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*Research Article*

# Idealization of Some Topological Concepts

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An approach is followed here to generate a new topology on a set X from an ideal $\mathcal{I}$ and a family $\mathcal{S}$ of subsets of X. The so-obtained topology is related to other known topologies on X. The cases treated here include the one when $\mathcal{S} = T^{\alpha}$ is taken, then the case when $\mathcal{S} = RO(X,T)$ is considered. The approach is open to apply to other choices of $\mathcal{S}$. As application, some known results are obtained as corollaries to those results appearing here. In the last part of this work, some ideal-continuity concepts are studied, which originate from some previously known terms and results.

## 1. Introduction

The interest in the idealized version of many general topological properties has grown drastically in the past 20 years. In this work, no particular paper will be referred to except where it is needed and encountered. However, many symbols, definitions, and concepts used here are as in [1].

## 2. Open Sets via a Family $\mathcal{S}$ and an Ideal $\mathcal{I}$

Recall that an ideal on a set X is a family $\mathcal{I}$ of subsets of X, that is, $\mathcal{I} \subseteq \wp(X)$ (the power set of X), such that $\mathcal{I}$ is closed under finite union, and if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$ (heredity property). An ideal topological space is a triple $(X, T, \mathcal{I})$, where X is a set, T is a topology on X, and $\mathcal{I}$ is an ideal on X. Let $(X, T, \mathcal{I})$ be an ideal topological space. The family $\mathcal{B} = \{U - I : U \in T \text{ and } I \in \mathcal{I}\}$forms a base for a topology $T^*(\mathcal{I})$ on X finer than T [1].

As a start, this concept will be put in a more general setting as follows.

*Definition 2.1.* Let $(X, T, \mathcal{I})$ be an ideal topological space, and let $\mathcal{S}$ be a family of subsets of X.

(a) A set $A \subseteq X$ is called an $\mathcal{I}\mathcal{S}$-open set if for each $x \in A$, there exist $U \in \mathcal{S}$ and $I \in \mathcal{I}$ such that $x \in U$ and $U - I \subseteq A$, or equivalently $U - A \in \mathcal{I}$. The family of all $\mathcal{I}\mathcal{S}$-open subsets is written $\mathcal{I}\mathcal{S}O(X, T)$.

(b) The topology on $X$ generated by the subbase $\mathcal{I}\mathcal{S}O(X, T)$ is denoted by $T\mathcal{I}\mathcal{S}O$.

*Remark 2.2.* (1) The family $\mathcal{I}\mathcal{S}O(X, T)$ needs not form, in general, a topology on $X$.

(2) In the case where $\mathcal{S} \subseteq T$, it is clear that $T\mathcal{I}\mathcal{S}O \subseteq T^*(\mathcal{I})$, and $T\mathcal{I}\mathcal{S}O = T^*(\mathcal{I})$ for the case $\mathcal{S} = T$.

**Proposition 2.3.** *Let $(X, T, \mathcal{I})$ be an ideal topological space and let $\mathcal{S} \subseteq \mathcal{P}(X)$. Let $\mathcal{S}(\mathcal{I}) = \{S - I : S \in \mathcal{S}$ and $I \in \mathcal{I}\}$. Let $T(\mathcal{S}(\mathcal{I}))$ be the topology generated by the subbase $\mathcal{S}(\mathcal{I})$, then $T(\mathcal{S}(\mathcal{I})) = T\mathcal{I}\mathcal{S}O$.*

*Proof.* Note that $\mathcal{S}(\mathcal{I}) \subseteq T\mathcal{I}\mathcal{S}O$, and therefore $T(\mathcal{S}(\mathcal{I})) \subseteq T\mathcal{I}\mathcal{S}O$. Now let $A \in T\mathcal{I}\mathcal{S}O$, then for each $x \in A$, there exists $S_x \in \mathcal{S}$ with $x \in S_x$ and $I_x \in \mathcal{I}$ such that $S_x - I_x \subseteq A$. This means that $A = \cup\{S_x - I_x : x \in A\}$ where $S_x - I_x \in \mathcal{S}(\mathcal{I})$ for each $x \in A$. Thus, $A \in T(\mathcal{S}(\mathcal{I}))$ since $\mathcal{S}(\mathcal{I}) \subseteq T(\mathcal{S}(\mathcal{I}))$. □

**Proposition 2.4.** *Let $(X, T, \mathcal{I})$ be an ideal topological space. If $T(\mathcal{S})$ denotes the topology on $X$ generated by the subbase $\mathcal{S}$, then $T\mathcal{I}\mathcal{S}O = T(\mathcal{S}(\mathcal{I})) = (T(\mathcal{S}))^*(\mathcal{I})$.*

