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Research Article

Orthogonal Polynomials of Compact Simple Lie Groups

Maryna Nesterenko,[1] Jiří Patera,[2] and Agnieszka Tereszkiewicz[3]

[1] *Department of Applied Research, Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Street, Kyiv-4 01601, Ukraine*
[2] *Centre de Recherches Mathématiques, Université de Montréal, C.P.6128-Centre Ville, Montréal, QC H3C 3J7, Canada*
[3] *Institute of Mathematics, University of Bialystok, Akademicka 2, 15-267 Bialystok, Poland*

Correspondence should be addressed to Maryna Nesterenko, maryna@imath.kiev.ua

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Recursive algebraic construction of two infinite families of polynomials in n variables is proposed as a uniform method applicable to every semisimple Lie group of rank n. Its result recognizes Chebyshev polynomials of the first and second kind as the special case of the simple group of type A_1. The obtained not Laurent-type polynomials are equivalent to the partial cases of the Macdonald symmetric polynomials. Recurrence relations are shown for the Lie groups of types A_1, A_2, A_3, C_2, C_3, G_2, and B_3 together with lowest polynomials.

1. Introduction

The majority of special functions and orthogonal polynomials introduced during the last decade are associated with Lie groups or their generalizations. In particular, special functions of mathematical physics are in fact matrix elements of representations of Lie groups [1] and recent multivariate generalizations of classical hypergeometric orthogonal polynomials are based on root systems of simple Lie groups/algebras [2–9]. In this connection a number of elegant results in theory of these polynomials, such as explicit (determinantal) computation of polynomials [10–12] and Pieri formulas [13, 14], were obtained, see also [15–17] and references therein.

Our primary objective at this stage is to establish a constructive method for finding orthogonal multivariate polynomials related to orbit functions of simple Lie groups of rank n, indeed, for actually seeing them. As far as we deal with the functions invariant/skew-invariant under the action of the corresponding Weyl group the obtained polynomials appear as building blocks in all multivariate polynomials associated with root systems. Unlike Gram-Schmidt type orthogonalization of the monomial basis with respect to Haar measure [3, 7, 8] or determinantal construction of polynomials [10–12] we make profit from decomposition of products of Weyl group orbits and from basic properties of the characters of irreducible finite dimensional representations.

Our method is purely algebraic and we propose three different ways to transform a C- or S-orbit function into a polynomial. The first one substitutes for each multivariable exponential term in an orbit function a monomial of as many variables. In $1D$ this results in Chebyshev polynomials written as Laurent polynomials with symmetrically placed positive and negative powers of the variable; and in the case of A_2 our results coincide with those from [18].

The second transformation, the "truly trigonometric" form, is based on the fact that, for many simple Lie algebras (see the list in (3.3) below), each C and S-orbit function consists of pairs of exponential terms that add up to either cosine or sine. Hence such a function is a sum of trigonometric terms (or a polynomial of one-dimensional Chebyshev polynomials). For the Chebyshev polynomials we obtain in this way their trigonometric form. Note from (3.3) that this method does not apply to the groups A_n for $n > 1$.

But this paper focuses on polynomials obtained by the third substitution of variables, mimicking Weyl's method for the construction of finite-dimensional representations from n fundamental representations. Thus the C polynomials have n variables that are the C-orbit functions, one for each fundamental weight ω_j. This approach results in a simple recursive construction that allows one to represent any orbit function/monomial symmetric function in non-Laurent polynomial form.

In addition to the general approach and associated tools we present a lot of explicit and practically useful data and discussions, namely, in Appendix A we compare the classical Chebyshev polynomials (Dickson polynomials) and orbit functions of A_1 with their recursion relations. Suitably normalized, the Chebyshev polynomials of the first and second kind coincide with the C and S polynomials. A table of the polynomials of each kind is presented. Appendices B, C, and D contain, respectively, the recursion relations for polynomials of the Lie algebras A_2, C_2, and G_2. In Appendix E recursion relations for A_3, B_3, and C_3 polynomials of both kinds are listed together with useful tools for solving these recursion relations.

2. Preliminaries and Conventions

This section serves to fix notations and terminology, additional details can be found for example in [19–26].

