let $x^* \in F$. Then,

$$\begin{aligned}\left\|y_{n,i} - x^*\right\|^2 &= \|\alpha_{n,i}(x_0 - x^*) + (1 - \alpha_{n,i})(T_i w_n - x^*)\|^2 \\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\|w_n - x^*\|^2 \\ &\le \|w_n - x^*\|^2 + \alpha_{n,i}\left(\|x_0 - x^*\|^2 - \|w_n - x^*\|^2\right) \\ &\le \|x_n - x^*\|^2 + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle x_n - x_0, x^*\rangle\right),\end{aligned} \tag{3.5}$$

which shows that $x^* \in C_{n,i}$, for all $n \ge 2$, for all $i = 1, 2, \ldots$. Thus, $F \subset C_{n,i}$, for all $n \ge 1$, for all $i = 1, 2, \ldots$. Hence, it follows that $F \subset C_n$, for all $n \ge 1$. Since $x_n = P_{C_n} x_0$, for all $n \ge 1$, and $x_{n+1} \in C_{n+1} \subset C_n$, for all $n \ge 1$, we have that

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\|, \quad \forall n \ge 1. \tag{3.6}$$

Also, as $F \subset C_n$ by (2.1), it follows that

$$\|x_n - x_0\| \le \|z - x_0\|, \quad z \in F, \ \forall n \ge 1. \tag{3.7}$$

From (3.6) and (3.7), we have that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Hence, $\{x_n\}_{n=0}^{\infty}$ is bounded and so are $\{z_n\}_{n=1}^{\infty}$, $\{Ax_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{Du_n\}_{n=1}^{\infty}$, $\{Bz_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{T_i w_n\}_{n=1}^{\infty}$, and $\{y_{n,i}\}_{n=1}^{\infty}$, $i = 1, 2, \ldots$. For $m > n \ge 1$, we have that $x_m = P_{C_m} x_0 \in C_m \subset C_n$. By (2.4), we obtain

$$\|x_m - x_n\|^2 \le \|x_n - x_0\|^2 - \|x_m - x_0\|^2. \tag{3.8}$$

Letting $m, n \to \infty$ and taking the limit in (3.8), we have that $x_m - x_n \to 0$, $m, n \to \infty$, which shows that $\{x_n\}_{n=0}^{\infty}$ is Cauchy. In particular, $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since, $\{x_n\}_{n=0}^{\infty}$ is Cauchy, we assume that $x_n \to z \in K$.

Since $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1}$, then

$$\left\|y_{n,i} - x_{n+1}\right\|^2 \leq \|x_n - x_{n+1}\|^2 + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle x_n - x_0, x_{n+1}\rangle\right) \longrightarrow 0, \tag{3.9}$$

and it follows that

$$\left\|y_{n,i} - x_n\right\| \leq \left\|y_{n,i} - x_{n+1}\right\| + \|x_n - x_{n+1}\|. \tag{3.10}$$

Thus,

$$\lim_{n\to\infty}\left\|y_{n,i} - x_n\right\| = 0, \quad i = 1, 2, \ldots. \tag{3.11}$$

Furthermore,

$$\begin{aligned}
\left\|y_{n,i} - x^*\right\|^2 &\leq \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\|T_i w_n - x^*\|^2 \\
&\leq \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\|w_n - x^*\|^2 \\
&\leq \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\|u_n - x^*\|^2 \\
&\leq \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\left\|T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) - T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_n B x^*)\right\|^2 \\
&\leq \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\|(z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*)\|^2 \\
&\leq \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\left[\|z_n - x^*\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bz_n - Bx^*\|^2\right] \\
&\leq \alpha_{n,i}\|x_0 - x^*\|^2 + \|x_n - x^*\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bz_n - Bx^*\|^2.
\end{aligned} \tag{3.12}$$