*Proof.* To show that $T(\mathcal{S}(\mathcal{I})) = (T(\mathcal{S}))^*(\mathcal{I})$, first note that $\mathcal{S} \subseteq T(\mathcal{S})$ and therefore $\mathcal{S}(\mathcal{I}) = \{S - I : S \in \mathcal{S}$ and $I \in \mathcal{I}\} \subseteq (T(\mathcal{S}))^*(\mathcal{I})$. This implies that $T(\mathcal{S}(\mathcal{I})) \subseteq (T(\mathcal{S}))^*(\mathcal{I})$. Next, consider the base $\mathcal{B} = \{U - I : U \in T(\mathcal{S})$ and $I \in \mathcal{I}\}$ for $(T(\mathcal{S}))^*(\mathcal{I})$. Let $B \in \mathcal{B}$, say $B = U - I$ for some $U \in T(\mathcal{S})$ and some $I \in \mathcal{I}$. If $x \in B$, then $x \in U$, and so there exist $S_1, \ldots, S_n \in \mathcal{S}$ such that $x \in (\bigcap_{k=1}^{n} S_k) - I \subseteq U - I$, where $(\bigcap_{k=1}^{n} S_k) - I = \bigcap_{k=1}^{n}(S_k - I) \in T(\mathcal{S}(\mathcal{I}))$, that is, $B = U - I \in T(\mathcal{S}(\mathcal{I}))$, and hence $\mathcal{B} \subseteq T(\mathcal{S}(\mathcal{I}))$. This shows that $(T(\mathcal{S}))^*(\mathcal{I}) \subseteq T(\mathcal{S}(\mathcal{I}))$.

If $(X, T)$ is a topological space, we let $\operatorname{int}(A)$ (resp., $\operatorname{cl}(A)$) denote the interior of $A$ (resp., the closure of $A$) in $(X, T)$. A subset $A$ of $(X, T)$ is called semiopen if $A \subseteq \operatorname{cl}(\operatorname{int}(A)$, and $A$ is called an $\alpha$-set if $A \subseteq \operatorname{int}(\operatorname{cl}(\operatorname{int}(A))$. The family of all $\alpha$-sets forms a topology $T^\alpha$ on $X$ finer than $T$. □

*Example 2.5.* Let $(X, T, \mathcal{I}_n)$ be an ideal topological space, where $\mathcal{I}_n$ is the ideal of all nowhere dense subsets of $(X, T)$. Let $\mathcal{S} = SO(X, T)$, the family of all semiopen subsets of $(X, T, \mathcal{I}_n)$. In this case, it is noted that $T(\mathcal{S})$ is the topology studied in [2] and is called the topology of semiopen sets and denoted by $TSO$. So $T(\mathcal{S}(\mathcal{I}_n)) = (T(\mathcal{S}))^*(\mathcal{I}_n) = (T(\mathcal{S}))^\alpha$, the topology of all $\alpha$-sets in the space $(X, T(\mathcal{S}))$.

Next, the case where $(X, T, \mathcal{I})$ is given and the family $\mathcal{S} = T^\alpha$ is considered. Recall that $T^\alpha$ is a topology on $X$ finer than $T$. It is known that $T^\alpha = T^*(\mathcal{I}_n)$. It is then clear that the family $\{U - I : U \in T$ and $I \in \mathcal{I}_n\}$ is a base for $T^\alpha$. In fact, $T^\alpha = \{U - I : U \in T$ and $I \in \mathcal{I}_n\}$, see [3].

*Definition 2.6.* Let $(X, T, \mathcal{I})$ be an ideal topological space, $A \subseteq X$, and $\mathcal{S} = T^\alpha$. $A$ is called an $\mathcal{I}$-$\alpha$-open subset ($\mathcal{I}\alpha O$-set, for short) if for each $x \in A$, there exist $U_x \in T^\alpha$ and $I_x \in \mathcal{I}$ such that $x \in U_x - I_x \subseteq A$, or equivalently $U_x - A \in \mathcal{I}$. The family of all $\mathcal{I}\alpha O$-sets of $(X, T, \mathcal{I})$ is denoted by $\mathcal{I}\alpha O(X, T)$.

The following is a consequence of [1, Theorem 3.1].

