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Research Article

p-Carleson Measures for a Class of Hardy-Orlicz Spaces

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An alternative interpretation of a family of weighted Carleson measures is used to characterize p-Carleson measures for a class of Hardy-Orlicz spaces admitting a nice weak factorization. As an application, we provide with a characterization of symbols of bounded weighted composition operators and Cesàro-type integral operators from these Hardy-Orlicz spaces to some classical holomorphic function spaces.

1. Introduction

Hardy-Orlicz spaces are the generalization of the usual Hardy spaces. We raise the question of characterizing those positive measures μ defined on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ such that these spaces embed continuously into the Lebesgue spaces $L^p(d\mu)$. More precisely, let denote by dV the Lebesgue measure on $\mathbb{B}^n$ and $d\sigma$ the normalized measure on the unit sphere $\mathbb{S}^n$ which is the boundary of $\mathbb{B}^n$. $H(\mathbb{B}^n)$ denotes the space of holomorphic functions on $\mathbb{B}^n$. Let Φ be continuous and nondecreasing function from $[0,\infty)$ onto itself. That is, Φ is a growth function. The Hardy-Orlicz space $\mathscr{H}^\Phi(\mathbb{B}^n)$ is the space of function f in $H(\mathbb{B}^n)$ such that the functions f_r, defined by $f_r(w) = f(rw)$ satisfy

$$\sup_{r<1} \inf\left\{\lambda > 0 : \int_{\mathbb{S}^n} \Phi\left(\frac{|f_r(x)|}{\lambda}\right) d\sigma(x) \le 1\right\} < \infty. \tag{1.1}$$

We denote the quantity on the left of the above inequality by $\|f\|_{\mathscr{H}^\Phi}^{\text{lux}}$ or simply $\|f\|_{\mathscr{H}^\Phi}$ when there is no ambiguity. Let us remark that $\|f\|_{\mathscr{H}^\Phi}^{\text{lux}} = \sup_{r<1}\|f_r\|_{L^\Phi}^{\text{lux}}$, where $\|f\|_{L^\Phi}^{\text{lux}}$ denotes the Luxembourg (quasi)-norm defined by

$$\|f\|_{L^\Phi}^{\text{lux}} := \inf\left\{\lambda > 0 : \int_{\mathbb{S}^n} \Phi\left(\frac{|f_r(x)|}{\lambda}\right) d\sigma(x) \le 1\right\} < \infty. \tag{1.2}$$

Given two growth functions Φ_1 and Φ_2, we consider the following question. For which positive measures μ on $\mathbb{B}^n$, the embedding map $I_\mu : \mathscr{H}^{\Phi_2}(\mathbb{B}^n) \to L^{\Phi_1}(d\mu)$, is continuous? When Φ_1 and Φ_2 are power functions, such a question has been considered and completely answered in the unit disc and the unit ball in [1–6]. For more general convex growth functions, an attempt to solve the question appears in [7], in the setting of the unit disc where the authors provided with a necessary condition which is not always sufficient and a sufficient condition. The unit ball version of [7] is given in [8]. To be clear at this stage, let us first introduce some usual notations. For any $\xi \in \mathbb{S}^n$ and $\delta > 0$, let

$$\begin{aligned} B_\delta(\xi) &= \{w \in \mathbb{S}^n : |1 - \langle w, \xi\rangle| < \delta\}, \\ Q_\delta(\xi) &= \{z \in \mathbb{B}^n : |1 - \langle z, \xi\rangle| < \delta\}. \end{aligned} \tag{1.3}$$

These are the higher dimension analogues of Carleson regions. We take as Φ_1 the power functions, that is, $\Phi_1(t) = t^p$ for $1 \le p < \infty$. Thus, the question is now to characterize those positive measures μ on the unit ball such that there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \le C\left(\|f\|_{\mathscr{H}^\Phi}^{\text{lux}}\right)^p \quad \forall f \in \mathscr{H}^\Phi(\mathbb{B}^n). \tag{1.4}$$

We call such measures p-Carleson measures for $\mathscr{H}^\Phi(\mathbb{B}^n)$. We give a complete answer for a special class of Hardy-Orlicz spaces $\mathscr{H}^\Phi(\mathbb{B}^n)$ with $\Phi(t) = (t/\log(e+t))^s$, $0 < s \le 1$. For simplicity, we denote this space by $\mathscr{H}_s(\mathbb{B}^n)$.

We prove the following result.

Theorem 1.1. *Let $0 < s \le 1$ and $1 \le p < \infty$. Then the following assertions are equivalent.*

(i) *There exists a constant $K_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and $\delta > 0$,*

$$\mu(Q_\delta(\xi)) \le K_1 \frac{\delta^{n(p/s)}}{(\log(4/\delta))^p}. \tag{1.5}$$

(i) *There exists a constant $K_2 > 0$ such that*

$$\int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \le K_2 \|f\|_{\mathscr{H}_s}^p \quad \forall f \in \mathscr{H}_s(\mathbb{B}^n). \tag{1.6}$$

To prove the above result, we combine weak-factorization results for Hardy-Orlicz spaces (see [9, 10]) and some equivalent characterizations of weighted Carleson measures for which we provide an alternative interpretation. We also provide with some further applications of our characterization of the measures considered here to the boundedness of weighted Cesàro-type integral operators from our Hardy-Orlicz spaces to some holomorphic function spaces in Section 3.