Since $0 < c \leq \lambda_n \leq f < 2\beta$, we have that

$$\begin{aligned}
c(2\beta - f)\|Bz_n - Bx^*\|^2 &\leq \|x_n - x^*\|^2 - \left\|y_{n,i} - x^*\right\|^2 + \alpha_{n,i}\|x_0 - x^*\|^2 \\
&\leq \left\|y_{n,i} - x_n\right\|\left(\|x_n - x^*\| + \left\|y_{n,i} - x^*\right\|\right) + \alpha_{n,i}\|x_0 - x^*\|^2.
\end{aligned} \tag{3.13}$$

Hence, $\lim_{n\to\infty}\|Bz_n - Bx^*\| = 0$. From (3.1), we have that

$$\begin{aligned}
\left\|y_{n,i} - x^*\right\|^2 &\leq \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\|T_i w_n - x^*\|^2 \\
&= \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\|u_n - x^*\|^2 \\
&\leq \alpha_{n,i}\|x_0 - x^*\|^2 + \|u_n - x^*\|^2.
\end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \left\| T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n) - T_{\lambda_n}^{(F_2,\varphi_2)}(x^* - \lambda_n B x^*) \right\|^2 \\
&\leq \langle (z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*), u_n - x^* \rangle \\
&= \frac{1}{2}\Big[\|(z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*)\|^2 + \|u_n - x^*\|^2 \\
&\quad - \|(z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*) - (u_n - x^*)\|^2 \Big] \\
&\leq \frac{1}{2}\Big[\|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|(z_n - \lambda_n B z_n) - (x^* - \lambda_n B x^*) - (u_n - x^*)\|^2 \Big] \\
&= \frac{1}{2}\Big[\|z_n - x^*\|^2 + \|u_n - x^*\|^2 - \|u_n - z_n\|^2 \\
&\quad + 2\lambda_n \langle z_n - u_n, B z_n - B x^* \rangle - \lambda_n^2 \|B z_n - B x^*\|^2 \Big],
\end{aligned} \tag{3.15}$$

and, hence,

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \langle z_n - u_n, B z_n - B x^* \rangle - \lambda_n^2 \|B z_n - B x^*\|^2 \\
&\leq \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \|z_n - u_n\| \|B z_n - B x^*\|.
\end{aligned} \tag{3.16}$$

Putting (3.16) into (3.14), we have that

$$\left\| y_{n,i} - x^* \right\|^2 \leq \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n \|z_n - u_n\| \|B z_n - B x^*\| + \alpha_{n,i} \|x_0 - x^*\|^2. \tag{3.17}$$

It follows that

$$\begin{aligned}
\|z_n - u_n\|^2 &\leq \|z_n - x^*\|^2 - \left\| y_{n,i} - x^* \right\|^2 + 2\lambda_n \|z_n - u_n\| \|B z_n - B x^*\| + \alpha_{n,i} \|x_0 - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \left\| y_{n,i} - x^* \right\|^2 + 2\lambda_n \|z_n - u_n\| \|B z_n - B x^*\| + \alpha_{n,i} \|x_0 - x^*\|^2 \\
&\leq \left\| y_{n,i} - x_n \right\| \left(\|x_n - x^*\| + \left\| y_{n,i} - x^* \right\| \right) + 2\lambda_n \|z_n - u_n\| \|B z_n - B x^*\| \\
&\quad + \alpha_{n,i} \|x_0 - x^*\|^2.
\end{aligned} \tag{3.18}$$

Therefore, $\lim_{n\to\infty}\|z_n - u_n\| = 0$. Furthermore,

$$\begin{aligned}\left\|y_{n,i} - x^*\right\|^2 &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\|T_i w_n - x^*\|^2\\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\|u_n - x^*\|^2\\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\|z_n - x^*\|^2\\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\left\|T_{r_n}^{(F_1,\varphi_1)}(x_n - r_nAx_n) - T_{r_n}^{(F_1,\varphi_1)}(x^* - r_nAx^*)\right\|^2\\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\|(x_n - r_nAx_n) - (x^* - r_nAx^*)\|^2\\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\left[\|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ax^*\|^2\right]\\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ax^*\|^2.\end{aligned} \tag{3.19}$$