Let $\mathbb{R}^n$ be the Euclidean space spanned by the simple roots of a simple Lie group G. The basis of the simple roots and the basis of fundamental weights are hereafter referred to as the α-basis and ω-basis, respectively. Bases dual to α- and ω-bases are denoted by $\check{\alpha}$- and $\check{\omega}$-bases. In addition we use $\{e_1, \dots, e_n\}$, the orthonormal basis of $\mathbb{R}^n$. The root lattice Q and the weight lattice P of G are formed by all integer linear combinations of the α-basis and ω-basis, respectively. In P we define the cone of dominant weights P^+ and its subset of strictly dominant weights P^{++}.

Hereafter $W = W(G)$ is the Weyl group of size $|W_\lambda|$, and W_λ is the orbit containing the (dominant) point $\lambda \in P^+ \subset \mathbb{R}^n$. The fundamental region $F(G) \subset \mathbb{R}^n$ is the convex hull of the vertices $\{0, (\omega_1/q_1), \dots, (\omega_n/q_n)\}$, where q_j, $j = \overline{1, n}$ are comarks of the highest root.

Definition 2.1. The C-function $C_\lambda(x)$ is defined as

$$C_\lambda(x) := \sum_{\mu\in W_\lambda(G)} e^{2\pi i\langle \mu,x\rangle}, \quad x \in \mathbb{R}^n,\ \lambda \in P^+. \tag{2.1}$$

Occasionally it is useful to scale up C_λ of nongeneric λ by the stabilizer of λ in W.

Definition 2.2. The S-function $S_\lambda(x)$ is defined as

$$S_\lambda(x) := \sum_{\mu\in W_\lambda(G)} (-1)^{p(\mu)} e^{2\pi i\langle \mu,x\rangle}, \quad x \in \mathbb{R}^n,\ \lambda \in P^{++}, \tag{2.2}$$

where $p(\mu)$ is the number of elementary reflections necessary to obtain μ from λ.

In this paper, we always suppose that $\lambda,\mu \in P$ are given in ω-basis and $x \in \mathbb{R}^n$ is given in $\check{\alpha}$-basis, hence the orbit functions have the following forms:

$$\begin{aligned} C_\lambda(x) &= \sum_{\mu\in W_\lambda} e^{2\pi i \sum_{j=1}^n \mu_j x_j} = \sum_{\mu\in W_\lambda} \prod_{j=1}^n e^{2\pi i \mu_j x_j}, \\ S_\lambda(x) &= \sum_{\mu\in W_\lambda} (-1)^{p(\mu)} e^{2\pi i \sum_{j=1}^n \mu_j x_j} = \sum_{\mu\in W_\lambda} (-1)^{p(\mu)} \prod_{j=1}^n e^{2\pi i \mu_j x_j}. \end{aligned} \tag{2.3}$$

There is a fundamental relation between the C- and S-orbit functions for simple Lie group G of any type and rank, called the Weyl character formula:

$$\chi_\lambda(x) = \frac{S_{\lambda+\rho}(x)}{S_\rho(x)} = \sum_\mu m_\mu^\lambda C_\mu(x), \quad x \in \mathbb{R}^n,\ \lambda,\mu \in P^+,\ \rho = \sum_{k=1}^n \omega_k. \tag{2.4}$$

The positive integer m_μ^λ is the Kostka number [27, 28].

The rank of the underlying semisimple Lie group/algebra is the number of variables of the orbit functions. C and S functions are continuous and have continuous derivatives; they are, respectively, symmetric and antisymmetric with respect to the $(n-1)$-dimensional boundary of F [23–25]. Moreover, any pair of orbit functions from the same family is orthogonal on the corresponding fundamental region [20], these families of functions are complete, and $C_\lambda(x)$- and $S_\lambda(x)$-orbit functions are eigenfunctions of the n-dimensional Laplace operator.

3. Multivariate Orthogonal Polynomials Corresponding to Orbit Functions

In this section we consider several transformations

$$\mathfrak{T} : \mathbb{R}^n \longrightarrow \mathbb{C}^n; \quad (x_1,\ldots,x_n) \longmapsto (X_1,\ldots,X_n), \tag{3.1}$$

that represent the $C(x)$- and $S(x)$-orbit functions in polynomial form.

It directly follows from the orthogonality of the orbit functions that such polynomials are orthogonal on the domain $\widetilde{F}$ with the weight function $\det^{-1}(D(X)/D(x))$, where $\widetilde{F}$ is the image of the fundamental region F under the transformation $\mathfrak{T}$.