All over the text, C, C_j and, K_j, $j = 1, \ldots,$ will denote positive constant not necessarily the same at each occurrence.

This work can be also considered as an application of some recent results obtained by the author and his collaborators [9–11].

2. λ-Hardy p-logarithmic Carleson Measures

For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we let $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ so that $|z|^2 = \langle z, z\rangle = |z_1|^2 + \cdots + |z_n|^2$.

Recall that when Φ is a power function, the Hardy-Orlicz space $\mathcal{H}^{\Phi}(\mathbb{B}^n)$ is just the classical Hardy space. More precisely, for $0 < p < \infty$, let $\mathcal{H}^p(\mathbb{B}^n)$ denote the Hardy space which is the space of all $f \in H(\mathbb{B}^n)$ such that

$$\|f\|_p^p := \sup_{0<r<1} \int_{\mathbb{S}^n} |f(r\xi)|^p d\sigma(\xi) < \infty. \tag{2.1}$$

We denote by $\mathcal{H}^{\infty}(\mathbb{B}^n)$, the space of bounded analytic functions in $\mathbb{B}^n$.

Let ρ be a continuous increasing function from $[0, \infty)$ onto itself, and such that for some α on $[0, 1]$

$$\varrho(st) \leq s^{\alpha}\varrho(t) \tag{2.2}$$

for $s > 1$, with $st \leq 1$. We define the space BMO(ρ) by

$$\text{BMO}(\rho) = \left\{ f \in L^2(\mathbb{S}^n);\ \sup_{B} \inf_{R \in \mathcal{P}_N(B)} \frac{1}{(\varrho(\sigma(B)))^2 \sigma(B)} \int_B |f - R|^2 d\sigma < \infty \right\}, \tag{2.3}$$

where for $B = B_{\delta}(\xi_0)$, the space $\mathcal{P}_N(B)$ is the space of polynomials of order $\leq N$ in the $(2n-1)$ last coordinates related to an orthonormal basis whose first element is ξ_0 and second element $\Im\xi_0$. The integer N is taken larger than $2n\alpha - 1$. For C, the quantity appearing in the definition of BMO(ρ), we note $\|f\|_{\text{BMO}(\rho)} := \|f\|_2 + C$. The space BMO$A(\rho)$ is then the space of function $f \in \mathcal{H}^2(\mathbb{B}^n)$ such that

$$\sup_{r<1} \|f_r\|_{\text{BMO}(\rho)} < \infty. \tag{2.4}$$

Clearly, BMO$A(\rho)$ coincides with the space of holomorphic functions in $\mathcal{H}^2(\mathbb{B}^n)$ such that their boundary values lie in BMO(ρ). The space BMO$A(1)$ is the usual space of function with bounded mean oscillation BMOA while the space of function of logarithmic mean oscillation LMOA is given by $1/\rho(t) = \log 4/t$.

Let μ denote a positive Borel measure on $\mathbb{B}^n$. The measure μ is called an s-Carleson measure, if there is a finite constant $C > 0$ such that for any $\xi \in \mathbb{S}^n$ and any $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \leq C(\sigma(B_\delta(\xi)))^s. \tag{2.5}$$

When $s = 1$, μ is just called Carleson measure. The infinimum of all these constants C will be denoted $\|\mu\|_s$. We use the notation $\|\mu\|$ for $\|\mu\|_1$. In this section, we are interested in Carleson measure with weights involving the logarithmic function. Let μ be a positive Borel measure on $\mathbb{B}^n$ and $0 < s < \infty$. For ρ, a positive function defined on $(0,1)$, we say μ is a (ρ, s)-Carleson measure if there is a constant $C > 0$ such that for any $\xi \in \mathbb{S}^n$ and $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \leq C\frac{(\sigma(B_\delta(\xi)))^s}{\rho(\delta)}. \tag{2.6}$$

If $s = 1$, μ is called a ρ-Carleson measure.

We will restrict here to the case $\rho(t) = (\log(4/t))^p(\log\log(e^4/t))^q$, $0 < p,\ q < \infty$ studied by the author in [11] (see also [12] for a special case in one dimension). But here we go beyond the interpretation provided in [11].

2.1. λ-Hardy ρ-Carleson Measures

In this section, we recall some results of [11] and the notion of λ-Hardy Carleson measures. We then provide with an alternative interpretation of the results of [11] that will be useful to our characterization. From now on, the notation $K_1 \approx K_2$, where K_1 and K_2 are two positive constants, will mean there exists an absolute positive constant M such that

$$M^{-1}K_2 \leq K_1 \leq MK_2, \tag{2.7}$$

and in this case, we say K_1 and K_2 are comparable or equivalent. The notation $K_1 \lesssim K_2$ means $K_1 \leq MK_2$ for some absolute positive constant M. Let set

$$K_a(z) = \frac{\left(1 - |a|^2\right)^n}{|1 - \langle a, z\rangle|^{2n}}. \tag{2.8}$$

We first recall the following higher dimension version of the theorem of Carleson [1] and its reproducing kernel formulation.