Since $0 < a \le r_n \le b < 2\alpha$, we have that

$$\begin{aligned}a(2\alpha - b)\|Ax_n - Ax^*\|^2 &\le \|x_n - x^*\|^2 - \left\|y_{n,i} - x^*\right\|^2 + \alpha_{n,i}\|x_0 - x^*\|^2\\ &\le \left\|y_{n,i} - x_n\right\|\left(\|x_n - x^*\| + \left\|y_{n,i} - x^*\right\|\right) + \alpha_{n,i}\|x_0 - x^*\|^2.\end{aligned} \tag{3.20}$$

Hence, $\lim_{n\to\infty}\|Ax_n - Ax^*\| = 0$. From (3.1), we have that

$$\begin{aligned}\left\|y_{n,i} - x^*\right\|^2 &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\|T_i w_n - x^*\|^2\\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\|u_n - x^*\|^2\\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + \|z_n - x^*\|^2.\end{aligned} \tag{3.21}$$

On the other hand,

$$\begin{aligned}\|z_n - x^*\|^2 &\le \left\|T_{r_n}^{(F_1,\varphi_1)}(x_n - r_nAx_n) - T_{r_n}^{(F_1,\varphi_1)}(x^* - r_nAx^*)\right\|^2\\ &\le \langle (x_n - r_nAx_n) - (x^* - r_nAx^*), z_n - x^*\rangle\\ &= \frac{1}{2}\Big[\|(x_n - r_nAx_n) - (x^* - r_nAx^*)\|^2 + \|z_n - x^*\|^2\\ &\quad - \|(x_n - r_nAx_n) - (x^* - r_nAx^*) - (z_n - x^*)\|^2\Big]\\ &\le \frac{1}{2}\left[\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(x_n - r_nAx_n) - (x^* - r_nAx^*) - (z_n - x^*)\|^2\right]\\ &= \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|z_n - x_n\|^2\\ &\quad + 2r_n\langle x_n - z_n, Ax_n - Ax^*\rangle - r_n^2\|Ax_n - Ax^*\|^2\Big],\end{aligned} \tag{3.22}$$

and, hence,

$$\begin{aligned}\|z_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2r_n\langle x_n - z_n, Ax_n - Ax^*\rangle - r_n^2\|Ax_n - Ax^*\|^2 \\ &\le \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2r_n\|x_n - z_n\|\|Ax_n - Ax^*\|.\end{aligned} \tag{3.23}$$

Putting (3.23) into (3.21), we have that

$$\|y_{n,i} - x^*\|^2 \le \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2r_n\|x_n - z_n\|\|Ax_n - Ax^*\| + \alpha_{n,i}\|x_0 - x^*\|^2. \tag{3.24}$$

It follows that

$$\begin{aligned}\|x_n - z_n\|^2 &\le \|x_n - x^*\|^2 - \|y_{n,i} - x^*\|^2 + 2r_n\|x_n - z_n\|\|Ax_n - Ax^*\| + \alpha_{n,i}\|x_0 - x^*\|^2 \\ &\le \|y_{n,i} - x_n\|\left(\|x_n - x^*\| + \|y_{n,i} - x^*\|\right) + 2r_n\|x_n - z_n\|\|Ax_n - Ax^*\| + \alpha_{n,i}\|x_0 - x^*\|^2.\end{aligned} \tag{3.25}$$

Therefore, $\lim_{n\to\infty}\|x_n - z_n\| = 0$. But $y_{n,i} = \alpha_{n,i}x_0 + (1-\alpha_{n,i})T_i w_n$ implies that

$$\|y_{n,i} - T_i w_n\| = \alpha_{n,i}\|x_0 - T_i w_n\| \longrightarrow 0. \tag{3.26}$$

Furthermore, we have that

$$\|x_n - T_i w_n\| \le \|y_{n,i} - T_i w_n\| + \|y_{n,i} - x_n\| \longrightarrow 0. \tag{3.27}$$