**Proposition 2.7.** *Let $(X, T, \mathcal{I})$ be an ideal topological space, then the family $\mathcal{B} = \{U - I : U \in T^\alpha$ and $I \in \mathcal{I}\}$ is a base for the topology $(T^\alpha)^*(\mathcal{I})$ on $X$.*

**Proposition 2.8.** *If $(X, T, \mathcal{I})$ is an ideal topological space, then $\mathcal{I}\alpha O(X, T) = (T^{\alpha})^*(\mathcal{I})$. So $T \subseteq T^{\alpha} \subseteq (T^{\alpha})^*(\mathcal{I}) = \mathcal{I}\alpha O(X, T)$.*

*Proof.* It is enough to note that the family $\mathcal{B} = \{U - I : U \in T^{\alpha} \text{ and } I \in \mathcal{I}\}$ is a base for $(T^{\alpha})^*(\mathcal{I})$ as well as for $\mathcal{I}\alpha O(X, T)$.

Let $(X, T, \mathcal{I})$ be an ideal topological space. When dealing with the topology $\mathcal{I}\alpha O(X, T) = (T^{\alpha})^*(\mathcal{I})$, it is found to have a base consisting of elements of the form $B = V - I$ with $V \in T^{\alpha}$ and $I \in \mathcal{I}$. But as it is well known, the set $V$ takes the form $U - I_1$ for some $U \in T$ and $I_1 \in \mathcal{I}_n$. So $B = V - I = (U - I_1) - I = U - (I_1 \cup I)$ where $U \in T$. It is now clear that in a more general setting, one needs to deal with a situation where two ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are considered on (X, T). At this point, let $\mathcal{I}_1 \vee \mathcal{I}_2 = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1 \text{ and } I_2 \in \mathcal{I}_2\}$, where it is easy to see that $\mathcal{I}_1 \vee \mathcal{I}_2$ is itself an ideal on $(X, T)$. □

**Proposition 2.9.** *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two ideals on a space $(X, T)$, then $T^*(\mathcal{I}_1 \vee \mathcal{I}_2) = T^*(\mathcal{I}_1) \vee T^*(\mathcal{I}_2)$ (see [1, Corollary 3.5]) (and recall that $T \vee T^{/}$ is the supremum of the two topologies $T$ and $T^{/}$ which is the topology generated by the subbase $T \cup T^{/}$).*

*Proof.* Since $\mathcal{I}_1 \subseteq \mathcal{I}_1 \vee \mathcal{I}_2$ and $\mathcal{I}_2 \subseteq \mathcal{I}_1 \vee \mathcal{I}_2$, then (by Theorem 2.3(b) of [1]) it follows that $T^*(\mathcal{I}_1) \subseteq T^*(\mathcal{I}_1 \vee \mathcal{I}_2)$ and $T^*(\mathcal{I}_2) \subseteq T^*(\mathcal{I}_1 \vee \mathcal{I}_2)$. Therefore, $T^*(\mathcal{I}_1) \vee T^*(\mathcal{I}_2) \subseteq T^*(\mathcal{I}_1 \vee \mathcal{I}_2)$. Next, if $\mathcal{B}$ is a base for $T^*(\mathcal{I}_1 \vee \mathcal{I}_2)$ and $B \in \mathcal{B}$, then $B = U - (I_1 \cup I_2)$ for some $U \in T$, $I_1 \in \mathcal{I}_1$, and $I_2 \in \mathcal{I}_2$. So $B = U - (I_1 \cup I_2) = U \cap (X - I_1 \cup I_2) = (U \cap (X - I_1) \cap (U \cap (X - I_2)) = B_1 \cap B_2$. Where $B_1$ is a basic open set in $T^*(\mathcal{I}_1)$ and $B_2$ is a basic open set in $T^*(\mathcal{I}_2)$. Thus $B_1 \cap B_1 \in T^*(\mathcal{I}_1) \vee T^*(\mathcal{I}_2)$ and so $\mathcal{B} \subseteq T^*(\mathcal{I}_1) \vee T^*(\mathcal{I}_2)$ which means that $T^*(\mathcal{I}_1 \vee \mathcal{I}_2) \subseteq T^*(\mathcal{I}_1) \vee T^*(\mathcal{I}_2)$. □

**Corollary 2.10.** *(see [1, Corollary 3.4]) Let $(X, T, \mathcal{I})$ be an ideal topological space, then $(T^*(\mathcal{I}))^*(\mathcal{I}) = T^*(\mathcal{I})$. In particular, $(T^{\alpha})^{\alpha} = T^{\alpha}$.*

## 3. $\mathcal{I}RO$ Sets

Let $(X, T)$ be a topological space. A subset $A \subseteq X$ is called regular open if $A = \mathrm{int}(\mathrm{cl}(A))$. The family of all regular open subsets of $(X, T)$ is denoted by $RO(X, T)$. It is a known fact that $RO(X, T)$ is a base for a topology $T_s$ on $X$, finer than $T$, and is called the semiregularization of $(X, T)$.