Theorem 2.1. *For a positive Borel measure μ on $\mathbb{B}^n$, and $0 < p < \infty$, the following are equivalent*

(i) *The measure μ is a Carleson measure.*

(ii) *There is a constant $K_1 > 0$ such that, for all $f \in \mathcal{H}^p(\mathbb{B}^n)$,*

$$\int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \leq K_1\|f\|_p^p. \tag{2.9}$$

(iii) *There is a constant $K_2 > 0$ such that, for all $a \in \mathbb{B}^n$,*

$$\int_{\mathbb{B}^n} K_a(w)\,d\mu(w) \le K_2 < \infty. \tag{2.10}$$

We note that the constants K_1, K_2 in Theorem 2.1 are both comparable to $\|\mu\|$. The proof of this theorem can be found in [13].

We now recall some basic facts about λ-Hardy measures.

Definition 2.2. Let $0 < p, q < \infty$ and $\lambda = q/p$. We say a positive measure μ on $\mathbb{B}^n$ is a λ-Hardy Carleson measure if there exists a constant $C > 0$ such that for all $f \in \mathcal{H}^p(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} |f(z)|^q\,d\mu(z) \le C\|f\|^q_{\mathcal{H}^p}. \tag{2.11}$$

The following high dimension Peter Duren's characterization of λ-Hardy Carleson measures is useful for our purpose.

Proposition 2.3. *Let $0 < p, q < \infty$ and $\lambda = q/p > 1$. Let μ be a positive measure on $\mathbb{B}^n$. Then the following assertions are equivalent.*

(i) *There exists a constant $K_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and any $0 < \delta < 1$,*

$$\mu(Q_\delta(\xi)) \le K_1(\sigma(B_\delta(\xi)))^\lambda. \tag{2.12}$$

(ii) *There exists a constant $K_2 > 0$ such that*

$$\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n} K_a^\lambda(z)\,d\mu(z) < K_2 < \infty. \tag{2.13}$$

(iii) *There exists a constant $K_3 > 0$ such that for all $f \in \mathcal{H}^p(\mathbb{B}^n)$,*

$$\int_{\mathbb{B}^n} |f(z)|^q\,d\mu(z) \le K_3\|f\|^q_{\mathcal{H}^p}. \tag{2.14}$$

The constants K_1, K_2, and K_3 in the above proposition are equivalent. That (i) $\Leftrightarrow$ (ii) can be found in [11]. The equivalence (i) $\Leftrightarrow$ (iii) can be found in [14] for example. We have the following elementary consequence.

Corollary 2.4. *Let $0 \le p, q < \infty$, $p \ne 0$ and let μ be a positive measure on $\mathbb{B}^n$. Then the following assertion are equivalent.*

(i) *There exists a constant $K_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and any $0 < \delta < 1$,*

$$\mu(Q_\delta(\xi)) \le K_1(\sigma(B_\delta(\xi)))^{1+(q/p)}. \tag{2.15}$$

(ii) *There exists a constant* $K_2 > 0$ *such that*

$$\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n} K_a^{1+(q/p)}(z)\,d\mu(z) \le K_2 < \infty. \tag{2.16}$$

(iii) *There exists a constant* $K_3 > 0$ *such that for all* $f \in \mathscr{L}^p(\mathbb{B}^n)$,

$$\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n} K_a(z)\left|f(z)\right|^q d\mu(z) \le K_3\|f\|_{\mathscr{L}^p}^q. \tag{2.17}$$

(iv) *There exists a constant* $K_4 > 0$ *such that for all* $f \in \mathscr{L}^p(\mathbb{B}^n)$ *and any* $g \in \mathscr{L}^r(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} \left|f(z)\right|^q \left|g(z)\right|^r d\mu(z) \le K_4\|f\|_{\mathscr{L}^p}^q\|g\|_{\mathscr{L}^r}^r. \tag{2.18}$$

Proof. The equivalence (i)⇔(ii) is a special case of Proposition 2.3. Note that (iii) is equivalent in saying that for any $f \in \mathscr{L}^p(\mathbb{B}^n)$, the measure $(|f(z)|^q d\mu(z))/\|f\|_{\mathscr{L}^p}^q$ is a Carleson measure which is equivalent to (iv). The implication (iv)⇒(i) follows from the usual arguments. Thus, it only remains to prove that (ii)⇒(iii). First by Proposition 2.3, (ii) is equivalent in saying that there exists a constant $K_2' > 0$ such that for any $f \in \mathscr{L}^p(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} \left|f(z)\right|^{p+q} d\mu(z) \le K_2'\|f\|_{\mathscr{L}^p}^{p+q}. \tag{2.19}$$

It follows from the hypotheses, the latter, and Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{B}^n} K_a(z)\left|f(z)\right|^q d\mu(z) &\le \left(\int_{\mathbb{B}^n} K_a(z)^{1+(q/p)} d\mu(z)\right)^{p/(p+q)}\left(\int_{\mathbb{B}^n}\left|f(z)\right|^{p+q} d\mu(z)\right)^{q/(p+q)}, \\ &\le K_2K_2'\|f\|_{\mathscr{L}^p}^q.\end{aligned} \tag{2.20}$$

Thus (ii)⇒(iii). The proof is complete. □

Next, we recall the following result proved in [11].

