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Research Article

The Stability Cone for a Matrix Delay Difference Equation

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We construct a stability cone, which allows us to analyze the stability of the matrix delay difference equation $x_n = Ax_{n-1} + Bx_{n-k}$. We assume that A and B are $m \times m$ simultaneously triangularizable matrices. We construct m points in $\mathbb{R}^3$ which are functions of eigenvalues of matrices A, B such that the equation is asymptotically stable if and only if all the points lie inside the stability cone.

1. Introduction

Parameters of linear systems are subject to time changes. That is why in order to construct such systems it is desirable to know if they are not only stable but also able to estimate the distance of the system from the boundary of the stability region in the parameter space. Therefore, it makes sense to investigate the geometry of the subset of stable polynomials in the space of characteristic polynomials of linear systems (in the *canonical space* [1]). This idea has already been applied to the investigation of geometry of the subset of stable polynomials in a two-dimensional subspace of the canonical space [2, 3], the stability simplex for general difference equations [4], connections of the convexity of the coefficients sequence with stability of difference equations [5], and stability ovals for matrix difference equations of the form $x_n = x_{n-1} + Bx_{n-k}$ with the delay k [6].

Consider the matrix equation

$$x_n = Ax_{n-1} + Bx_{n-k}, \quad n = 0, 1, 2 \ldots, \tag{1.1}$$

where $k \in \mathbb{Z}_+$ is the delay. Equations of the form (1.1) have been used for investigations of a delayed discrete-time Hopfield neural network [7, 8]. The suitable representation of the solution of (1.1) with commutative matrices A, B and nonsingular A is given in [9]. But [9] does not solve the stability problems of (1.1). The stability of (1.1) was investigated in [7, 8, 10, 11] with special 2×2 matrices A, B. In [12], the stability of (1.1) was investigated without any restriction on the dimension but with the special matrix $A = \alpha I$, $\alpha \in \mathbb{R}$, $0 \leqslant \alpha \leqslant 1$, where I is the identity matrix.

In this paper we give a geometric solution to the problem of asymptotic stability of (1.1) in any dimension with simultaneously triangularizable matrices A, B. It is known that commuting matrices are simultaneously triangularizable [13]. As usual, we say that (1.1) is stable if its zero solution is stable. Our solution is based on constructing the stability ovals which, in turn, form a stability cone. At the same time we give an algorithm for checking the stability of the scalar equation

$$x_n = a x_{n-1} + b x_{n-k}, \quad n = 0, 1, 2 \ldots \tag{1.2}$$

with complex coefficients a, b.

The paper is organized as follows. In the second section, we recall the results on the stability of the scalar equation (1.2) with real nonnegative a and any real b [14, 15]. Further in that section we construct the stability oval for (1.2) with real nonnegative coefficient a and complex coefficient b. In Section 3, we consider a wider class of equations of the form (1.2) with a, b being complex numbers. In Section 4, we state a system of inequalities allowing us to check the stability of the scalar equation (1.2) with two complex coefficients. In Section 5, we give a geometrical criterion for the asymptotic stability of matrix equation (1.1) with simultaneously triangularizable matrices. Besides, we establish nongeometric necessary and sufficient conditions for the stability of matrix equation (1.1) in terms of inequalities. In Section 6, we use the stability ovals and cones for analysis of some numerical examples.

2. The Stability Oval for (1.2) with Real Nonnegative a and Complex b

We start by stating the results from [14, 15] in the form which is suitable for us. Since the case $k = 1$ is obvious, we consider only the case $k > 1$.

Theorem 2.1 (see [14, 15]). *In* (1.2) *let a and b be real, $a \geqslant 0$, $k > 1$.*

(1) *If $a \geqslant k/(k-1)$, then* (1.2) *is unstable.*

(2) *If $0 \leqslant a < k/(k-1)$, then* (1.2) *is asymptotically stable if and only if*

$$-\sqrt{a^2 + 1 - 2a\cos\omega_1} < b < 1 - a, \tag{2.1}$$

where $\omega_1 \in [0, \pi/k]$ is the root of the equation

$$a = \frac{\sin k\omega}{\sin(k-1)\omega}. \tag{2.2}$$

Stability region of (1.2) is shown in Figure 1.

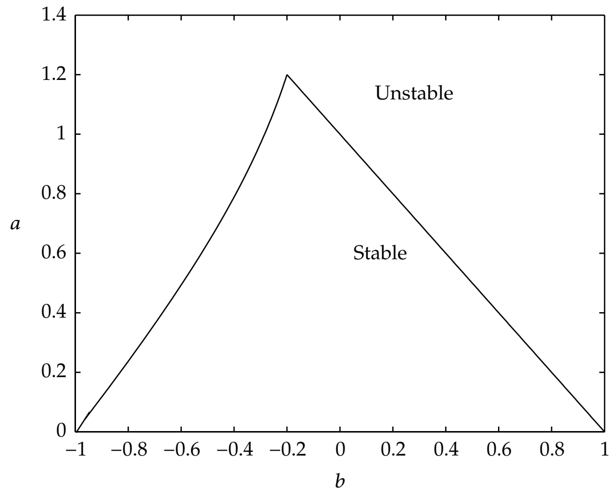


Figure 1: The domain of asymptotic stability of (1.2), $a \geqslant 0$, $k = 6$.

Our first new result about the case of nonnegative real a and complex b in (1.2) is the following.

Theorem 2.2. *Let $a \geqslant 0$ be a real number and b a complex number, $k > 1$.*

(1) *If $a \geqslant k/(k-1)$, then* (1.2) *is unstable.*

(2) *If $0 \leqslant a < k/(k-1)$, then* (1.2) *is asymptotically stable if and only if b lies inside the oval bounded by*

$$b = \exp(ik\omega) - a\exp(i(k-1)\omega), \quad -\omega_1 \leqslant \omega \leqslant \omega_1, \tag{2.3}$$

where $\omega_1 \in [0, \pi/k]$ is the root of (2.2).