(i) The first type of transformation is rather straightforward

$$X_j := e^{2\pi i x_j}, \quad x_j \in \mathbb{R}, \ j = 1, 2, \ldots, n. \tag{3.2}$$

Polynomial summands are products $\prod_{j=1}^{n} X_j^{\mu_j}$, where $\mu_j \in \mathbb{Z}$ are components of the orbit points relative to a suitable basis. Under this transformation orbit function, $C_\lambda(x)$ and $S_\lambda(x)$, given by (2.3), become Laurent polynomials in n variables X_j, where $j = \{1, 2, \ldots, n\}$.

The exponential substitution polynomials are complex-valued in general, admit negative powers, and have all their coefficients equal to one in C polynomials, and 1 or -1 in S polynomials.

(ii) The W-orbits of the Lie groups

$$A_1, \quad B_n \ (n \geq 3), \quad C_n \ (n \geq 2), \quad D_{2n} \ (n \geq 2), \quad E_7, \quad E_8, \quad F_4, \quad G_2 \tag{3.3}$$

have an additional property $\pm\mu \in W_\lambda(L)$ for all $\mu \in W_\lambda(L)$, then the pair of corresponding terms of the function of $W_\lambda(L)$ can be combined so that $C_\lambda(x)$ and $S_\lambda(x)$ become linear combinations of cosines and sines:

$$\begin{aligned} e^{2\pi i\langle \mu, x\rangle} + e^{-2\pi i\langle \mu, x\rangle} &= 2\cos\bigl(2\pi\langle \mu, x\rangle\bigr) \in C_\lambda(x), \\ e^{2\pi i\langle \mu, x\rangle} - e^{-2\pi i\langle \mu, x\rangle} &= 2i\sin\bigl(2\pi\langle \mu, x\rangle\bigr) \in S_\lambda(x), \end{aligned} \tag{3.4}$$

that admits a truly trigonometric substitution of variables. Note that in the case of A_1, this is precisely the trigonometric substitution made for Chebyshev polynomials of the first and second kind.

Remark 3.1. Chebyshev polynomials of one variable play crucial role for the orbit functions of the above-mentioned Lie groups as far as they allow us to calculate the polynomial coefficients explicitly.

Really, as far as we suppose that λ are given in ω-basis and x is given in $\check{\alpha}$-basis, then, using common trigonometric identities, cosines and sines can be expressed through the cosines and sines of $2\pi k x_i$, $k \in \mathbb{N}$, $i = 1, \ldots, n$. What immediately represents our orbit functions as polynomials of Chebyshev polynomials of the first $T_k(x_i)$ and second $U_k(x_i)$ kind with well-known formulas for coefficients.

(iii) The third transformation, which we propose here, works uniformly for simple Lie algebras of all types. We choose C functions of the n fundamental weights as the n new variables:

$$X_j := X_j(x) := C_{\omega_j}(x), \quad j = 1, 2, \ldots, n, \ x \in \mathbb{R}^n \tag{3.5}$$

completed by one more variable, the lowest S function,

$$\mathbf{S} := S_\rho(x), \quad x \in \mathbb{R}^n, \ \rho = (1, 1, \dots, 1) = \sum_{m=1}^{n} \omega_m. \tag{3.6}$$

The recursive construction of *C-polynomials* begins by multiplying the variables X_j and C-functions and decomposing their products into sums of C-polynomials. A judicious choice of the sequence of products allows one to find ever higher degree C-polynomials.

First, *generic recursion relations* are found as the decomposition of products $X_j C_{(a_1,a_2,\dots,a_n)}$ with "sufficiently large" $a_1, a_2, \dots, a_n$ (i.e., all C functions in the decomposition should correspond to generic points). Then the rest of necessary recursions ("*additional*") are constructed. An efficient way to find the decompositions is to work with products of Weyl group orbits, rather than with orbit functions. Their decomposition has been studied, and many examples have been described in [29]. Note that these recursion relations are always linear and the corresponding matrix is triangular. The procedure is exemplified in Appendices A–E for Simple Lie groups of ranks 1, 2, and 3.

Results of the recursive procedures can be summarized as follows (see [30] for the proof).