In pursuing the approach used in the first section, it is now the time to consider the case where $\mathcal{S} = RO(X, T)$.

*Definition 3.1.* Let $(X, T, \mathcal{I})$ be an ideal topological space, and let $A \subseteq X$. The set A is called an ideal regular-open set, or $\mathcal{I}RO$ set for short, if for each $x \in A$, there exist a regular open set $R_x \in RO(X, T)$ and $I_x \in \mathcal{I}$ such that $x \in R_x - I_x \subseteq A$, or equivalently $R_x - A \in \mathcal{I}$. The family of all $\mathcal{I}RO$ sets of $(X, T, \mathcal{I})$ is denoted by $\mathcal{I}RO(X, T)$.

**Proposition 3.2.** *Let $(X, T, \mathcal{I})$ be an ideal topological space, then the family $\mathcal{I}RO(X, T)$ is a base for a topology $M$ on X.*

*Proof.* It is clear that $X \in \mathcal{I}RO(X, T)$. So it is enough to show that if $B_1$, $B_2 \in \mathcal{I}RO(X, T)$ then $B_1 \cap B_2 \in \mathcal{I}RO(X, T)$. To this end, let $x \in B_1 \cap B_1$, then there exist $R_1$, $R_2 \in RO(X, T)$ such that $x \in R_1 \cap R_2$, $I_1 = R_1 - B_1 \in \mathcal{I}$, and $I_2 = R_2 - B_2 \in \mathcal{I}$. Now, $R_1 \cap R_2 \in RO(X, T)$ and $R_1 \cap R_2 - (B_1 \cap B_2) = (R_1 \cap R_2) \cap (X - B_1 \cap B_2) = (R_1 \cap R_2) \cap ((X - B_1) \cup (X - B_2)) = ((R_1 \cap R_2) \cap (X - B_1)) \cup ((R_1 \cap R_2) \cap (X - B_2)) \subseteq I_1 \cap I_2 \in \mathcal{I}$. So $B_1 \cap B_2 \in \mathcal{I}RO(X, T)$ as claimed. □

**Proposition 3.3.** *Let $(X, T, \mathcal{I})$ be an ideal topological space. If $T_s$ denotes the topology on $X$ generated by the base $RO(X, T)$, then $M = (T_s)^*(\mathcal{I})$ (where $M$ is the topology constructed in Proposition 3.2).*

*Proof.* We need to appeal to Proposition 2.4, with $\mathcal{S} = RO(X, T)$ and so $T(\mathcal{S}) = T_s$, while $T\mathcal{I}\mathcal{S}O = T\mathcal{I}\mathcal{R}O = M = (T_s)^*(\mathcal{I})$.

In general, for an ideal topological space $(X, T, \mathcal{I})$, the two topologies $T$ and $(T_s)^*$ need not be comparable. □

*Example 3.4.* (a) Let $(X, T, \mathcal{I})$ be an ideal topological space where $X = \mathbb{R}$ (the set of real numbers), $T$ is the left ray topology on $\mathbb{R}$, and $\mathcal{I}$ is the ideal $\mathcal{I}_f$ of all finite subsets of X. It is easy to see that $T_s = \{\emptyset, \mathbb{R}\}$. Then $(T_s)^*(\mathcal{I})$ is the cofinite topology on $X$ [1, Example 2.5], and clearly $T$ and $(T_s)^*$ are incomparable.

(b) Consider the space $(X, T, \mathcal{I})$ where $X = \mathbb{R}$, $T$ is the cofinite topology on $\mathbb{R}$, and $\mathcal{I}$ is the ideal $\mathcal{I}_c$ of all countable subsets of X. Again $T_s = \{\emptyset, \mathbb{R}\}$ while $(T_s)^*(\mathcal{I})$ is the cocountable topology on X. Here, $T \subsetneq (T_s)^*(\mathcal{I})$.

*Definition 3.5* (see [4]). A subset $A$ of a space $(X, T)$ is called $\omega$-regular open if for each $x \in A$, there exists a regular open set $R_x \in RO(X, T)$ such that $x \in R_x$ and $R_x - A$ is countable. The family of all $\omega$-regular open subsets of $(X, T)$ is denoted by $\omega RO(X, T)$.

The next result is an immediate consequence of definitions.