(3) *If $0 \leqslant a < k/(k-1)$ and b is outside of the stability oval* (2.3), *then* (1.2) *is unstable.*

(4) *If $0 \leqslant a < k/(k-1)$ and b lies on the boundary* (2.3) *of the stability oval, then* (1.2) *is stable (nonasymptotically).*

Proof. We will use the D-decomposition method (parameter plane method) [16, 17]. A characteristic polynomial for (1.2) is the following:

$$f(z) = z^k - az^{k-1} - b. \tag{2.4}$$

For any fixed value of a, the complex plane of the parameter b is divided into some regions by a curve $f(\exp(i\omega)) = 0$, that is,

$$\exp(ik\omega) - a\exp(i(k-1)\omega) - b = 0, \quad -\pi \leqslant \omega \leqslant \pi. \tag{2.5}$$

Hence, D-decomposition occurs in the plane of the complex parameter b by means of a curve

$$b(\omega) = \exp(ik\omega) - a\exp(i(k-1)\omega), \quad -\pi \leqslant \omega \leqslant \pi. \tag{2.6}$$

The example of the D-decomposition for $k = 6$, $a = 0.7$ is shown in Figure 2. From (2.6) we obtain

$$|b(\omega)|^2 = 1 + a^2 - 2a\cos\omega. \tag{2.7}$$

Therefore, $|b|$ increases monotonically when ω runs from 0 to π. Similarly, $|b|$ increases monotonically when ω runs from 0 to $(-\pi)$. Let us construct an increasing sequence $(\omega_i)_{i=0}^{r}$ of all values $\omega \in [0,\pi]$, such that $\mathrm{Im}(b(\omega_i)) = \mathrm{Im}(b(-\omega_i)) = 0$. Here $1 \leqslant r \leqslant k$, $\omega_0 = 0$, $\omega_r = \pi$. Each pair of the curves (2.6) formed by motion ω on intervals $[\omega_i, \omega_{i+1}]$, $[-\omega_{i+1}, -\omega_i]$ creates a new region of D-decomposition of the complex plane of parameter b. This region necessarily contains some real values of b as the function $|b(\omega)|$ is monotone on $[0,\pi]$ and on $[-\pi, 0]$. Due to the properties of the regions, if some inner point of this region is an asymptotically stable point, then the whole region consists of asymptotically stable points. If $a \geqslant k/(k-1)$, then according to Theorem 2.1 there are no stable points on the axis Im $(b) = 0$. Hence, there are no stable points on the complex plane of parameter b. Let now $a < k/(k-1)$. Let $\omega_1 \in [0, \pi/k]$ be the root of (2.2). Then the unique D-decomposition region containing the real straight line segment (2.1) is the oval with the boundary (2.3). Parts 1–3 of Theorem 2.2 are proved. Direct checking shows that the derivative of a characteristic polynomial (2.4) is not equal to zero on the boundary of the stability oval. Therefore, when the parameter b runs along the boundary of the stability oval, all the corresponding roots z of the characteristic equation, satisfying $|z| = 1$, are simple. Hence, (1.2) is stable (nonasymptotically). Theorem 2.2 is proved. □

Example 2.3. Let $k = 6$ and, also, (1) $a = 0.2$, (2) $a = 0.75$, and (3) $a = 1.1$. Let $b = 0.33\exp(i\alpha)$, $0 \leqslant \alpha < 2\pi$. Let us analyze the asymptotic behavior of the solutions of (1.2) for all values of α. We construct three stability ovals for three values of a and also the circle $b = 0.33\exp(i\alpha)$ (Figure 3). Theorem 2.2 and Figure 3 give the following result. (1) For $a = 0.2$, the equation is asymptotically stable for any value of α. (2) For $a = 0.75$, the equation is asymptotically stable for $2.0918 \cong \alpha_0 < \alpha < 2\pi - \alpha_0 \cong 4.1914$, it is unstable for $\alpha \notin [\alpha_0, 2\pi - \alpha_0]$, and it is stable (nonasymptotically) for $\alpha = \alpha_0$ and $\alpha = 2\pi - \alpha_0$. (3) Finally, for $a = 1.1$, the equation is unstable for any value of α.

Example 2.4. Let $k = 6$ and, also, (1) $a = 0.2$, (2) $a = 0.75$, and (3) $a = 1.1$. Let $b = r\exp(i\,19\pi/20)$, $r \geqslant 0$. Let us find the asymptotic behavior of (1.2) for all values of r. We construct the beam $b = r\exp(i\,19\pi/20)$ together with three stability ovals (Figure 3). Theorem 2.2 and Figure 3 give the following result.

(1) For $a = 0.2$, the equation is asymptotically stable for $0 \leqslant r < r_1 \cong 0.8276$, it is unstable for $r > r_1$, and it is stable (nonasymptotically) for $r = r_1$.

(2) For $a = 0.75$, the equation is asymptotically stable for $0 \leqslant r < r_2 \cong 0.4080$, it is unstable for $r > r_2$, and it is stable (nonasymptotically) for $r = r_2$.

(3) Finally, for $a = 1.1$, the equation is stable for $0.1063 \cong r_{31} < r < r_{32} \cong 0.1944$, it is unstable for $r \notin [r_{31}, r_{32}]$, and it is stable (nonasymptotically) for $r = r_{31}$, $r = r_{32}$.