Theorem 2.5. *Let* $0 \le p, q < \infty$, $s \ge 1$, *and let* μ *be a positive Borel measure on* $\mathbb{B}^n$. *Then the following conditions are equivalent.*

(i) *There is* $K_1 > 0$ *such that for any* $\xi \in \mathbb{S}^n$ *and* $0 < \delta < 1$,

$$\mu(Q_\delta(\xi)) \le K_1 \frac{(\sigma(B_\delta(\xi)))^s}{(\log(4/\delta))^p(\log\log(e^4/\delta))^q}. \tag{2.21}$$

(ii) *There is* $K_2 > 0$ *such that*

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)^p\left(\log\log\frac{e^4}{1-|a|}\right)^q\int_{\mathbb{B}^n}K_a(z)^s d\mu(z)\leq K_2<\infty. \tag{2.22}$$

(iii) *There is* $K_3 > 0$ *such that for any* $f \in BMOA$,

$$\sup_{a\in\mathbb{B}^n}\left(\log\log\frac{e^4}{1-|a|}\right)^q\int_{\mathbb{B}^n}K_a(z)^s\left|f(z)\right|^p d\mu(z)\leq K_3\|f\|_{BMOA}^p. \tag{2.23}$$

(iv) *There is* $K_4 > 0$ *such that for any* $g \in LMOA$,

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)^q\int_{\mathbb{B}^n}K_a(z)^s\left|g(z)\right|^q d\mu(z)\leq K\|g\|_{LMOA}^q. \tag{2.24}$$

(v) *There is* $K_5 > 0$ *such that for any* $f \in BMOA$ *and any* $g \in LMOA$,

$$\sup_{a\in\mathbb{B}^n}\int_{B^n}K_a(z)^s\left|f(z)\right|^p\left|g(z)\right|^q d\mu(z)\leq K_5\|f\|_{BMOA}^p\|\|_{LMOA}^q. \tag{2.25}$$

Definition 2.6. Let $0 < p, q < \infty$ and $\lambda = q/p$. Let ρ be a positive function defined on $[0,\infty)$. We say a positive measure μ on $\mathbb{B}^n$ is a λ-Hardy ρ-Carleson measure if for any $f \in \mathscr{H}^p(\mathbb{B}^n)$, the measure

$$d\tilde{\mu}(z)=\frac{\left|f(z)\right|^q}{\|f\|_p^q}d\mu(z) \tag{2.26}$$

is a ρ-Carleson measure.

We have the following characterization of λ-Hardy ρ-Carleson measure which is in fact an alternative interpretation of Theorem 2.5.

Theorem 2.7. *Let* $0 \leq p, q, r, s < \infty$, $s \neq 0$, *and let* μ *be a positive Borel measure on* $\mathbb{B}^n$. *Then the following conditions are equivalent.*

(i) *There is* $K_1 > 0$ *such that for any* $\xi \in \mathbb{S}^n$ *and* $0 < \delta < 1$,

$$\mu(Q_\delta(\xi))\leq K_1\frac{(\sigma(B_\delta(\xi)))^{1+(r/s)}}{(\log(4/\delta))^p(\log\log(e^4/\delta))^q}. \tag{2.27}$$

(ii) *There is* $K_2 > 0$ *such that for any* $f \in BMOA$, *and any* $h \in \mathscr{H}^s(\mathbb{B}^n)$,

$$\sup_{a\in\mathbb{B}^n}\left(\log\log\frac{e^4}{1-|a|}\right)^q\int_{\mathbb{B}^n}K_a(z)|h(z)|^r\left|f(z)\right|^p d\mu(z)\leq K_2\|h\|_{\mathscr{H}^s}^r\|f\|_{BMOA}^p. \tag{2.28}$$

(iii) *There is* $K_3 > 0$ *such that for any* $g \in LMOA$, *and any* $h \in \mathcal{H}^s(\mathbb{B}^n)$,

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)^p\int_{\mathbb{B}^n}K_a(z)|h(z)|^r|g(z)|^q d\mu(z) \leq K_2\|h\|_{\mathcal{H}^s}^r\|g\|_{LMOA}^q. \tag{2.29}$$

(iv) *There is* $K_4 > 0$ *such that for any* $f \in BMO$, *any* $g \in LMOA$, *and any* $h \in \mathcal{H}^s(\mathbb{B}^n)$,

$$\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n}K_a(z)|h(z)|^r|f(z)|^p|g(z)|^q d\mu(z) \leq K_3\|h\|_{\mathcal{H}^s}^r\|f\|_{BMOA}^p\|g\|_{LMOA}^q. \tag{2.30}$$