**Proposition 3.6.** *Let $(X, T)$ be a topological space, then a subset $A \subseteq X$ is an $\omega$-regular open subset of $(X, T)$ if and only if $A$ is ideal regular open with $\mathcal{I} = \mathcal{I}_c$. Thus, $\omega RO(X, T) = (T_s)^*(\mathcal{I}_c)$.*

**Corollary 3.7** (see [4, Theorem 2.1]). *Let $(X, T)$ be a topological space, then $(X, \omega RO(X, T))$ is a topological space.*

*A topological space $(X, T)$ is called nearly Lindelöf (see [5]) if every cover of $(X, T)$ by regular open subsets has a countable subcover.*

*It is clear that if $(X, T)$ is nearly Lindelöf, then $(X, T_s)$ is Lindelöf. In fact, the family $RO(X, T)$ forms a base for $(X, T_s)$, and in this case, every basic open cover of $(X, T_s)$ will have a countable subcover. On the other hand, by the well-known fact [6, Lemma 1.1] that the two spaces $(X, T)$ and $(X, T_s)$ have the same regular open sets, it follows that $(X, T)$ is nearly Lindelöf if and only if $(X, T_s)$ is nearly Lindelöf.*

**Corollary 3.8** (see [5, Proposition 1.3]). *A space $(X, T)$ is nearly Lindelöf if and only if $(X, T_s)$ is Lindelöf.*

*Recall that a subset $A$ of a space $(X, T)$ is called $\omega$-open (see [7]) if for each $x \in A$, there exists $U_x \in T$ such that $x \in U_x$ and $U_x - A$ is countable, that is, $U_x - I_x \subseteq A$ for some $I_x \in \mathcal{I}_c$. The family of all $\omega$-open subsets of a space $(X, T)$ is denoted by $T_\omega$.*

*The definitions imply directly the following result.*

**Proposition 3.9.** *If $(X, T)$ is a topological space, then $T_\omega = T^*(\mathcal{I}_c)$.*

**Corollary 3.10** (see [7, Proposition 2.5]). *Let $(X, T)$ be a topological space. The family $T_\omega$ is a topology on $X$ with $T \subseteq T_\omega$.*

**Proposition 3.11** (see [7, Proposition 4.5]). *A space $(X, T)$ is Lindelöf if and only if the space $(X, T_\omega) = (X, T^*(\mathcal{I}_c))$ is Lindelöf.*

The following result can now be stated.

**Proposition 3.12.** *The following statements are equivalent for a space* $(X, T)$:

(a) $(X, T)$ *is nearly Lindelöf,*

(b) $(X, T_s)$ *is Lindelöf,*

(c) $(X, (T_s)^*(\mathcal{I}_c))$ *is Lindelöf.*

*Proof.* (a)⇔(b) Follow by Corollary 3.8.

(b)⇔(c) By applying Proposition 3.11 to the space $(X, T_s)$, it follows that $(X, T_s)$ is Lindelöf if and only if $(X, (T_s)_\omega) = (X, (T_s)^*(\mathcal{I}_c))$ is Lindelöf. □

**Corollary 3.13** (see [4, Theorem 3.1]). *The following statements are equivalent for any space* $(X, T)$:

(a) $(X, T)$ *is nearly Lindelöf,*

(b) *Every* $\omega$*-regular open cover of* $(X, T)$ *admits a countable subcover.*

*Proof.* Now, $(X, T)$ is nearly Lindelöf if and only if $(X, (T_s)^*(\mathcal{I}_c))$ is Lindelöf (Proposition 3.12). On the other hand, $(X, (T_s)^*(\mathcal{I}_c)) = \omega RO(X, T)$ (Proposition 3.6), and therefore $(X, \mathrm{T})$ is nearly Lindelöf if and only if $(X, \omega RO(X, T))$ is Lindelöf if and only if every $\omega$-regular open cover of $(X, T)$ has a countable subcover. □

**Corollary 3.14** (see [4, Proposition 3.1]). *A space* $(X, T)$ *is nearly Lindelöf if and only if for every family of* $\omega$*-regular closed subsets* $\{F_\alpha : \alpha \in \Delta\}$ *that satisfies the countable intersection property has a nonempty intersection.*

*Proof.* Again $(X, T)$ is nearly Lindelöf if and only if $(X, \omega RO(X, T))$ is Lindelöf (note that $\omega RO(X, T) = (T_s)^*(\mathcal{I}_c)$). Now, the result follows by a well-known fact concerning Lindelöf spaces and the fact that a subset $F$ is $\omega$-regular closed if it is the complement of an $\omega$-regular open set.