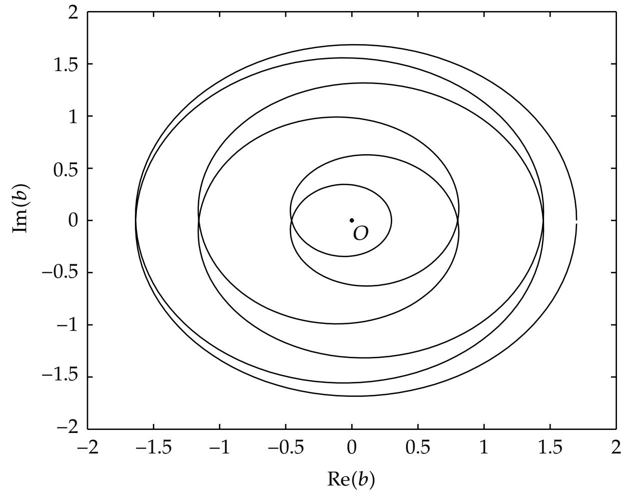


Figure 2: D-decomposition of the complex plane of the parameter b for $k = 6$, $a = 0.7$.

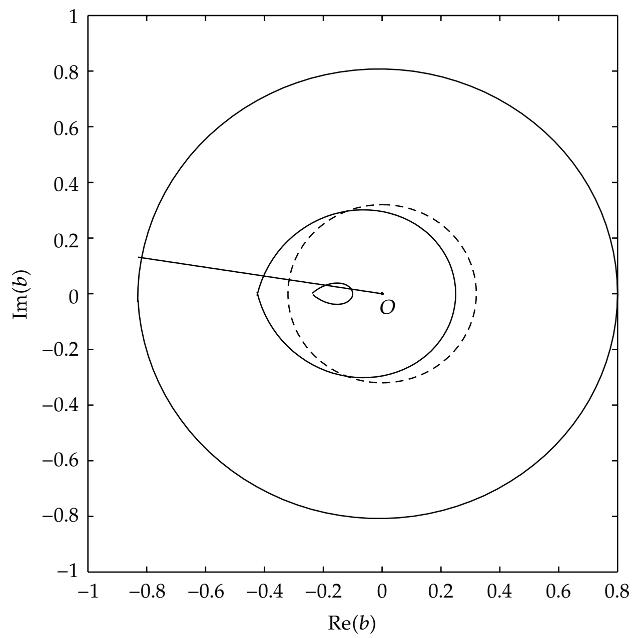


Figure 3: Stability ovals for $k = 6$, $a = 0.2$, and $a = 0.75$; $a = 1.1$. A circle $b = 0.33\exp(i\alpha)$ and a beam $b = r\exp(i\ 19\pi/20)$ are constructed for Examples 2.3 and 2.4.

3. The Stability Cone for (1.2) with Complex Coefficients

The family of stability ovals depending on $a \in k/(k-1)$ forms a surface which we call the stability cone.

Definition 3.1. The stability cone for delay k is a surface in a three-dimensional space $(\mathrm{Re}(b), \mathrm{Im}(b), z)$ *with* $0 \leqslant z \leqslant k/(k-1)$*, such that its intersection with any plane* $z = a$ $(0 \leqslant a \leqslant k/(k-1))$ *is the stability oval* (2.3).

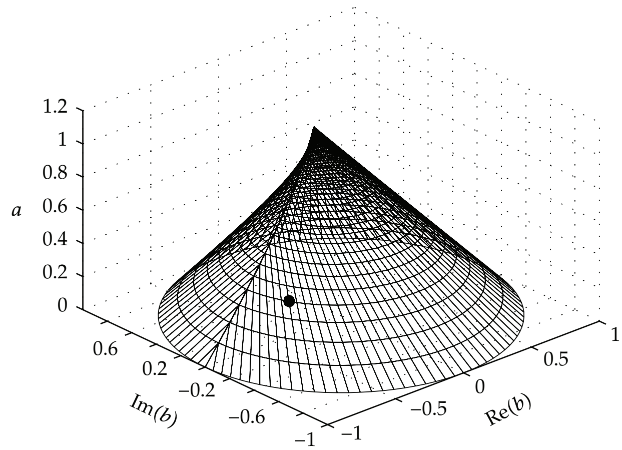


Figure 4: Stability cone for $k = 6$. The point on the cone is constructed for Example 3.3.

The stability cone is the image of the two-dimensional domain in the space (ω, a)

$$0 \leqslant a \leqslant \frac{\sin k\omega}{\sin (k-1)\omega},$$
$$-\frac{\pi}{k} \leqslant \omega \leqslant \frac{\pi}{k} \tag{3.1}$$

under the mapping into $\mathbb{R}^3$ by the functions

$$\operatorname{Re}(b) = \cos k\omega - a\cos(k-1)\omega,$$
$$\operatorname{Im}(b) = \sin k\omega - a\sin(k-1)\omega, \tag{3.2}$$
$$z = a.$$

The stability cone for $k = 6$ is presented in Figure 4.

Let us now study the problem of the stability of scalar equation (1.2) with complex coefficients a, b. We consider the equation

$$x_n = \rho_1 \exp(i\alpha_1)x_{n-1} + \rho_2 \exp(i\alpha_2)x_{n-k}, \tag{3.3}$$

with real nonnegative ρ_1, ρ_2 and real α_1, α_2. We set $x_n = y_n \exp(in\alpha_1)$. Then (3.3) becomes

$$y_n = \rho_1 y_{n-1} + \rho_2 \exp(i(\alpha_2 - k\alpha_1))y_{n-k}. \tag{3.4}$$

Obviously, (3.4) is stable (asymptotically stable) if and only if (3.3) is stable (asymptotically stable). The stability problem of (3.4) can be solved due to Theorem 2.2. Thus, we obtain the following theorem.

Theorem 3.2. *Consider* (3.3)*. Put*

$$a = \rho_1, \qquad b = \rho_2 \exp\left(i(\alpha_2 - k\alpha_1)\right). \tag{3.5}$$

Construct the point $M = (\mathrm{Re}(b), \mathrm{Im}(b), a)$ *in* $\mathbb{R}^3$.