Proposition 3.2. *Any irreducible C-function and any character χ_λ of a simple Lie group G can be represented as a polynomial of C-functions of the fundamental weights $\omega_1, \dots, \omega_n$, that is, a polynomial in the variables $X_1, X_2, \dots, X_n$.*

The recursive construction of *S-polynomials* starts by multiplying the variables $\mathbf{S}$ and X_j and decomposing their products into sums of S polynomials. However, the higher the rank of the underlying Lie algebra, the recursive procedure for S polynomials becomes more laborious, what caused by the presence of negative terms in S polynomials. Fortunately, there is an alternative to the recursive procedure. Once the C polynomials have been calculated, they can be used in Weyl character formula for finding S polynomials as sums of C polynomials multiplied by the variable $\mathbf{S}$. In practice, polynomials $S_\lambda/\mathbf{S}$ should be used instead of S_λ.

Remark 3.3. There are two easy and practical checks on recursion relations applicable to all simple Lie algebras. The first one is the equality of numbers of exponential terms in S- or C-functions on both sides of a recursion relation (the numbers of exponential terms are calculated using the sizes of Weyl group orbits). The second check is the equality of congruence numbers.

Remark 3.4. Polynomial forms of C and S functions introduced in this section are partial cases of the Macdonald symmetric polynomials.

All C- and S-orthogonal polynomials (and, therefore, the Macdonald polynomials) inherit from orbit functions important discretization properties. A uniform discretization of these polynomials follows from their invariance with respect to the affine Weyl group of G and from the well-established discretization of the fundamental region $F(G)$ [20]. One more advantage is the cubature formula introduced in [31].

For the application reason in Appendices A–E we present recurrence relations and lowest polynomials for the simple Lie groups A_1, A_2, C_2, G_2, A_3, B_3, and C_3. All cases contain both generic and additional recursions or, instead of cumbersome additional recursions, we present all their solutions in form of lowest polynomials. The skipped explicit formulas are available in [30, 32].

The content of Appendices A–E is also motivated by the fact that calculation of additional recurrences is not suitable for complete computer automatization. However, as soon as additional recurrences (or their solutions) were obtained, all other calculations concerning polynomials and their applications become very algorithmic and can easily be done by computer algebra packages for Lie theory.

4. Conclusion

There is an alternative way to our construction of the polynomials in all but in the A_n cases. The crucial substitution (3.5) can be replaced by

$$X_k := \chi_{\omega_k}(x), \quad k = 1, 2, \ldots, n. \tag{4.1}$$

In (4.1) the variables are characters of irreducible representations with highest weights given as the fundamental weights, while in (3.5) the variables are C functions of the fundamental weights. Only for A_n the two coincide, $C_{\omega_k}(x) = \chi_{\omega_k}(x)$ for all $k = 1, \ldots, n$ and for all $x \in \mathbb{R}^n$. Already for the rank two cases other than A_2 there is a difference. Indeed, (4.1) reads as follows:

$$\begin{aligned} C_2\colon X_1 &= \chi_{\omega_1}(x) = C_{\omega_1}(x), \qquad X_2 = \chi_{\omega_2}(x) = C_{\omega_2}(x) + 2, \\ G_2\colon X_1 &= \chi_{\omega_1}(x) = C_{\omega_1}(x) + C_{\omega_2}(x) + 2, \qquad X_2 = \chi_{\omega_2}(x) = C_{\omega_2}(x) + 1. \end{aligned} \tag{4.2}$$

Since products of characters decompose into their sum, the recursive construction can proceed, but the polynomials will be different.

For simplicity of formulation, we insisted throughout this paper that the underlying Lie group be simple. The extension to compact semisimple Lie groups and their Lie algebras is straightforward. Thus, orbit functions are products of orbit functions of simple constituents, and different types of orbit functions can be mixed.

Polynomials formed from other orbit functions (E-, E^--, E^+-, S^+-, S^--, C^+-, C^--functions) by the same substitution of variables should be equally interesting once $n > 1$. These functions have been studied in [20, 25, 26, 33].

Appendices

A. Orbit Functions of A_1, Their Polynomial Forms, and Chebyshev Polynomials

A number of multivariate generalizations of classical Chebyshev polynomials are available in the literature [34–39]; the aim of this section is to show in all details how Chebyshev polynomials appear as particular case of the multivariate polynomials proposed in this paper. First we recall that well-known classical Chebyshev polynomials can be obtained independently using only the properties of C- and S-orbit functions of the Lie group A_1, see [40] for details.

The C-polynomials generated by our approach are naturally normalized in a different way than the classical polynomials (they coincide with the form of Dickson polynomials).