Furthermore,

$$\begin{aligned}\|y_{n,i} - x^*\|^2 &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\|T_i w_n - x^*\|^2 \\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\|w_n - x^*\|^2 \\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\|P_K(u_n - s_nDu_n) - P_K(x^* - s_nDx^*)\|^2 \\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\|(u_n - s_nDu_n) - (x^* - s_nDx^*)\|^2 \\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + (1-\alpha_{n,i})\left[\|u_n - x^*\|^2 + s_n(s_n - 2\gamma)\|Du_n - Dx^*\|^2\right] \\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + \|x_n - x^*\|^2 + (1-\alpha_{n,i})s_n(s_n - 2\gamma)\|Du_n - Dx^*\|^2.\end{aligned} \tag{3.28}$$

Thus,

$$\begin{aligned}(1-d_i)h(2\gamma - j)\|Du_n - Dx^*\|^2 &\le (1-\alpha_{n,i})s_n(2\gamma - s_n)\|Du_n - Dx^*\|^2 \\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + \|x_n - x^*\|^2 - \|y_{n,i} - x^*\|^2 \\ &\le \alpha_{n,i}\|x_0 - x^*\|^2 + \|y_{n,i} - x_n\|\left(\|x_n - x^*\| + \|y_{n,i} - x^*\|\right).\end{aligned} \tag{3.29}$$

Since $0 < h \leq s_n \leq j < 2\gamma$, condition (iii) and $\|y_{n,i} - x_n\| \to 0$ as $n \to \infty$, we have that $\lim_{n\to\infty}\|Du_n - Dx^*\| = 0$. Now, using (2.2), we obtain

$$\begin{aligned}\|w_n - x^*\|^2 &\leq \|P_K(u_n - s_nDu_n) - P_K(x^* - s_nDx^*)\|^2 \\ &\leq \langle (u_n - s_nDu_n) - (x^* - s_nDx^*), w_n - x^* \rangle \\ &= \frac{1}{2}\Big[\|(u_n - s_nDu_n) - (x^* - s_nDx^*)\|^2 + \|w_n - x^*\|^2 \\ &\quad - \|(u_n - s_nDu_n) - (x^* - s_nDx^*) - (w_n - x^*)\|^2\Big] \\ &\leq \frac{1}{2}\Big[\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|(u_n - s_nDu_n) - (x^* - s_nDx^*) - (w_n - x^*)\|^2\Big] \\ &= \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|w_n - x^*\|^2 - \|w_n - u_n\|^2 + 2s_n\langle u_n - w_n, Du_n - Dx^* \rangle \\ &\quad - s_n^2\|Du_n - Dx^*\|^2\Big].\end{aligned} \tag{3.30}$$

Thus,

$$\|w_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|w_n - u_n\|^2 + 2s_n\|w_n - u_n\|\|Du_n - Dx^*\|. \tag{3.31}$$

Using this last inequality, we obtain from (3.1)

$$\begin{aligned}\left\|y_{n,i} - x^*\right\|^2 &\leq \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\|T_iw_n - x^*\|^2 \\ &\leq \alpha_{n,i}\|x_0 - x^*\|^2 + (1 - \alpha_{n,i})\|w_n - x^*\|^2 \\ &\leq \alpha_{n,i}\|x_0 - x^*\|^2 + \|x_n - x^*\|^2 - (1 - \alpha_{n,i})\|w_n - u_n\|^2 \\ &\quad + 2s_n(1 - \alpha_{n,i})\|w_n - u_n\|\|Du_n - \mathrm{D}x^*\|.\end{aligned} \tag{3.32}$$