(1) *Equation* (3.3) *is asymptotically stable if and only if the point* M *lies inside the cone* (3.2) *(if* $a = 0$ *and* $(\mathrm{Re}(b))^2 + (\mathrm{Im}(b))^2 < 1$*, then the point* M *is assumed to be the inner point of the cone).*

(2) *If the point* M *lies outside the cone* (3.2) *or on its top* $\mathrm{Re}(b) = -1/(k-1)$, $\mathrm{Im}\,(b) = 0$, $a = k/(k-1)$, *then* (3.3) *is unstable.*

(3) *If the point* M *lies on the boundary of a cone* (3.2)*, but not on its top, then* (3.3) *is stable (nonasymptotically).*

Example 3.3. Let us test the stability of the equation with complex coefficients

$$x_n = \sigma \exp\left(\frac{i\pi}{5}\right) x_{n-1} + 0.7 \exp\left(\frac{i\pi}{3}\right) x_{n-k} \tag{3.6}$$

with a real parameter σ and a delay $k = 6$. Put first $\sigma \geqslant 0$. By Theorem 3.2, according to (3.5) we calculate

$$b = 0.7 \exp\left(i\left(\frac{\pi}{3} - 6 \cdot \frac{\pi}{5}\right)\right) = 0.7 \exp\left(\frac{i17\pi}{15}\right). \tag{3.7}$$

The vertical line $(\mathrm{Re}(b), \mathrm{Im}(b), \sigma)$, $(0 \leqslant \sigma < \infty)$ in $\mathbb{R}^3$ intersects the boundary of the stability cone at $z = \sigma_0 \cong 0.3442$ (Figure 4). To study the negative values of σ we rewrite (3.6):

$$x_n = -\sigma \exp\left(\frac{i6\pi}{5}\right) x_{n-1} + 0.7 \exp\left(\frac{i\pi}{3}\right) x_{n-k}. \tag{3.8}$$

By Theorem 3.2, according to (3.5) we calculate

$$b = 0.7 \exp\left(i\left(\frac{\pi}{3} - 6 \cdot \frac{6\pi}{5}\right)\right) = 0.7 \exp\left(\frac{i17\pi}{15}\right). \tag{3.9}$$

As the results (3.9), (3.7) coincide, we obtain the following answer: (3.6) is asymptotically stable for $(-\sigma_0) < \sigma < \sigma_0 \cong 0.3442$, and it is unstable for $\sigma \notin [-\sigma_0, \ \sigma_0]$. According to part 2 of Theorem 3.2 for $\sigma = \sigma_0$ or $\sigma = -\sigma_0$, (3.6) is stable (nonasymptotically).

Example 3.4. We test (3.6) with a real parameter σ for stability. Unlike the previous example let now the delay be odd: $k = 7$. For positive σ by Theorem 3.2, according to (3.5) we obtain

$$b = 0.7 \exp\left(i\left(\frac{\pi}{3} - 7 \cdot \frac{\pi}{5}\right)\right) = 0.7 \exp\left(\frac{i14\pi}{15}\right). \tag{3.10}$$

The vertical line $(\mathrm{Re}(b), \mathrm{Im}(b), \sigma)$ in $\mathbb{R}^3$ intersects the boundary of the stability cone at $z = \sigma_0 \cong 0.3377$. To study the negative values of σ similarly to the previous example, we obtain

$$b = 0.7 \exp\left(i\left(\frac{\pi}{3} - 7 \cdot \frac{6\pi}{5}\right)\right) = 0.7 \exp\left(\frac{i29\pi}{15}\right). \tag{3.11}$$

The results (3.10), (3.11) do not coincide. For (3.11) the vertical line $(\mathrm{Re}(b), \mathrm{Im}(b), \sigma)$, $(0 \leqslant \sigma < \infty)$ intersects the boundary of the stability cone at $z = \sigma_1 \cong 0.3002$. We obtain the following answer: (3.6) with $k = 7$ is asymptotically stable if $(-0.3002) \cong -\sigma_1 < \sigma < \sigma_0 \cong 0.3377$, it is unstable if $\sigma \notin [-\sigma_1,\ \sigma_0]$, and it is stable (nonasymptotically) for $\sigma = \sigma_0$ or $\sigma = -\sigma_1$.

Comparison of Examples 3.3 and 3.4 reveals the difference in the behavior of (3.3) for even and odd values of the delay k. Let us compare the stability of (1.1) and the following equations:

$$x_n = -Ax_{n-1} + Bx_{n-k}, \tag{3.12}$$

$$x_n = -Ax_{n-1} - Bx_{n-k}. \tag{3.13}$$

Substituting $x_n = (-1)^n y_n$ reduces (1.1) to

$$y_n = -Ay_{n-1} + (-1)^k By_{n-k}. \tag{3.14}$$

Equations (1.1) and (3.14) are simultaneously stable or unstable. Therefore, we have the following symmetry property of the stability region for (1.1).

Theorem 3.5. *For even delays k the (asymptotic) stability of* (1.1) *implies the (asymptotic) stability of* (3.12) *and vice versa. For odd k the (asymptotic) stability of* (1.1) *implies the (asymptotic) stability of* (3.13) *and vice versa.*

Similar properties of symmetry have been specified in [11, 15] for the scalar equation (1.2) with real a, b.

4. A System of Inequalities for Checking the Stability of (3.3)

In the previous sections we used some geometric procedures. In this section we construct a system of inequalities in order to check the stability of (3.3). Henceforth we assume $0 \leqslant \arg(z) < 2\pi$ for a complex variable z.

Theorem 4.1.