The orbit functions of A_1 are of two types:

$$C_m(x) = e^{2\pi imx} + e^{-2\pi imx} = 2\cos(2\pi mx), \quad x \in \mathbb{R},\ m \in \mathbb{Z}^{>0},$$
$$S_m(x) = e^{2\pi imx} - e^{-2\pi imx} = 2i\sin(2\pi mx), \quad x \in \mathbb{R},\ m \in \mathbb{Z}^{>0}. \tag{A.1}$$

We introduce new variables X and S as follows:

$$X := C_1(x) = e^{2\pi ix} + e^{-2\pi ix} = 2\cos(2\pi x),$$
$$S := S_1(x) = e^{2\pi ix} - e^{-2\pi ix} = 2i\sin(2\pi x). \tag{A.2}$$

Polynomials can now be constructed recursively in the degrees of X and S by calculating the decompositions of products of appropriate orbit functions. "Generic" recursion relations are those where one of the first degree polynomials, X or S, multiplies the generic polynomial C_m or S_m, that is, $m \geq 1$. Omitting the dependence on x from the symbols, we have the generic recursion relations

$$\begin{array}{ll} XC_m = C_{m+1} + C_{m-1}, & SC_m = S_{m+1} - S_{m-1}, \\ XS_m = S_{m+1} + S_{m-1}, & SS_m = C_{m+1} - C_{m-1}, \end{array} \quad m \geqslant 1. \tag{A.3}$$

When solving recursion relations for C polynomials, we need to start from the lowest ones; several results are in Table 1. Hence we conclude that $C_m = 2T_m$, for $m = 0, 1, \ldots$.

The character $\chi_m(x)$ of an irreducible representation of A_1 of dimension $m+1$ is known explicitly for all $m \geqslant 0$. There are two ways to write the character: as the ratio of S-functions, and as the sum of C-functions. Explicitly, that is

$$S_m(X) = \chi_m(x) = \frac{S_{m+1}(x)}{S(x)} = C_m(x) + C_{m-2}(x) + \cdots + \begin{cases} C_2(x) + C_0 & \text{if } m \text{ even}, \\ C_3(x) + C_1(x) & \text{if } m \text{ odd}. \end{cases} \tag{A.4}$$

Note that (A.4) is the Chebyshev polynomial of the second kind $U_m(x)$.

Remark A.1. The main argument in favor of our normalization of Chebyshev polynomials is that polynomials C_m from Table 1 are Dickson polynomials (it is well known that they are equivalent to Chebyshev polynomials over the complex numbers). It is easy to prove (see e.g., [40]) that Weyl group of A_n is equivalent to S_{n+1}, therefore it is natural to consider multivariate C-polynomials of A_n as n-dimensional generalizations of Dickson polynomials (as permutation polynomials). Also our form of Dickson-Chebyshev polynomials makes them the lowest special case of (2.4) without additional adjustments and it appears more "natural" because, for example, the equality $C_2^2 = C_4 + 2$ would not hold for T_2 and T_4.

Table 1: The irreducible C and S polynomials of A_1 of degrees up to 8.

C polynomials		S polynomials	
# = 0		# = 0	
C_2	X^2-2	S_0	1
C_4	X^4-4X^2+2	S_2	X^2-1
C_6	$X^6-6X^4+9X^2-2$	S_4	X^4-3X^2+1
C_8	$X^8-8X^6+20X^4-16X^2+2$	S_6	$X^6-5X^4+6X^2-1$
# = 1		# = 1	
C_1	X	S_1	X
C_3	X^3-3X	S_3	X^3-2X
C_5	X^5-5X^3+5X	S_5	X^5-4X^3+3X
C_7	$X^7-7X^5+14X^3-7X$	S_7	$X^7-6X^5+10X^3-4X$

B. Recursion Relations for A_2 Orbit Functions and Polynomials

The variables of the A_2 polynomials are the C functions of the lowest dominant weights $\omega_1=(1,0)$ and $\omega_2=(0,1)$:

$$X_1 := C_{(1,0)}(x_1,x_2), \qquad X_2 := C_{(0,1)}(x_1,x_2), \qquad S := S_{(1,1)}(x_1,x_2). \tag{B.1}$$

We omit writing (x_1,x_2) at the symbols of orbit functions for simplicity of notations.

In addition to the obvious polynomials X_1, X_2, X_1^2, X_1X_2, and X_2^2, we recursively find the rest of the A_2-polynomials. The degree of the polynomial $C_{(a,b)}$ equals $a+b$. The degree of $S_{(a,b)}$ is also $a+b$ provided $ab\neq 0$, otherwise the S-polynomials are zero.