This implies that

$$\begin{aligned}(1 - \alpha_{n,i})\|w_n - u_n\|^2 &\leq \alpha_{n,i}\|x_0 - x^*\|^2 + \|x_n - x^*\|^2 - \left\|y_{n,i} - x^*\right\|^2 \\ &\quad + 2s_n(1 - \alpha_{n,i})\|w_n - u_n\|\|Du_n - Dx^*\| \\ &\leq \alpha_{n,i}\|x_0 - x^*\|^2 + \left\|y_{n,i} - x_n\right\|\left(\|x_n - x^*\| + \left\|y_{n,i} - x^*\right\|\right) \\ &\quad + 2s_n(1 - \alpha_{n,i})\|w_n - u_n\|\|Du_n - Dx^*\|.\end{aligned} \tag{3.33}$$

Since $\lim_{n\to\infty}\alpha_{n,i} = 0$, $\|y_{n,i} - x_n\| \to 0$ as $n \to \infty$, and $\|Du_n - Dx^*\| \to 0$ as $n \to \infty$, we have that $\lim_{n\to\infty}\|w_n - u_n\| = 0$. Also since $\lim_{n\to\infty}\|w_n - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - z\| = 0$, we have that $\lim_{n\to\infty}\|w_n - z\| = 0$. Now,

$$\|w_n - T_iw_n\| \leq \|x_n - T_iw_n\| + \|w_n - x_n\|. \tag{3.34}$$

Hence, $\lim_{n\to\infty}\|w_n - T_i w_n\| = 0$, $i = 1, 2, \ldots$. By $\lim_{n\to\infty}\|w_n - z\| = 0$ and $\lim_{n\to\infty}\|w_n - T_i w_n\| = 0$, $i = 1, 2, \ldots$, we have that $z \in \bigcap_{i=1}^{\infty} F(T_i)$.

Since $z_n := T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n)$, $n \geq 1$, we have, for any $y \in K$, that

$$F_1(z_n, y) + \varphi_1(y) - \varphi_1(z_n) + \langle A x_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0. \tag{3.35}$$

Furthermore, replacing n by n_j in the last inequality and using (A2), we obtain

$$\varphi_1(y) - \varphi_1\left(z_{n_j}\right) + \left\langle A x_{n_j}, y - z_{n_j}\right\rangle + \frac{1}{r_{n_j}}\left\langle y - z_{n_j}, z_{n_j} - x_{n_j}\right\rangle \geq F_1\left(y, z_{n_j}\right). \tag{3.36}$$

Let $z_t := ty + (1-t)z$ for all $t \in (0,1]$ and $y \in K$. This implies that $z_t \in K$. Then, we have that

$$\begin{aligned}
\left\langle z_t - z_{n_j}, A z_t\right\rangle &\geq \varphi_1\left(z_{n_j}\right) - \varphi_1(z_t) + \left\langle z_t - z_{n_j}, A z_t\right\rangle - \left\langle z_t - z_{n_j}, A x_{n_j}\right\rangle \\
&\quad - \left\langle z_t - z_{n_j}, \frac{z_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle + F_1\left(z_t, z_{n_j}\right) \\
&= \varphi_1\left(z_{n_j}\right) - \varphi_1(z_t) + \left\langle z_t - z_{n_j}, A z_t - A z_{n_j}\right\rangle \\
&\quad + \left\langle z_t - z_{n_j}, A z_{n_j} - A x_{n_j}\right\rangle - \left\langle z_t - z_{n_j}, \frac{z_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle + F_1\left(z_t, z_{n_j}\right).
\end{aligned} \tag{3.37}$$

Since $\|x_{n_j} - z_{n_j}\| \to 0$, $j \to \infty$, we obtain $\|A x_{n_j} - A z_{n_j}\| \to 0$, $j \to \infty$. Furthermore, by the monotonicity of A, we obtain $\langle z_t - z_{n_j}, A z_t - A z_{n_j}\rangle \geq 0$. Then, by (A4), we obtain (noting that $z_{n_j} \to z$)

$$\langle z_t - z, A z_t\rangle \geq \varphi_1(z) - \varphi_1(z_t) + F_1(z_t, z), \quad j \longrightarrow \infty. \tag{3.38}$$