(v) *There is* $K_5 > 0$ *such that for any* $f \in BMOA$, $g \in LMOA$, *and any* $h \in \mathcal{H}^s(\mathbb{B}^n)$ *and* $l \in \mathcal{H}^m(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n}|f(z)|^p|g(z)|^q|h(z)|^r|l(z)|^m d\mu(z) \leq K_5\|h\|_{\mathcal{H}^s}^r\|l\|_{\mathcal{H}^m}^m\|f\|_{BMOA}^p\|g\|_{LMOA}^q. \tag{2.31}$$

Proof. (i)⇔(iv): we first observe with Theorem 2.5 that (i) is equivalent in saying that there is a constant C_1 such that for any $f \in \mathrm{BMO}A$ and any $g \in \mathrm{LMO}A$,

$$\sup_{a\in\mathbb{B}^n}\int_{\mathbb{B}^n}K_a(z)^{1+(r/s)}|f(z)|^p|g(z)|^q d\mu(z) \leq C_1\|f\|_{\mathrm{BMO}A}^p\|g\|_{\mathrm{LMO}A}^q. \tag{2.32}$$

By Corollary 2.4, the latter is equivalent to (iv).

(ii)⇔(iii)⇔(iv): by rewriting (ii) as

$$\sup_{a\in\mathbb{B}^n}\left(\operatorname{loglog}\frac{e^4}{1-|a|}\right)^q\int_{\mathbb{B}^n}K_a(z)|f(z)|^p d\tilde{\mu}(z) \leq K_2\|h\|_{\mathcal{H}^s}^r\|f\|_{\mathrm{BMO}A}^p, \tag{2.33}$$

where $d\tilde{\mu}(z) = (|h(z)|^r/\|h\|_{\mathcal{H}^s}^r)d\mu(z)$, it follows directly from Theorem 2.5 that (ii)⇔(iii)⇔(iv).

That (iv)⇔(v) is a consequence of Theorem 2.1. The proof is complete. □

2.2. *p*-Carleson Measures for Hardy-Orlicz Spaces

In this section, we characterize p-Carleson measures of some special Hardy-Orlicz spaces. For this, we will need a weak factorization result of functions in these spaces which follows from the one in [10].

Proposition 2.8. *Let* $0 < s \leq 1$. *Let* $\mathcal{H}_s(\mathbb{B}^n)$ *denote the Hardy-Orlicz space corresponding to the function* $\Phi(t) = (t/\log(e+t))^s$. *Then the following assertions hold.*

(i) *The product of two functions, one in* $\mathcal{H}^s(\mathbb{B}^n)$ *and the other one in BMOA, is in* $\mathcal{H}_s(\mathbb{B}^n)$. *Moreover,*

$$\|fg\|_{\mathcal{H}_s} \lesssim \|f\|_{\mathcal{H}^s}\|g\|_{BMOA}. \tag{2.34}$$

(ii) *Any function f in the unit ball of $\mathcal{H}_s(\mathbb{B}^n)$ admits the following representation (weak factorization):*

$$f = \sum_j f_j g_j, \qquad f_j \in \mathcal{H}^s(\mathbb{B}^n), \; g_j \in BMOA \tag{2.35}$$

with

$$\sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^s} \|g_j\|_{BMOA} \lesssim \|f\|_{\mathcal{H}_s}. \tag{2.36}$$

Let us remark that the space $\mathcal{H}_1(\mathbb{B}^n)$ is the predual of LMOA. The following theorem gives a characterization of p-Carleson measures of the Hardy-Orlicz spaces considered here.

Theorem 2.9. *Let $0 < s \leq 1$, $1 \leq p < \infty$. Let $\mathcal{H}_s(\mathbb{B}^n)$ be the Hardy-Orlicz space $\mathcal{H}^\Phi(\mathbb{B}^n)$ corresponding to the function $\Phi(t) = (t/\log(e+t))^s$. Then, for μ a positive measure on $\mathbb{B}^n$, the following assertions are equivalent.*

(i) *There exists a constant $K_1 > 0$ such that for any $\xi \in \mathbb{S}^n$ and any $0 < \delta < 1$,*

$$\mu(Q_\delta(\xi)) \leq K_1 \frac{(\sigma(B_\delta(\xi)))^{(p/s)}}{(\log(4/\delta))^p}. \tag{2.37}$$

(ii) *There exists a constant $K_2 > 0$ such that for any $f \in \mathcal{H}_s(\mathbb{B}^n)$,*

$$\int_{\mathbb{B}^n} |f(z)|^p d\mu(z) \leq K_2 \|f\|_{\mathcal{H}_s}^p. \tag{2.38}$$