Due to the A_2 outer automorphism, polynomials $C_{(a,b)}$ and $C_{(b,a)}$ are related by the interchange of variables $X_1 \leftrightarrow X_2$ (i.e. $C_{(a,b)}(X_1,X_2)=C_{(b,a)}(X_2,X_1)$).

In general, each term in an irreducible polynomial, equivalently each weight of an orbit, must belong to the same congruence class specified by the congruence number #. For A_2-weight (a,b), we have

$$\#(a,b) = (a+2b) \bmod 3. \tag{B.2}$$

Hence, irreducible orbit functions have a well-defined value of #. For A_2-orbit functions, we have $\#(C_{(a,b)}) = \#(S_{(a,b)}) = (a+2b) \bmod 3$. Consequently, there are three classes of polynomials corresponding to $\# = 0,1,2$. During multiplication, the congruence numbers add up mod 3. A product of irreducible orbits decomposes into the sum of orbits belonging to the same congruence class. The sizes of the irreducible orbits of $W(A_2)$ are found in [30]. The dimension $d_{(a,b)}$ of the representation of A_2 with the highest weight (a,b) is given by $d_{(a,b)} = (1/2)(a+1)(b+1)(a+b+2)$.

B.1. Recursion Relations for C-Function Polynomials of A_2

There are two 4-term generic recursion relations for C functions. They are obtained as the decomposition of the products of X and Y, each being a sum of three exponential functions, with a generic C-function which is the sum of $|W(A_2)|=6$ exponential terms,

$$X_1 C_{(a,b)} = C_{(a+1,b)} + C_{(a-1,b+1)} + C_{(a,b-1)}, \quad a,b \geqslant 2,$$
$$X_2 C_{(a,b)} = C_{(a,b+1)} + C_{(a+1,b-1)} + C_{(a-1,b)}, \quad a,b \geqslant 2. \tag{B.3}$$

Before generic recursion relations can be used, the special recursion relations for particular values $a, b \in \{0, 1\}$ need to be solved recursively starting from the lowest ones:

$$X_1^2 = C_{(2,0)} + 2X_2, \quad X_2^2 = C_{(0,2)} + 2X_1, \quad X_1 X_2 = C_{(1,1)} + 3,$$
$$X_1 C_{(1,1)} = C_{(2,1)} + 2C_{(0,2)} + 2X_1, \qquad X_2 C_{(1,1)} = C_{(1,2)} + 2C_{(2,0)} + 2X_2, \tag{B.4}$$

for $a \geqslant 2$:

$$X_1 C_{(a,1)} = C_{(a+1,1)} + C_{(a-1,2)} + 2C_{(a,0)}, \qquad X_2 C_{(a,1)} = C_{(a,2)} + 2C_{(a+1,0)} + C_{(a-1,1)},$$
$$X_1 C_{(a,0)} = C_{(a+1,0)} + C_{(a-1,1)}, \qquad X_2 C_{(a,0)} = C_{(a,1)} + C_{(a-1,0)}, \tag{B.5}$$

for $b \geqslant 2$:

$$X_1 C_{(1,b)} = C_{(2,b)} + 2C_{(0,b+1)} + C_{(1,b-1)}, \qquad X_2 C_{(1,b)} = C_{(1,b+1)} + C_{(2,b-1)} + 2C_{(0,b)},$$
$$X_1 C_{(0,b)} = C_{(1,b)} + C_{(0,b-1)}, \qquad X_2 C_{(0,b)} = C_{(0,b+1)} + C_{(1,b-1)}. \tag{B.6}$$

Using the symmetry of orbit functions with respect to the permutation of the components of dominant weights, we obtain analogous polynomials $C_{(a,0)}$ and $C_{(a,1)}$ for all $a \in \mathbb{N}$. Then the 4-term special recursion relations are solved yielding $C_{(2,b)}$ and $C_{(a,2)}$ for all $a, b \in \mathbb{N}$. After that, the generic recursion relations should be used.