Using (A1), (A4), and (3.38), we also obtain

$$\begin{aligned}
0 &= F_1(z_t, z_t) + \varphi_1(z_t) - \varphi_1(z_t) \\
&\leq t F_1(z_t, y) + (1-t) F_1(z_t, z) + t\varphi_1(y) + (1-t)\varphi_1(z) - \varphi_1(z_t) \\
&\leq t\left[F_1(z_t, y) + \varphi_1(y) - \varphi_1(z_t)\right] + (1-t)\langle z_t - z, A z_t\rangle \\
&= t\left[F_1(z_t, y) + \varphi_1(y) - \varphi_1(z_t)\right] + (1-t)t\langle y - z, A z_t\rangle,
\end{aligned} \tag{3.39}$$

and, hence,

$$0 \leq F_1(z_t, y) + \varphi_1(y) - \varphi_1(z_t) + (1-t)\langle y - z, A z_t\rangle. \tag{3.40}$$

Letting $t \to 0$, we have, for each $y \in K$, that

$$0 \leq F_1(z,y) + \varphi_1(y) - \varphi_1(z) + \langle y - z, Az \rangle. \tag{3.41}$$

This implies that $z \in \mathrm{GMEP}(F_1, A, \varphi_1)$. By following the same arguments, we can show that $z \in \mathrm{GMEP}(F_2, B, \varphi_2)$.

Following the arguments of [3, Theorem 3.1, pages 346-347], we can show that $z \in \mathrm{VI}(K, D)$. Therefore, $z \in \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, A, \varphi_1) \cap \mathrm{GMEP}(F_2, B, \varphi_2) \cap \mathrm{VI}(K, D)$.

Noting that $x_n = P_{C_n} x_0$, we have by (2.3),

$$\langle x_0 - x_n, y - x_n \rangle \leq 0, \tag{3.42}$$

for all $y \in C_n$. Since $F \subset C_n$ and by the continuity of inner product, we obtain, from the above inequality,

$$\langle x_0 - z, y - z \rangle \leq 0, \tag{3.43}$$

for all $y \in F$. By (2.3) again, we conclude that $z = P_F x_0$. This completes the proof. □

Corollary 3.2. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be generated by*

$$\begin{gathered} x_0 \in K, \\ y_n = \alpha_n x_0 + (1 - \alpha_n) T x_n, \\ C_{n+1} = \left\{ z \in C_n : \left\| y_n - z \right\|^2 \leq \| x_n - z \|^2 + \alpha_n \left(\| x_0 \|^2 + 2 \langle x_n - x_0, z \rangle \right) \right\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1. \end{gathered} \tag{3.44}$$

Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ such that $\lim_{n \to \infty} \alpha_n = 0$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{F(T)} x_0$.

Remark 3.3. Corollary 3.2 can be viewed as an improvement of Theorem 3.1 of Martinez-Yanes and Xu [39] because we relax the iterative step Q_n in the algorithm of Theorem 3.1 of [39].

4. Applications

Let C be a nonempty closed and convex cone in H and D an operator of C into H. We define the *polar* of C in H to be the set

$$K^* := \{ y^* \in H : \langle x, y^* \rangle \geq 0, \ \forall x \in C \}. \tag{4.1}$$

Then, the element $u \in C$ is called a solution of the *complementarity problem* if

$$Du \in K^*, \langle u, Du \rangle = 0. \tag{4.2}$$

The set of solutions of the complementarity problem is denoted by $C(C, D)$. We will assume that D satisfies the following conditions:

(E1) D is γ-inverse strongly monotone,

(E2) $C(C, D) \neq \emptyset$.

Also, we replace conditions (B1) and (B2) with

(D1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0, \tag{4.3}$$

(D2) C is a bounded set.