(1) *If $\rho_1 < 1 - \rho_2$, then* (3.3) *is asymptotically stable.*

(2) *If $1 - \rho_2 \leqslant \rho_1 < \min(1 + \rho_2, k/(k-1))$, then for the asymptotic stability of* (3.3) *it is necessary and sufficient to fulfill simultaneously the following conditions* $(H1)$, $(H2)$*:*

$$\rho_2 < \sqrt{\rho_1^2 + 1 - 2\rho_1 \cos\omega_1}, \tag{$H1$}$$

where $\omega_1 \in [0, \pi/k]$ *is the root of the equation*

$$\rho_1 = \frac{\sin k\omega}{\sin (k-1)\omega},$$
$$\left|\pi - \arg\left(\exp(i(\alpha_2 - k\alpha_1))\right)\right| < \pi - (k-1)\arccos\frac{1+\rho_1^2-\rho_2^2}{2\rho_1} - \arccos\frac{1-\rho_1^2-\rho_2^2}{2\rho_1\rho_2}. \quad (H2)$$

(3) *If* $\rho_1 \geqslant \min(1+\rho_2, k/(k-1))$, *then* (3.3) *is not asymptotically stable.*

Proof. The stability of (3.3) is equivalent to the stability of (3.4), so we will work with (3.4).

(1) For $\rho_1 < 1-\rho_2$ by Theorem 2.2 the stability oval exists and the circle of radius ρ_2 lies completely inside the oval. Therefore, Theorem 2.2 implies the asymptotic stability of (3.4). Part 1 of Theorem is proved.

(2) Let $1-\rho_2 \leqslant \rho_1 < \min(1 + æ_2, \mathrm{k}/(\mathrm{k}-1))$. Let us consider two cases.

Case 1 ($(1-\rho_2) \leqslant \rho_1 < 1$). In this case, the stability oval exists by Theorem 2.2, and the origin of the coordinates lies inside the oval. For (3.4) to be asymptotically stable, it is necessary and sufficient to satisfy the two following conditions. The first one is that the circle of radius ρ_2 should intersect the stability oval. It is equivalent to ($H1$). The second condition is that the argument of a point $\exp(i(\alpha_2 - k\alpha_1))$ should be between the arguments of the two crosspoints M_1, M_2 of a circle of radius ρ_2 with the stability oval (2.3). Let us assume that $\mathrm{Im}(M_1) > 0$, and let the parameter ω correspond to the point M_1. We obtain

$$\arg(M_1) = \arg\left(\exp(ik\omega) - \rho_1 \exp(i(k-1)\omega)\right) = (k-1)\omega + \arg\left(\exp(i\omega) - \rho_1\right) \quad (4.1)$$

from (2.3). But we also obtain

$$\cos\omega = \frac{1+\rho_1^2-\rho_2^2}{2\rho_1} \quad (4.2)$$

from (2.7). Equalities (4.1), (4.2) give

$$\arg(M_1) = (k-1)\arccos\frac{1+\rho_1^2-\rho_2^2}{2\rho_1} + \arccos\frac{1-\rho_1^2-\rho_2^2}{2\rho_1\rho_2}. \quad (4.3)$$

It follows from (4.3) that the second requirement is equivalent to ($H2$). Part 2 of Theorem 4.1 is proved in Case 1.

Case 2 ($1 \leqslant \rho_1 < \min(1+\rho_2, k/(k-1))$). By virtue of the inequality $\rho_1 < k/(k-1)$, the stability oval exists, and, since $\rho_1 \geqslant 1$, the origin does not lie inside the oval. The same requirements as in the previous case lead to the same conditions ($H1$), ($H2$). Part 2 of the Theorem is proved.

(3) Let $\rho_1 \geqslant \min(1+\rho_2, k/(k-1))$. We consider two cases.

Case 1 ($1+\rho_2 \leqslant \rho_1 < k/(k-1)$). In this case the stability oval exists by Theorem 2.2, and, by virtue of the inequality $\rho_1 \geqslant 1$, the origin does not lie inside the oval. Due to the inequality $\rho_2 \leqslant \rho_1 - 1$, no point of the circle of radius ρ_2 lies inside the oval and so (3.4) is not asymptotically stable by Theorem 2.2.

Case 2 ($\rho_1 \geqslant k/(k-1)$). In this case, (3.4) is unstable by Theorem 2.2.

Theorem 4.1 is proved. □

As we see, the text of Theorem 4.1 does not contain any geometric terms. However, Theorem 3.2 has a considerable advantage over Theorem 4.1 because of its simplicity and geometric visualization. That is why in the future examples we prefer describing the stability of matrix equation (1.1) in geometric terms.

5. The Stability Cone for the Matrix Equation with Simultaneously Triangularizable Matrices

In this section we consider (1.1) with simultaneously triangularizable matrices A, B.

Theorem 5.1. *Let $A, B, S \in \mathbb{R}^{m\times m}$ and $S^{-1}AS = A_T$ and $S^{-1}BS = B_T$, where A_T, B_T are the lower triangular matrices with elements, respectively, $\lambda_{js}, \mu_{js}(1 \leqslant j, s \leqslant m)$. Let*

$$b_j = |\mu_{jj}| \exp\left(i(\arg(\mu_{jj}) - k\arg(\lambda_{jj}))\right), \quad a_j = |\lambda_{jj}| \quad (1 \leqslant j \leqslant m), \tag{5.1}$$

and let the points M_j in $\mathbb{R}^3$ be constructed by

$$M_j = (\mathrm{Re}(b_j), \mathrm{Im}(b_j), a_j) \quad (1 \leqslant j \leqslant m). \tag{5.2}$$

Then (1.1) *is asymptotically stable if and only if for any j $(1 \leqslant j \leqslant m)$ the point M_j lies inside cone* (3.2).