Proof. We remark that if (2.38) holds in the unit ball of $\mathcal{H}_s(\mathbb{B}^n)$, then it holds for all $f \in \mathcal{H}_s(\mathbb{B}^n)$. Recall that by Proposition 2.8, every function f in the unit ball of $\mathcal{H}_s(\mathbb{B}^n)$ weakly factorizes as

$$f = \sum_{j=0}^{\infty} f_j g_j \tag{2.39}$$

and $\sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^s} \|g_j\|_{\mathrm{BMOA}} \lesssim \|f\|_{\mathcal{H}_s}$. It follows using the equivalent assertion (iv) of Theorem 2.7 that

$$\begin{aligned}\left(\int_{\mathbb{B}^n} |f(z)|^p d\mu(z)\right)^{1/p} &= \left(\int_{\mathbb{B}^n} \left|\sum_{j=0}^{\infty} f_j(z) g_j(z)\right|^p d\mu(z)\right)^{1/p} \\ &\leq \sum_{j=0}^{\infty} \left(\int_{\mathbb{B}^n} |f_j(z) g_j(z)|^p d\mu(z)\right)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty} \left(\int_{\mathbb{B}^n} |f_j(z)|^s |f_j(z)|^{p-s} |g_j(z)|^p d\mu(z) \right)^{1/p} \\
&\lesssim \sum_{j=0}^{\infty} \left(\|f_j\|_{\mathcal{H}^s}^s \|f_j\|_{\mathcal{H}^s}^{p-s} \|g_j\|_{\mathrm{BMO}A}^p \right)^{1/p} \\
&= \sum_{j=0}^{\infty} \|f_j\|_{\mathcal{H}^s} \|g_j\|_{\mathrm{BMO}A} \lesssim \|f\|_{\mathcal{H}_s}.
\end{aligned}
\tag{2.40}
$$

Now we prove that (ii)⇒(i). That (ii) holds implies in particular that for any $f \in \mathcal{H}^s(\mathbb{B}^n)$ and any $g \in \mathrm{BMO}A$,

$$
\int_{\mathbb{B}^n} |f(z)|^p |g(z)|^p d\mu(z) \leq K_2 \|f\|_{\mathcal{H}^s}^p \|g\|_{\mathrm{BMO}A}^p. \tag{2.41}
$$

We observe with Corollary 2.4 that (2.41) is equivalent in saying that for any $g \in \mathrm{BMO}A$, the measure

$$
d\tilde{\mu}(z) = \frac{|g(z)|^p}{\|g\|_{\mathrm{BMO}A}^p} d\mu(z) \tag{2.42}
$$

is a (p/s)-Carleson measure or equivalently,

$$
\sup_{a \in \mathbb{B}^n} \int_{\mathbb{B}^n} K_a(z)^{(p/s)} |g(z)|^p d\mu(z) \leq K_3 \|g\|_{\mathrm{BMO}A}^p. \tag{2.43}
$$

By Theorem 2.5, the latter is equivalent to

$$
\sup_{a \in \mathbb{B}^n} \left(\log \frac{4}{1-|a|} \right)^p \int_{\mathbb{B}^n} K_a(z)^{(p/s)} d\mu(z) \leq K_4, \tag{2.44}
$$

which is equivalent to (i). The proof is complete. □

3. Some Applications

We provide in this section with some applications of p-Carleson measures of the above Hardy-Orlicz spaces to the boundedness of multiplication operators, composition operators, and Cesàro integral-type operators. Let us first introduce the generalized Bergman spaces in the unit ball. We recall that for $f \in H(\mathbb{B}^n)$, its radial derivative Rf is the holomorphic function defined by

$$
Rf(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z). \tag{3.1}
$$

Let $\alpha \in \mathbb{R}$, $1 \leq p < \infty$ with $\alpha + p > -1$. The generalized Bergman space $\mathcal{A}^p_\alpha(\mathbb{B}^n)$ consists of holomorphic function f such that

$$\|f\|^p_{p,\alpha} := \int_{\mathbb{B}^n} |Rf(z)|^p \left(1 - |z|^2\right)^{\alpha+p} dV(z) < \infty. \tag{3.2}$$

Clearly, $\mathcal{A}^p_\alpha(\mathbb{B}^n)$ is a Banach under

$$\|f\|^p_{p,\alpha} := |f(0)| + \int_{\mathbb{B}^n} |Rf(z)|^p \left(1 - |z|^2\right)^{\alpha+p} dV(z) < \infty. \tag{3.3}$$

These spaces have been studied in [15]. When $\alpha > -1$, the space $\mathcal{A}^p_\alpha(\mathbb{B}^n)$ corresponds to the usual weighted Bergman space which consists of holomorphic function f in $\mathbb{B}^n$ such that

$$\|f\|^p_{p,\alpha} := \int_{\mathbb{B}^n} |f(z)|^p \left(1 - |z|^2\right)^{\alpha} dV(z) < \infty. \tag{3.4}$$

For $\alpha = -1$ and $p = 2$, the corresponding space is just the Hardy space $\mathcal{H}^2(\mathbb{B}^n)$.