Let $(X, T, \mathcal{I})$ be an ideal topological space. The ideal $\mathcal{I}$ is called completely codense [8] if $\mathcal{I} \cap PO(X, T) = \{\emptyset\}$, where $PO(X, T)$ is the family of all preopen subsets of $(X, T)$ and $A \subseteq X$ is called preopen if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$. □

**Proposition 3.15.** *Let* $(X, T, \mathcal{I})$ *be an ideal topological space, and assume that* $\mathcal{I}$ *is completely codense, then* $(X, T, \mathcal{I})$ *is nearly Lindelöf if and only if* $(X, T^*(\mathcal{I}))$ *is nearly Lindelöf.*

*Proof.* By a remark on [9, page 3], it follows that $RO(X, T) = RO(X, T^*(\mathcal{I}))$. This implies that $T_s = (T^*(\mathcal{I}))_s$. Then $(X, T)$ is nearly Lindelöf, if and only if $(X, T_s)$ is Lindelöf if and only if $(X, (T^*(\mathcal{I}))_s)$ is Lindelöf if and only if $(X, T^*(\mathcal{I}))$ is nearly Lindelöf.

Let $(X, T, \mathcal{I})$ be an ideal topological space. The topology $T$ is compatible with the ideal $\mathcal{I}$, written $T \sim \mathcal{I}$, [1], if whenever a subset $A \subseteq X$ satisfies for each $x \in A$, there exists $U_x \in T$ with $x \in U_x$ and $U_x \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. □

**Proposition 3.16.** *Let* $(X, T, \mathcal{I})$ *be an ideal topological space, then* $T \sim \mathcal{I}$ *if and only if* $T^*(\mathcal{I}) \sim \mathcal{I}$.

*Proof.* Let $T^*(\mathcal{I}) \sim \mathcal{I}$. Let $A \subseteq X$ be satisfying for each $x \in A$, there exists $U_x \in T$ with $x \in U_x$ and $U_x \cap A \in \mathcal{I}$. Our assumption and the fact that $T \subseteq T^*(\mathcal{I})$ imply $A \in \mathcal{I}$, and so $T \sim \mathcal{I}$. Conversely, assume that $T \sim \mathcal{I}$, then $T^*(\mathcal{I}) = \mathcal{B} = \{U - I : U \in T \text{ and } I \in \mathcal{I}\}$ [1, Theorem 4.4]. Let $A \subseteq X$ satisfy for each $x \in A$ there exists $B_x = U_x - I_x \in \mathcal{B}$ with $x \in B_x$ and

$B_x \cap A = (U_x - I_x) \cap A = I_1 \in \mathcal{I}$. Then $x \in U_x \in T$ with $U_x \cap A \subseteq I_x \cup I_1$ and so $U_x \cap A \in \mathcal{I}$. But $T \sim \mathcal{I}$, so $A \in \mathcal{I}$ which means that $T^*(\mathcal{I}) \sim \mathcal{I}$. □

**Proposition 3.17.** *A space $(X, T, \mathcal{I}_c)$ is hereditarily Lindelöf if and only if $(X, T^*(\mathcal{I}_c))$ is hereditarily Lindelöf.*

*Proof.* It is known that $(X, T)$ is hereditarily Lindelöf if and only if $T \sim \mathcal{I}_c$ [1, Theorem 4.10] if and only if $T^*(\mathcal{I}_c) \sim \mathcal{I}_c$ if and only if $(X, T^*(\mathcal{I}_c))$ is hereditarily Lindelöf. □

## 4. Continuity via Ideals

As a start, two known concepts of ideal continuity are stated.

*Definition 4.1* (see [10]). Let $(X, T, \mathcal{I})$ be an ideal topological space. Let $f : (X, T, \mathcal{I}) \to (Y, M)$ be a given function.

(a) The function $f$ is called $\mathcal{I}$-continuous if for every $V \in M$, there exist $U \in T$ and $I \in \mathcal{I}$ such that $f^{-1}(V) = U - I$.

(b) The function $f$ is called pointwise $\mathcal{I}$-continuous if for each $x \in X$ and for each $V \in M$ with $f(x) \in V$, there exists $U \in T$ such that $x \in U$ and $U - f^{-1}(V) \in \mathcal{I}$.

It is noted in [10] that every $\mathcal{I}$-continuous function is pointwise $\mathcal{I}$-continuous. An example [10, page 327] is provided to show that the converse is not true in general.