Theorem 4.1. *Let C be a nonempty closed and convex subset of a real Hilbert space H. For each $m = 1, 2$, let F_m be a bifunction from $C \times C$ satisfying (A1)–(A4), $\varphi_m : C \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function with assumption (B1) or (B2), A an α-inverse-strongly monotone mapping of C into H, and B a β-inverse-strongly monotone mapping of C into H, and, for each $i = 1, 2, \ldots$, let $T_i : C \to C$ be a nonexpansive mapping such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let D be a γ-inverse-strongly monotone mapping of K into H. Suppose that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, A, \varphi_1) \cap \mathrm{GMEP}(F_2, B, \varphi_2) \cap C(C, D) \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1, 2, \ldots)$, and $\{x_n\}_{n=0}^{\infty}$ be generated by $x_0 \in C$, $C_{1,i} = C$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$,*

$$\begin{gathered}
z_n = T_{r_n}^{(F_1, \varphi_1)}(x_n - r_n A x_n), \\
u_n = T_{\lambda_n}^{(F_2, \varphi_2)}(z_n - \lambda_n B z_n), \\
w_n = P_C(u_n - s_n D u_n), \\
y_{n,i} = \alpha_{n,i} x_0 + (1 - \alpha_{n,i}) T_i w_n, \\
C_{n+1,i} = \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \leq \|x_n - z\|^2 + \alpha_{n,i}\left(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle \right) \right\}, \\
C_{n+1} = \bigcap_{i=1}^{\infty} C_{n+1,i}, \\
x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1.
\end{gathered} \tag{4.4}$$

Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0, 1)$ $(i = 1, 2, \ldots)$, $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$, and $\{\lambda_n\}_{n=1}^{\infty} \subset [0, 2\beta]$ satisfy

(i) $0 < a \leq r_n \leq b < 2\alpha$,

(ii) $0 < c \leq \lambda_n \leq f < 2\beta$,

(iii) $\lim_{n \to \infty} \alpha_{n,i} = 0$,

(iv) $0 < h \leq s_n \leq j < 2\gamma$.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_F x_0$.

Proof. Using Lemma 7.1.1 of [52], we have that $\mathrm{VI}(C,D) = C(C,D)$. Hence, by Theorem 3.1 we obtain the desired conclusion. □

Next we study the problem of finding a minimizer of a continuously Fréchet differentiable convex functional in a Hilbert space.

Theorem 4.2. *For each $m = 1,2$, let F_m be a bifunction from $H \times H$ satisfying (A1)–(A4), $\varphi_m : H \to \mathbb{R} \cup \{+\infty\}$ a proper lower semicontinuous and convex function with assumption (B1) or (B2), A an α-inverse-strongly monotone mapping of H into itself, and B a β-inverse-strongly monotone mapping of H into itself, and, for each $i = 1,2,\ldots$, let $T_i : H \to H$ be a nonexpansive mapping such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that f is a functional on H which satisfies the following conditions:*

(1) *f is a continuously Fréchet differentiable convex functional on H and ∇f is $(1/\gamma)$-Lipschitz continuous,*

(2) *$(\nabla f)^{-1}0 = \{z \in H : f(z) = \min_{y\in H} f(y)\} \neq \emptyset$.*

Suppose that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{GMEP}(F_1, A, \varphi_1) \cap \mathrm{GMEP}(F_2, B, \varphi_2) \cap (\nabla f)^{-1}0 \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$, $\{y_{n,i}\}_{n=1}^{\infty}$ $(i = 1,2,\ldots)$, and $\{x_n\}_{n=0}^{\infty}$ be generated by $x_0 \in K$, $C_{1,i} = K$, $C_1 = \bigcap_{i=1}^{\infty} C_{1,i}$, $x_1 = P_{C_1} x_0$,

$$
\begin{aligned}
z_n &= T_{r_n}^{(F_1,\varphi_1)}(x_n - r_n A x_n),\\
u_n &= T_{\lambda_n}^{(F_2,\varphi_2)}(z_n - \lambda_n B z_n),\\
w_n &= u_n - s_n \nabla f(u_n),\\
y_{n,i} &= \alpha_{n,i} w_n + (1-\alpha_{n,i}) T_i w_n,\\
C_{n+1,i} &= \left\{ z \in C_{n,i} : \left\| y_{n,i} - z \right\|^2 \le \|x_n - z\|^2 + \alpha_{n,i}\Big(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle \Big) \right\},\\
C_{n+1} &= \bigcap_{i=1}^{\infty} C_{n+1,i},\\
x_{n+1} &= P_{C_{n+1}} x_0, \quad n \ge 1.
\end{aligned}
\tag{4.5}
$$