If for some j $(1 \leqslant j \leqslant m)$ the point M_j lies outside cone (3.2)*, then* (1.1) *is unstable.*

Proof. In (1.1) we substitute $x_n = Sy_n$ and multiply the equation by S^{-1}. We obtain

$$y_n = A_T y_{n-1} + B_T y_{n-k} \tag{5.3}$$

with lower triangular matrices A_T, B_T. By virtue of the nondegeneracy of matrix S, the stability of (5.3) is equivalent to the stability of (1.1). Let us assume that $y_n = (y_n^{(1)}, \dots, y_n^{(m)})^T$. The system (5.3) consists of m scalar equations

$$y_n^{(j)} = \lambda_{jj} y_{n-1}^{(j)} + \mu_{jj} y_{n-k}^{(j)} + \sum_{s=1}^{j-1} \lambda_{js} y_{n-1}^{(s)} + \sum_{s=1}^{j-1} \mu_{js} y_{n-k}^{(s)} \quad (1 \leqslant j \leqslant m). \tag{5.4}$$

As usual, $\sum_{s=1}^{0} = 0$. Equation (5.4) is called exponentially stable if there are real $C > 0$, $q \in (0,1)$, such that for any solution $y_n^{(j)}$ the estimate

$$\left|y_n^{(j)}\right| \leqslant Cq^n \max_{(-k)\leqslant u\leqslant 1,\, 1\leqslant s\leqslant j} \left|y_u^{(s)}\right| \tag{5.5}$$

holds. The exponential stability is equivalent to the asymptotic stability for the equations under consideration. It is more convenient to prove the exponential stability. Let the points (5.2) ($1 \leqslant j \leqslant m$) lie inside the cone (3.2). Due to Theorem 3.2, all equations of the form

$$y_n^{(j)} = \lambda_{jj} y_{n-1}^{(j)} + \mu_{jj} y_{n-k}^{(j)} \quad (1 \leqslant j \leqslant m) \tag{5.6}$$

are exponentially stable. Let us prove by induction on j that (5.4) are exponentially stable. For $j = 1$, (5.4) coincides with (5.6), so it is exponentially stable. Let, for any $r < j$, (5.4) with r instead of j be exponentially stable. Then (5.4) is represented in the form

$$y_n^{(j)} = \lambda_{jj} y_{n-1}^{(j)} + \mu_{jj} y_{n-k}^{(j)} + g_n^{(j)}, \tag{5.7}$$

where $|g_n^{(j)}|$ has an estimate of the form (5.5) by the induction assumption. Assuming $z_n = (y_n^{(j)}, y_{n-1}^{(j)}, \ldots, y_{n-k}^{(j)})^T$, we represent (5.7) in the form

$$z_n = G z_{n-1} + h_n, \tag{5.8}$$

where $G \in \mathbb{R}^{k \times k}$, G is a stable matrix, and $|h_n|$ has an estimate of the form (5.5). From (5.8) we obtain $z_n = G^n z_0 + \sum_{r=1}^{n} G^{n-r} h_r$, which implies the exponential stability of (5.4). The induction is finished, and asymptotic stability of (1.1) is proved.

Let us assume that some point (5.2) does not lie strictly inside the cone. Then, for the initial data in (5.4), we assume that $y_{-n}^{(s)} = 0$ for any s, n, such that $1 \leqslant s \leqslant j$, $1 \leqslant n \leqslant k$. Thus, (5.4) becomes (5.6). If the point (5.2) lies on the cone boundary, then (5.6) has a trajectory which does not tend to zero because the characteristic polynomial of (5.6) has a root z such that $|z| = 1$. If some point (5.2) lies outside the cone, then by Theorem 3.2 the equation has unlimited trajectories. Theorem 5.1 is proved. $\square$

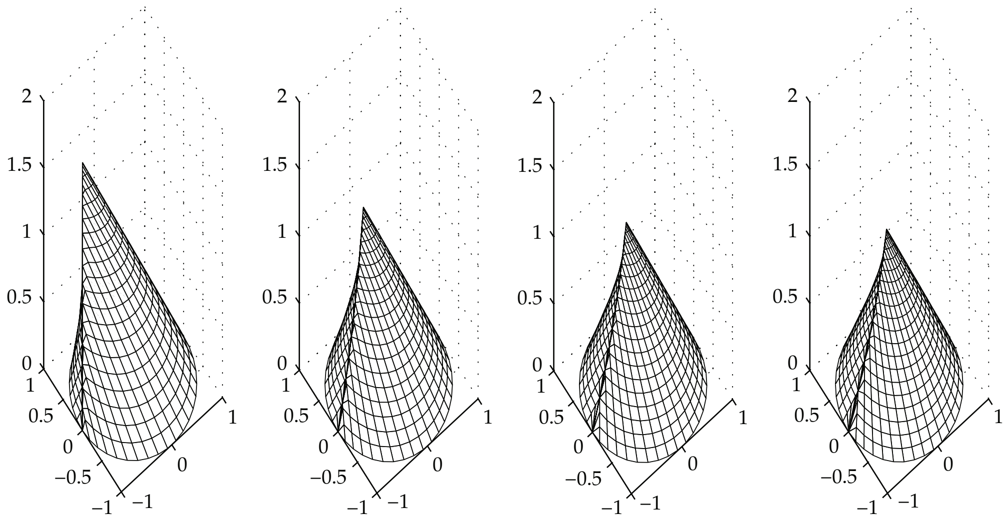
Remark 5.2. If no points (5.2) lie outside the stability cone, but some of them lie on the cone boundary, then (1.1) can be stable (nonasymptotically) or unstable.

Remark 5.3. The stability cones (Figure 5) are constructed for each delay k independently of the dimension m in (1.1). If $k \to \infty$, then the intersection of all stability cones is the right circular cone with the base radius 1 and the height 1. The interior of this cone is the "absolute stability domain," that is, the stability domain for any delay.

The next theorem, which is the evident consequence of Theorems 4.1 and 5.1, will establish the asymptotic stability criterion in the form of inequalities for matrix equation (1.1).