Let u be a holomorphic function in $\mathbb{B}^n$. We denote by $\mathcal{M}_u$ the multiplication operator by u defined on $H(\mathbb{B}^n)$ by

$$\mathcal{M}_u(f)(z) = u(z)f(z), \quad f \in H(\mathbb{B}^n). \tag{3.5}$$

We recall that if φ is a holomorphic self map of $\mathbb{B}^n$, then the composition operator C_φ is defined on $H(\mathbb{B}^n)$ by

$$C_\varphi(f) := f \circ \varphi. \tag{3.6}$$

For u a holomorphic function in $\mathbb{B}^n$, the weighted composition operator uC_φ is the composition operator followed by the multiplication by u. That is,

$$uC_\varphi(f) = \mathcal{M}_u(f \circ \varphi) = u(f \circ \varphi). \tag{3.7}$$

For b a holomorphic function in $\mathbb{B}^n$, the Cesàro-type integral operator T_b is defined by

$$T_b(f)(z) = \int_0^1 f(tz)Rg(tz)\frac{dt}{t}, \quad g, f \in \mathrm{H}(\mathbb{B}^n). \tag{3.8}$$

Combining this operator with the weighted composition operator, we obtain a more general operator $T_{u,\varphi,b} = T_b(\mathcal{M}_u(f \circ \varphi)) = T_b(u(f \circ \varphi))$ given by

$$T_{u,\varphi,b}(f)(z) = \int_0^1 u(tz)(f \circ \varphi)(tz)Rg(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}^n). \tag{3.9}$$

When $\varphi(z) = z$ for all $z \in \mathbb{B}^n$, we write $T_{u,\varphi,b} = T_{u,b}$. The multiplication operator, the composition operator, the Cesàro-type integral, and their products have been intensively studied by many authors on various holomorphic function spaces. We refer to the following and the references therein [11, 12, 16–30]. As an application of the characterization of p-Carleson measures for the Hardy-Orlicz spaces of the previous section, we consider boundedness criteria of the above operators from Hardy-Orlicz spaces to (generalized) weighted Bergman spaces and weighted BMOA spaces in the unit ball. We have the following result.

Theorem 3.1. *Let $0 < s \leq 1$, $1 \leq p < \infty$ and, $\alpha > -1$. Then uC_φ is bounded from $\mathcal{H}_s(\mathbb{B}^n)$ to $\mathcal{A}^p_\alpha(\mathbb{B}^n)$ if and only if*

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{(1-|a|)}\right)^p\int_{\mathbb{B}^n}\left(\frac{\left(1-|a|^2\right)^n}{\left|1-\langle\varphi(z),a\rangle\right|^{2n}}\right)^{(p/s)}|u(z)|^p\left(1-|z|^2\right)^\alpha dV(z) < \infty. \tag{3.10}$$

Proof. Clearly, that uC_φ is bounded from $\mathcal{H}_s(\mathbb{B}^n)$ to $\mathcal{A}^p_\alpha(\mathbb{B}^n)$ is equivalent in saying that there is a constant $C > 0$ such that for any $f \in \mathcal{H}_s(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n}\left|f\circ\varphi(z)\right|^p|u(z)|^p\left(1-|z|^2\right)^\alpha dV(z) \leq C\|f\|^p_{\mathcal{H}_s}. \tag{3.11}$$

Let us write $dV_\alpha(z) = (1-|z|^2)^\alpha dV(z)$, $dV_{\alpha,u}(z) = |u(z)|^p dV_\alpha(z)$. If $\mu = V_{\alpha,u}\circ\varphi^{-1}$, then an easy change of variables gives that (3.11) is equivalent to

$$\int_{\mathbb{B}^n}\left|f(z)\right|^p d\mu(z) \leq C\|f\|^p_{\mathcal{H}_s}. \tag{3.12}$$

The latter inequality is equivalent in saying that the measure μ is a p-Carleson measure for $\mathcal{H}_s(\mathbb{B}^n)$. It follows from Theorem 2.9 and the equivalent definitions in Theorem 2.7 that (3.11) is equivalent to

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{(1-|a|)}\right)^p\int_{\mathbb{B}^n}\left(\frac{\left(1-|a|^2\right)^n}{|1-\langle w,a\rangle|^{2n}}\right)^{p/s} d\mu(w) < \infty. \tag{3.13}$$

Changing the variables back, we finally obtain that uC_φ is bounded from $\mathcal{H}^{\Phi_s}(\mathbb{B}^n)$ to $\mathcal{A}^p_\alpha(\mathbb{B}^n)$ if and only if

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{(1-|a|)}\right)^p\int_{\mathbb{B}^n}\left(\frac{\left(1-|a|^2\right)^n}{\left|1-\langle\varphi(z),a\rangle\right|^{2n}}\right)^{(p/s)}|u(z)|^p\left(1-|z|^2\right)^\alpha dV(z) < \infty. \tag{3.14}$$

The proof is complete. □

Remarking that one has

$$R(T_b f)(z) = f(z) Rb(z) \quad \text{for any } g, f \in H(\mathbb{B}^n), \tag{3.15}$$

we prove in the same way the following result.

Theorem 3.2. *Let $0 < s \leq 1$, $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$ with $\alpha + p > -1$. Then $T_{u,\varphi,b}$ is bounded from $\mathcal{H}_s(\mathbb{B}^n)$ to $\mathcal{A}^p_\alpha(\mathbb{B}^n)$ if and only if*

$$\sup_{a\in\mathbb{B}^n} \left(\log \frac{4}{1-|a|}\right)^p \int_{\mathbb{B}^n} \left(\frac{\left(1-|a|^2\right)^n}{\left|1-\langle \varphi(z), a\rangle\right|^{2n}} \right)^{p/s} d\mu(z) < \infty, \tag{3.16}$$

where $d\mu(z) = |u(z)|^p |Rb(z)|^p (1-|z|^2)^{\alpha+p} dV(z)$.