B.2. The Character of A_2

In the A_2 case the general formula (2.4) is specialized

$$S_{(a,b)}(X_1, X_2) = \chi_{(a,b)}(x, y) = \frac{S_{(a+1,b+1)}(x, y)}{S_{(1,1)}(x, y)} = C_{(a,b)}(x, y) + \sum_{\lambda} m_\lambda C_\lambda(x, y). \tag{B.7}$$

The summation extends over the dominant weights that have positive multiplicities m_λ in the case of $\chi_{(a,b)}$. The coefficients (dominant weight multiplicities) are tabulated in [27] for the 50 first $\chi_{(a,b)}$ in each congruence class of A_2. The first few characters for the congruence class # = 0 are

$$\chi_{(0,0)} = C_{(0,0)} = 1,$$
$$\chi_{(1,1)} = C_{(1,1)} + 2C_{(0,0)},$$
$$\chi_{(3,0)} = C_{(3,0)} + C_{(1,1)} + C_{(0,0)},$$
$$\chi_{(0,3)} = C_{(0,3)} + C_{(1,1)} + C_{(0,0)},$$

$$\begin{aligned}
\chi_{(2,2)} &= C_{(2,2)} + C_{(0,3)} + C_{(3,0)} + 2C_{(1,1)} + 3C_{(0,0)}, \\
\chi_{(1,4)} &= C_{(1,4)} + C_{(2,2)} + 2C_{(0,3)} + C_{(3,0)} + 2C_{(1,1)} + 2C_{(0,0)}, \\
\chi_{(3,3)} &= C_{(3,3)} + C_{(4,1)} + C_{(1,4)} + 2C_{(2,2)} + 2C_{(0,3)} + 2C_{(3,0)} + 3C_{(1,1)} + 4C_{(0,0)}, \\
\chi_{(6,0)} &= C_{(6,0)} + C_{(4,1)} + C_{(2,2)} + C_{(0,3)} + C_{(3,0)} + C_{(1,1)} + C_{(0,0)}.
\end{aligned} \tag{B.8}$$

The equalities must satisfy two relatively simple conditions: (i) the dominant weights on both sides must have the same congruence number (B.2), and (ii) the number of exponential terms in a character $\chi_{(a,b)}$ is known to be the dimension of the irreducible representation (a,b). Therefore, the sizes of the orbit functions on the right side have to add up to the dimension.

For # = 1:

$$\begin{aligned}
\chi_{(1,0)} &= C_{(1,0)}, \\
\chi_{(0,2)} &= C_{(0,2)} + C_{(1,0)}, \\
\chi_{(2,1)} &= C_{(2,1)} + C_{(0,2)} + 2C_{(1,0)}, \\
\chi_{(1,3)} &= C_{(1,3)} + C_{(2,1)} + 2C_{(0,2)} + 2C_{(1,0)}, \\
\chi_{(4,0)} &= C_{(4,0)} + C_{(2,1)} + C_{(0,2)} + C_{(1,0)}, \\
\chi_{(0,5)} &= C_{(0,5)} + C_{(1,3)} + C_{(2,1)} + C_{(0,2)} + C_{(1,0)}.
\end{aligned} \tag{B.9}$$

For # = 2, it suffices to interchange the component of all dominant weights in the equalities for # = 1. Thus no independent calculation is needed, see Table 2 for the solution.

C. Recursion Relations for C_2 Orbit Functions

There are two congruence classes of C_2 orbit functions/polynomials. For C_2 weight (a,b) (dominant or not), we have

$$\#(a,b) = a \bmod 2. \tag{C.1}$$

The dimension $d_{(a,b)}$ of an irreducible representation of C_2 with the highest weight (a,b) is given by

$$d_{(a,b)} = \frac{1}{6}(a+1)(b+1)(2a+b+3)(a+b+2). \tag{C.2}$$

In multiplying the polynomials, congruence numbers add up mod 2. Character in the case of C_2 is given by (2.4), where the C and S functions are those of C_2, as are the coefficients m_λ (also tabulated in [27]).

Table 2: The irreducible C- and S-polynomials of A_2 of degree up to 4. From any polynomial $C_{(a,b)}$ or $S_{(a,b)}$ we obtain $C_{(b,a)}$ or $S_{(b,a)}$, respectively, by interchanging X_1 and X_2.