Theorem 5.4. *Let $A, B, S \in \mathbb{R}^{m \times m}$ and $S^{-1}AS = A_T$ and $S^{-1}BS = B_T$, where A_T, B_T are the lower triangular matrices with elements λ_{js}, μ_{js} ($1 \leqslant j, s \leqslant m$), respectively. Let*

$$\rho_{1j} = |\lambda_{jj}|, \qquad \alpha_{1j} = \arg(\lambda_{jj}), \qquad \rho_{2j} = |\mu_{jj}|, \qquad \alpha_{2j} = \arg(\mu_{jj}) \quad (1 \leqslant j \leqslant m). \tag{5.9}$$



(a) (b) (c) (d)

Figure 5: Stability cones for $k = 2, 3, 4, 5$.

Construct a set $AS \subseteq P = \{j \in \mathbb{Z}_+ : 1 \leqslant j \leqslant m\}$ *by the following rules (cf. Theorem 4.1).*

(1) *If* $\rho_{1j} < 1 - \rho_{2j}$*, then* $j \in AS$*.*

(2) *If* $1 - \rho_{2j} \leqslant \rho_{1j} < \min(1 + \rho_{2j}, k/(k-1))$*, then for* $j \in AS$ *it is necessary and sufficient to fulfill simultaneously the following conditions* (H1j), (H2j)*:*

$$\rho_{2j} < \sqrt{\rho_{1j}^2 + 1 - 2\rho_{1j} \cos \omega_{1j}}, \tag{H1j}$$

where $\omega_{1j} \in [0, \pi/k]$ *is the root of the equation*

$$\rho_{1j} = \frac{\sin k\omega}{\sin (k-1)\omega},$$
$$\left|\pi - \arg\left(\exp\left(i\left(\alpha_{2j} - k\alpha_{1j}\right)\right)\right)\right| < \pi - (k-1)\arccos\frac{1 + \rho_{1j}^2 - \rho_{2j}^2}{2\rho_{1j}} - \arccos\frac{1 - \rho_{1j}^2 - \rho_{2j}^2}{2\rho_{1j}\rho_{2j}}. \tag{H2j}$$

(3) *If* $\rho_{1j} \geqslant \min(1 + \rho_{2j}, k/(k-1))$*, then* $j \notin AS$*.*
Equation (1.1) *is asymptotically stable if and only if* $AS = P$*.*

6. Examples of the Stability Oval and the Cone for Matrix Equations

Example 6.1. Consider the equation

$$x_n = 1.0309 A x_{n-1} + (0.9680B)^s x_{n-6}, \tag{6.1}$$

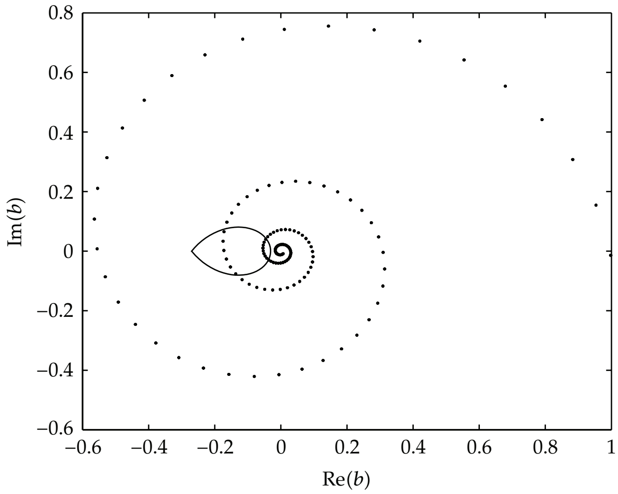


Figure 6: The stability oval and points $M^{(s)}$ for Example 6.1.

where

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \tag{6.2}$$

with $\alpha = 0.0314$, $\beta = 0.1745$. Let us find out for what values of $s \in \mathbb{Z}_+$ (6.1) is stable. Matrices A, B are commuting; therefore, they are simultaneously triangularizable. The eigenvalues of matrices A, B are $\lambda_{1,2} = \exp(\pm 0.0314i)$ and $\mu_{1,2} = \exp(\pm 0.1745i)$ correspondingly. In Figure 6 the stability oval is shown, which is the section of the stability cone (3.2) on the level $z = 1.0309$. By Theorem 5.1 we have to know whether the points

$$M^{(s)} = \left(0.9680^s \cos(0.1745s - 0.1884), 0.9680^s \sin(0.1745s - 0.1884)\right) \tag{6.3}$$

lie inside the cone. Figure 6 illustrates that points $M^{(s)}$ enter the oval of stability twice ($s = 53, s = 87$) and leave it twice ($s = 58,\ s = 94$). The conclusion is that the system (6.1), (6.2) is stable for $53 \leqslant s \leqslant 58$ and for $87 \leqslant s \leqslant 94$ and is unstable for all the other values of s.

Example 6.2. Consider the equation

$$x_n = \frac{1}{3}\left(A^s x_{n-1} + A x_{n-6}\right), \tag{6.4}$$

where

$$A = \begin{pmatrix} 0 & 1.0150 \\ -1.0150 & 2.0300 \end{pmatrix}. \tag{6.5}$$

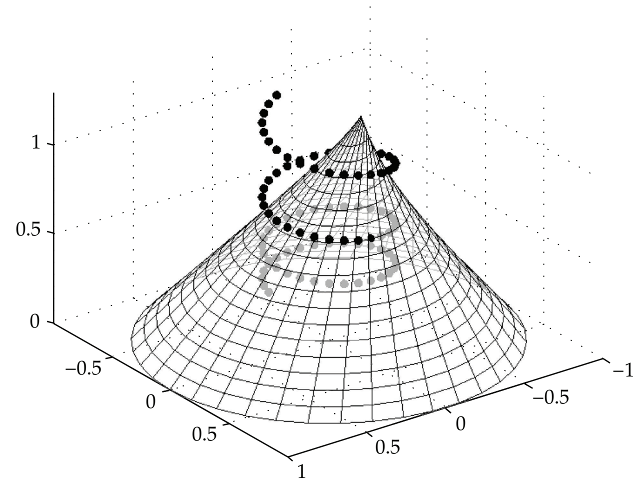


Figure 7: The stability cone and points $M^{(s)}$ for Example 6.2. Grey points $M^{(s)}$ $(1 \leqslant s \leqslant 49)$ are located inside the cone therefore (6.4) is stable. Dark points $M^{(s)}$ $(s \geqslant 50)$ are located outside therefore (6.4) is unstable.