Assume that $\{\alpha_{n,i}\}_{n=1}^{\infty} \subset (0,1)$ $(i = 1,2,\ldots)$, $\{r_n\}_{n=1}^{\infty} \subset [0,2\alpha]$, and $\{\lambda_n\}_{n=1}^{\infty} \subset [0,2\beta]$ satisfy

(i) *$0 < a \le r_n \le b < 2\alpha$,*

(ii) *$0 < c \le \lambda_n \le f < 2\beta$,*

(iii) *$\lim_{n\to\infty} \alpha_{n,i} = 0$,*

(iv) *$0 < h \le s_n \le j < 2\gamma$.*

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_F x_0$.

Proof. We know from condition (i) and Lemma 2.2 that ∇f is an γ-inverse-strongly monotone operator from H into H. Using Theorem 3.1, we have the desired conclusion. □

We now study a kind of multiobjective optimization problem with nonempty set of solutions:

$$\min\, h_1(x), \quad \min\, h_2(x), \quad x \in K, \tag{4.6}$$

where K is a nonempty closed convex subset of a real Hilbert space H, and $h_i : K \to \mathbb{R}$, $i = 1,2$, is a convex and a lower semicontinuous functional. Let us denote the set of solutions to (4.6) by Ω and assume that $\Omega \neq \emptyset$.

We will denote the set of solutions of the following two optimization problems by Ω_1 and Ω_2, respectively:

$$\min_{x\in K} h_1(x), \quad \min_{x\in K} h_2(x). \tag{4.7}$$

Clearly, if we find a solution $x \in \Omega_1 \cap \Omega_2$, then one must have $x \in \Omega$.

Now, for each $i = 1,2$, let $F_i : K \times K \to \mathbb{R}$ be defined by $F_i(x,y) := h_i(y) - h_i(x)$. Let us now find the following equilibrium problem: find $x \in K$ such that

$$F_i(x,y) \geq 0, \quad i = 1,2, \tag{4.8}$$

for all $y \in K$. It is obvious that F_i satisfies conditions (A1)–(A4) and $\mathrm{EP}(F_i) = \Omega_i$, $i = 1,2$, where $\mathrm{EP}(F_i)$ is the set of solutions to (4.8). By Theorem 3.1, we have the following theorem.

Theorem 4.3. *Let K be a nonempty closed and convex subset of a real Hilbert space H. For each $i = 1,2$, let h_i be a lower semicontinuous and convex function such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Let $\{z_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{x_n\}_{n=0}^{\infty}$ be generated by $x_0 \in K$, $C_1 = K$, $x_1 = P_{C_1}x_0$,*

$$\begin{gathered} h_1(y) - h_1(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in K, \\ h(y) - h(u_n) + \frac{1}{\lambda_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \quad \forall y \in K, \\ y_n = \alpha_n x_0 + (1-\alpha_n)u_n, \\ C_{n+1} = \left\{z \in C_n : \left\|y_n - z\right\|^2 \leq \|x_n - z\|^2 + \alpha_n\left(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle\right)\right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \geq 1. \end{gathered} \tag{4.9}$$

Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, $\{r_n\}_{n=1}^{\infty} \subset (0,\infty)$, and $\{\lambda_n\}_{n=1}^{\infty} \subset (0,\infty)$ satisfy

(i) $\liminf_{n\to\infty} r_n > 0$,

(ii) $\liminf_{n\to\infty} \lambda_n > 0$,

(iii) $\lim_{n\to\infty} \alpha_n = 0$.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{\Omega_1\cap\Omega_2}x_0$.

Remark 4.4. Our results in this paper also hold for infinite family of uniformly continuous quasi-nonexpansive mappings in a real Hilbert space.

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