*Definition 4.2.* Let $f : (X, T, \mathcal{I}) \to (Y, M)$ be given. The function $f$ is called $\mathcal{I}^*$-continuous if $f : (X, T^*(\mathcal{I})) \to (Y, M)$ is continuous.

**Proposition 4.3.** *A function $f : (X, T, \mathcal{I}) \to (Y, M)$ is $\mathcal{I}^*$-continuous if and only if $f$ is pointwise $\mathcal{I}$-continuous.*

*Proof.* Let $f : (X, T, \mathcal{I}) \to (Y, M)$ be $\mathcal{I}^*$-continuous, that is, $f : (X, T^*(\mathcal{I})) \to (Y, M)$ is continuous. Let $x \in X$, $V \in M$, and $f(x) \in V$, then $x \in f^{-1}(V) \in T^*(\mathcal{I})$. So there exists a basic open set $U - I$, for some $U \in T$ and $I \in \mathcal{I}$, such that $x \in U - I \subseteq f^{-1}(V)$. Equivalently, $x \in U$ and $U$-$f^{-1}(V) \in \mathcal{I}$. Thus, $f$ is pointwise $\mathcal{I}$-continuous. Conversely, let $f$ be pointwise $\mathcal{I}$-continuous. If $V \in M$, then for each $x \in f^{-1}(V)$, there exists $U_x \in T$ such that $U_x - f^{-1}(V) \in \mathcal{I}$, or equivalently, $x \in U_x - I_x \subseteq f^{-1}(V)$ for some $I_x \in \mathcal{I}$. It follows that $f^{-1}(V) = \cup\{U_x - I_x : x \in f^{-1}(V)\}$ and so $f^{-1}(V) \in T^*(\mathcal{I})$. Thus, $f$ is $\mathcal{I}^*$-continuous. □

The statement of the lemma on [10, page 326] can be formulated as in the next result.

**Proposition 4.4.** *Let $f : (X, T, \mathcal{I}) \to (Y, M)$ be a given function. Assume that $\mathcal{I}$ is codense (this means that $T \cap \mathcal{I} = \{\emptyset\}$) and that $(Y, M)$ is regular. If $f$ is $\mathcal{I}^*$-continuous, then $f$ is continuous and hence $\mathcal{I}$-continuous.*

**Proposition 4.5.** *Let $f : (X, T, \mathcal{I}) \to (Y, M)$ be a given function. Assume that $T \sim \mathcal{I}$, then $f$ is $\mathcal{I}$-continuous if and only if $f$ is $\mathcal{I}^*$-continuous.*

*Proof.* Let $f$ be $\mathcal{I}$-continuous, then as stated at the beginning of this section, $f$ is $\mathcal{I}^*$-continuous. Conversely, let $f$ be $\mathcal{I}^*$-continuous, then by [1, Theorem 4.4], $T^*(\mathcal{I}) = \{U - I : U \in T \text{ and } I \in \mathcal{I}\}$. So if $V \in M$, then $f^{-1}(V) \in T^*(\mathcal{I})$ and so $f^{-1}(V) = U - I$ for some $U \in T$ and $I \in \mathcal{I}$. Thus, $f$ is $\mathcal{I}$-continuous. □

*Definition 4.6* (see [4]). A function $f : (X, T) \to (Y, M)$ is $\delta_\omega$-continuous if for each $x \in X$ and each regular open set $V \in RO(Y, M)$ with $f(x) \in V$, there exists an $\omega$-regular open set $U$ of $(X, T)$ such that $x \in U$ and $f(U) \subseteq V$.

Recall that $A$ is an $\omega$-regular open subset of $(X, T)$ if $A \in \omega RO(X, T) = (T_s)^*(\mathcal{I}_c)$. So the next result is now clear.

**Proposition 4.7.** *A function $f : (X, T) \to (Y, M)$ is $\delta_\omega$-continuous if and only if $f : (X, T_s, \mathcal{I}_c) \to (Y, M_s)$ is $\mathcal{I}^*$-continuous (i.e., $f : (X, (T_s)^*(\mathcal{I}_c)) \to (Y, M_s)$ is continuous).*