Let us find out for what values of $s \in \mathbb{Z}_+$ (6.4) is stable. The eigenvalues of A are $\lambda_{1,2} = 1.0150\exp(\pm 0.0374i)$. Stability ovals are symmetric about the real axis. Therefore, by Theorem 5.1, only points

$$M^{(s)} = \left(0.3383\cos(0.0374(1-6s)), 0.3383\sin(0.0374(1-6s)), \frac{1.0150^s}{3}\right) \tag{6.6}$$

(see Figure 7) should be checked. Figure 7 displays that points $M^{(s)}$ for $1 \leqslant s \leqslant 49$ are inside the stability cone (3.2) and for $s \geqslant 50$ points are outside of the stability cone. The conclusion is that (6.4) is stable for $1 \leqslant s \leqslant 49$ and it is unstable for $s \geqslant 50$.

7. Conclusion

The stability analysis for (1.1) in $\mathbb{R}^m$ can be reduced to the pole placement problem for a polynomial of degree km. Our geometric approach allows us to reduce the dimension. To use the approach, we need to know the eigenvalues of A, B. This is the problem of finding the roots of a polynomial of degree m. Using these eigenvalues, we get a finite sequence of points in $\mathbb{R}^3$ such that their position with respect to the stability cone allows us to make a conclusion about the stability of (1.1).

In our future work we intend to analyze the stability of equation $x_n = Ax_{n-m} + Bx_{n-k}$ with two delays m, k with simultaneously triangularizable matrices A, B. The scalar version of this equation was examined in [2, 3, 18]. The stability cone for the matrix differential equation $\dot{x}(t) = Ax(t) + Bx(t-\tau)$ was introduced in [19].

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References

[1] A. T. Fam and J. S. Meditch, "A canonical parameter space for linear systems design," *IEEE Transactions on Automatic Control*, vol. 23, no. 3, pp. 454–458, 1978.

[2] Y. P. Nikolaev, "The geometry of D-decomposition of a two-dimensional plane of arbitrary coefficients of the characteristic polynomial of a discrete system," *Automation and Remote Control*, vol. 65, no. 12, pp. 1904–1914, 2004.

[3] M. M. Kipnis and R. M. Nigmatulin, "Stability of trinomial linear difference equations with two delays," *Automation and Remote Control*, vol. 65, pp. 1710–1723, 2004.

[4] M. M. Kipnis and D. A. Komissarova, "A note on explicit stability conditions for autonomous higher order difference equations," *Journal of Difference Equations and Applications*, vol. 13, no. 5, pp. 457–461, 2007.

[5] V. M. Gilyazev and M. M. Kipnis, "Convexity of a sequence of coefficients and the stability of discrete systems," *Automation and Remote Control*, vol. 70, pp. 1856–1861, 2009.

[6] I. S. Levitskaya, "A note on the stability oval for $x_{n+1} = x_n + Ax_{n-k}$," *Journal of Difference Equations and Applications*, vol. 11, no. 8, pp. 701–705, 2005.

[7] S. Guo, X. Tang, and L. Huang, "Stability and bifurcation in a discrete system of two neurons with delays," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 4, pp. 1323–1335, 2008.

[8] E. Kaslik and S. Balint, "Bifurcation analysis for a two-dimensional delayed discrete-time Hopfield neural network," *Chaos, Solitons & Fractals*, vol. 34, no. 4, pp. 1245–1253, 2007.

[9] J. Diblík and D. Y. Khusainov, "Representation of solutions of discrete delayed system $x(k+1) = Ax(k)+Bx(k-m)+f(k)$ with commutative matrices," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 63–76, 2006.

[10] H. Matsunaga, "Stability regions for a class of delay difference systems," in *Differences and Differential Equations*, vol. 42 of *Fields Institute Communications*, pp. 273–283, American Mathematical Society, Providence, RI, USA, 2004.

[11] H. Matsunaga and C. Hajiri, "Exact stability sets for a linear difference system with diagonal delay," *Journal of Mathematical Analysis and Applications*, vol. 369, no. 2, pp. 616–622, 2010.

[12] E. Kaslik, "Stability results for a class of difference systems with delay," *Advances in Difference Equations*, vol. 2009, Article ID 938492, 13 pages, 2009.

[13] R. Horn and C. Johnson, *Matrix Theory*, Cambridge University Press, Cambrige, UK, 1986.

[14] S. A. Kuruklis, "The asymptotic stability of $x(n+1) - ax(n) + bx(n-k) = 0$," *Journal of Mathematical Analysis and Applications*, vol. 188, no. 3, pp. 719–731, 1994.

[15] V. G. Papanicolaou, "On the asymptotic stability of a class of linear difference equations," *Mathematics Magazine*, vol. 69, no. 1, pp. 34–43, 1996.

[16] E. N. Gryazina and B. T. Polyak, "Stability regions in the parameter space: D-decomposition revisited," *Automatica*, vol. 42, no. 1, pp. 13–26, 2006.

[17] D. D. Šiljak, "Parameter space methods for robust control design: a guided tour," *IEEE Transactions on Automatic Control*, vol. 34, no. 7, pp. 674–688, 1989.

[18] F. M. Dannan, "The asymptotic stability of $x(n+K) + ax(n) + bx(n-1) = 0$," *Journal of Difference Equations and Applications*, vol. 10, no. 6, pp. 589–599, 2004.

[19] T. Khokhlova, M. Kipnis, and V. Malygina, "The stability cone for a delay differential matrix equation," *Applied Mathematics Letters*, vol. 24, no. 5, pp. 742–745, 2011.