Let us consider now the operator $T_{u,b}$. We have the following:

Theorem 3.3. *Let $0 < s \leq 1, 0 \leq p, q < \infty$, and $\alpha > -1$. Let $1/\rho(t) = (\log(4/t))^p (\operatorname{loglog}(e^4/t))^q$. Then $T_{u,b}$ is bounded from $\mathcal{H}_s(\mathbb{B}^n)$ to $BMOA(\rho)$, if and only if*

$$\sup_{a\in\mathbb{B}^n} \left(\log \frac{4}{1-|a|}\right)^{2(p+1)} \left(\operatorname{loglog} \frac{e^4}{1-|a|}\right)^{2q} \int_{\mathbb{B}^n} \left(\frac{\left(1-|a|^2\right)^n}{|1-\langle z, a\rangle|^{2n}} \right)^{1+(2/s)} d\mu(z) < \infty, \tag{3.17}$$

with $d\mu(z) = |u(z)|^2 |Rb(z)|^2 (1-|z|^2) dV(z)$.

Proof. We recall that a function h is in BMO$A(\rho)$ if and only if the measure $|Rh(z)|^2(1 - |z|^2)dV(z)$ is a $(1/\rho^2)$-Carleson measure (see [31]). That is

$$\sup_{a\in\mathbb{B}^n} \left(\log \frac{4}{1-|a|}\right)^{2p} \left(\operatorname{loglog} \frac{e^4}{1-|a|}\right)^{2q} \int_{\mathbb{B}^n} \frac{\left(1-|a|^2\right)^n}{|1-\langle z, a\rangle|^{2n}} |Rh(z)|^2 \left(1-|z|^2\right) dV(z) < \infty. \tag{3.18}$$

It follows that $T_{u,b}$ is bounded from $\mathcal{H}_s(\mathbb{B}^n)$ to BMO$A(\rho)$ if and only if for any $f \in \mathcal{H}_s(\mathbb{B}^n)$,

$$\sup_{a\in\mathbb{B}^n} \left(\log \frac{4}{1-|a|}\right)^{2p} \left(\operatorname{loglog} \frac{e^4}{1-|a|}\right)^{2q} \int_{\mathbb{B}^n} \frac{\left(1-|a|^2\right)^n}{|1-\langle z, a\rangle|^{2n}} \left|f(z)\right|^2 d\mu(z) \leq C\|f\|^2_{\mathcal{H}_s}, \tag{3.19}$$

$d\mu(z) = |u(z)|^2|Rb(z)|^2(1-|z|^2)dV(z)$. By the equivalent definition in Theorem 2.7, this is equivalent in saying that for any $f_1 \in \mathrm{BMO}A$, $f_2 \in \mathrm{LMO}A$, and any $g \in \mathcal{H}^m(\mathbb{B}^n)$,

$$\int_{\mathbb{B}^n} |f(z)|^2 |g(z)|^m |f_1(z)|^{2p} |f_2(z)|^{2q} d\mu(z) \leq C\|f\|^2_{\mathcal{H}_s} \|f_1\|^{2p}_{\mathrm{BMO}A} \|f_2\|^{2q}_{\mathrm{LMO}A} \|g\|^m_m, \tag{3.20}$$

which is equivalent in saying that the measure

$$d\tilde{\mu}(z) = \frac{|f_1(z)|^{2p}|f_2(z)|^{2q}}{\|f_1\|^{2p}_{\mathrm{BMO}A}\|f_2\|^{2q}_{\mathrm{LMO}A}} \frac{|g(z)|^m}{\|g\|^m_m} |u(z)|^2|Rb(z)|^2\left(1-|z|^2\right)dV(z) \tag{3.21}$$

is a 2-Carleson measure for $\mathcal{H}_s(\mathbb{B}^n)$. It follows from the equivalent definitions of Theorems 2.7 and 2.9 that the latter is equivalent to

$$\sup_{a\in\mathbb{B}^n}\left(\log\frac{4}{1-|a|}\right)^{2(p+1)}\left(\operatorname{loglog}\frac{e^4}{1-|a|}\right)^{2q}\int_{\mathbb{B}^n}\left(\frac{\left(1-|a|^2\right)^n}{|1-\langle z,a\rangle|^{2n}}\right)^{1+(2/s)} d\mu(z) < \infty. \tag{3.22}$$

The proof is complete. □

The methods used in this text are quite specific to the case considered here, that is, the embedding $I_\mu : \mathcal{H}_s(\mathbb{B}^n) \to L^p(\mathbb{B}^n)$. We remark that even in the case $0 < s \leq p < 1$, the condition (i) of Theorem 2.9 is still necessary. The proof given here does not allow to say if it is sufficient. In general, the characterization of those positive measures μ on $\mathbb{B}^n$ such that the embedding map $I_\mu : \mathcal{H}^{\Phi_1}(\mathbb{B}^n) \to \mathcal{H}^{\Phi_2}(\mathbb{B}^n)$ ($\Phi_1 \neq \Phi_2$ if Φ_1 and Φ_2 are convex growth functions) is bounded, is still open.

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