*Proof.* Let $f : (X, T) \to (Y, M)$ be $\delta_\omega$-continuous. If $V \in RO(Y, M)$, then by definition, $f^{-1}(V)$ is a union of $\omega$-regular open subsets of $(X, T)$, that is, $f^{-1}(V)$ is a union of elements of $(T_s)^*(\mathcal{I}_c)$, and therefore $f^{-1}(V) \in (T_s)^*(\mathcal{I}_c)$. But $RO(Y, M)$ is a base for $M_s$, and therefore $f : (X, (T_s)^*(\mathcal{I}_c)) \to (Y, M_s)$ is continuous. For the converse, let $f : (X, T_s, \mathcal{I}_c) \to (Y, M_s)$ be $\mathcal{I}^*$-continuous. Let $x \in X$, $V \in RO(Y, M)$, and $f(x) \in V$, then $x \in f^{-1}(V) \in (T_s)^*(\mathcal{I}_c)$ and therefore $U = f^{-1}(V)$ is an $\omega$-regular open set containing $x$ and $f(U) \subseteq V$. Thus, $f$ is $\delta_\omega$-continuous. □

**Corollary 4.8** (see [4, Theorem 4.1]). *Let $f : (X, T) \to (Y, M)$ be a $\delta_\omega$-continuous surjection. Assume that $(X, T)$ is nearly Lindelöf, then $(Y, M)$ is nearly Lindelöf.*

*Proof.* Assume that $f : (X, T) \to (Y, M)$ is a $\delta_\omega$-continuous surjection. This means, by Proposition 4.7, that $f : (X, T_s, \mathcal{I}_c) \to (Y, M_s)$ is an $\mathcal{I}^*$-continuous surjection which means that $f : (X, (T_s)^*(\mathcal{I}_c)) \to (Y, M_s)$ is a continuous surjection. Assume now that $(X, T)$ is nearly Lindelöf, then, by Proposition 3.12, $(X, (T_s)^*(\mathcal{I}_c))$ is Lindelöf. So $(Y, M_s)$ is Lindelöf, being the continuous image of a Lindelöf space. Finally, $(Y, M)$ is nearly Lindelöf by Corollary 3.8. □

*Definition 4.9* (see [4]). A function $f : (X, T) \to (Y, M)$ is called $\omega R$-continuous if $f^{-1}(V)$ is $\omega$-regular open in $(X, T)$ for each open set $V$ in $(Y, M)$.

The following result is an immediate consequence of the definitions involved.

**Proposition 4.10.** *A function $f : (X, T) \to (Y, M)$ is $\omega R$-continuous if and only if the function $f : (X, T_s, \mathcal{I}_c) \to (Y, M)$is $\mathcal{I}^*$-continuous (i.e., $f : (X, (T_s)^*(\mathcal{I}_c)) \to (Y, M)$ is continuous).*

**Corollary 4.11** (see [4, Theorem 4.2]). *Let $f : (X, T) \to (Y, M)$ be an $\omega R$-continuous surjection. If $(X, T)$ is nearly Lindelöf, then $(Y, M)$ is Lindelöf.*

*Proof.* Let $f : (X, T) \to (Y, M)$ be an $\omega R$-continuous surjection, then $f : (X, (T_s)^*(\mathcal{I}_c)) \to (Y, M)$ is continuous. If $(X, T)$ is assumed to be nearly Lindelöf, then $(X, (T_s)^*(\mathcal{I}_c))$ is Lindelöf, by Proposition 3.12. Therefore $(Y, M)$, being the continuous image of a Lindelöf space, is Lindelöf. □

*Definition 4.12* (see [4]). A function $f : (X, T) \to (Y, M)$ is called completely continuous if $f^{-1}(V)$ is a regular open set in $(X, T)$ for each open set $V$ in $(Y, M)$.

**Proposition 4.13.** *If a function $f : (X, T) \to (Y, M)$ is completely continuous, then the function $f : (X, T_s) \to (Y, M)$ is continuous.*

*Proof.* The easy proof is omitted. □

**Corollary 4.14.** *Let $f : (X, T) \to (Y, M)$ be a completely continuous surjection. If $(X, T)$ is nearly Lindelöf, then $(Y, M)$ is Lindelöf.*

*Proof.* The proof is a consequence of Propositions 3.12 and 4.13. □

The following example shows that the converse of Proposition 4.13 is not true in general.

*Example 4.15.* Let $X$ be a set and $A$ a proper nonempty subset of $X$. Consider the topology $T = \{U : U \subseteq A\} \cup \{X\}$ on $X$. Let $Y = \{0, 1\}$ with the topology $M = \{\emptyset, Y, \{0\}\}$. Let $f : (X, T) \to (Y, M)$ be the function defined by $f(x) = 1$ if $x \in X - A$ and $f(x) = 0$ if $x \in A$. Then $V = \{0\} \in M$ such that $f^{-1}(V) = A \in T_s - RO(X, T)$, and so $f$ is continuous but not completely continuous.

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