	C polynomials		S polynomials
# = 0		# = 0	
$C_{(1,1)}$	$X_1X_2 - 3$	$S_{(1,1)}$	$X_1X_2 - 1$
$C_{(3,0)}$	$X_1^3 - 3X_1X_2 + 3$	$S_{(0,3)}$	$X_2^3 - 2X_1X_2 + 1$
$C_{(2,2)}$	$X_1^2X_2^2 - 2X_1^3 - 2X_2^3 + 4X_1X_2 - 3$	$S_{(2,2)}$	$X_1^2X_2^2 - X_1^3 - X_2^3$
# = 1		# = 1	
$C_{(1,0)}$	X_1	$S_{(1,0)}$	X_1
$C_{(0,2)}$	$X_2^2 - 2X_1$	$S_{(0,2)}$	$X_2^2 - X_1$
$C_{(2,1)}$	$X_1^2X_2 - 2X_2^2 - X_1$	$S_{(2,1)}$	$X_1^2X_2 - X_2^2 - X_1$
$C_{(1,3)}$	$X_1X_2^3 - 3X_1^2X_2 - X_2^2 + 5X_1$	$S_{(1,3)}$	$X_1X_2^3 - 2X_1^2X_2 - X_2^2 + 2X_2$
$C_{(4,0)}$	$X_1^4 - 4X_1^2X_2 + 2X_2^2 + 4X_1$	$S_{(4,0)}$	$X_1^4 - 3X_1^2X_2 + X_2^2 + X_1 + X_2$

We denote the variables of the C_2-polynomials by

$$X_1 := C_{(1,0)}(x,y), \quad X_2 := C_{(0,1)}(x,y), \quad \text{and} \quad S := S_{(1,1)}(x,y), \quad S_{(a,b)}(X_1,X_2) = \chi_{(a,b)}(x,y) \tag{C.3}$$

often omitting (x,y) from the symbols. The variable S cannot be built out of X_1 and X_2. Although the variables are denoted by the same symbols as in the case of A_2 (and also G_2 below), they are very different. Thus X_1 and X_2 contain 4 exponential terms and S contains 8 terms. The congruence number # of X_1 and S is 1, while that of X_2 is 0.

C.1. Recursion Relations for C Functions of C_2

The two generic recursion relations for C-functions of C_2 are

$$\begin{aligned} X_1C_{(a,b)} &= C_{(a+1,b)} + C_{(a-1,b+1)} + C_{(a+1,b-1)} + C_{(a-1,b)}, \quad a,b \geqslant 2, \\ X_2C_{(a,b)} &= C_{(a,b+1)} + C_{(a+2,b-1)} + C_{(a-2,b+1)} + C_{(a,b-1)}, \quad a \geqslant 3,\ b \geqslant 2. \end{aligned} \tag{C.4}$$

The special recursion relations for C-functions involving low values of a and b have to be solved first starting from the lowest ones:

$$\begin{aligned} X_1C_{(a,1)} &= C_{(a+1,1)} + C_{(a-1,2)} + 2C_{(a+1,0)} + C_{(a-1,1)}, \quad a \geqslant 2, \\ X_1C_{(a,0)} &= C_{(a+1,0)} + C_{(a-1,1)} + C_{(a-1,0)}, \quad a \geqslant 2, \\ X_1C_{(1,b)} &= C_{(2,b)} + 2C_{(0,b+1)} + C_{(2,b-1)} + 2C_{(0,b)}, \quad b \geqslant 2, \\ X_1C_{(0,b)} &= C_{(1,b)} + C_{(1,b-1)}, \quad b \geqslant 2, \\ X_1C_{(1,1)} &= C_{(2,1)} + 2C_{(0,2)} + 2C_{(2,0)} + 2X_2, \\ X_1X_2 &= C_{(1,1)} + 2X_1; \qquad X_1^2 = C_{(2,0)} + 2X_2 + 4, \end{aligned}$$

$$
\begin{aligned}
X_2 C_{(a,1)} &= C_{(a,2)} + 2C_{(a+2,0)} + C_{(a-2,2)} + 2C_{(a,0)}, \quad a \geqslant 3, \\
X_2 C_{(a,0)} &= C_{(a,1)} + C_{(a-2,1)}, \quad a \geqslant 3, \\
X_2 C_{(2,b)} &= C_{(2,b+1)} + C_{(4,b-1)} + 2C_{(0,b+1)} + C_{(2,b-1)}, \quad b \geqslant 2, \\
X_2 C_{(1,b)} &= C_{(1,b+1)} + C_{(3,b-1)} + C_{(1,b-1)} + C_{(1,b)}, \quad b \geqslant 2, \\
X_2 C_{(0,b)} &= C_{(0,b+1)} + C_{(2,b-1)} + C_{(0,b-1)}, \quad b \geqslant 2, \\
X_2 C_{(2,1)} &= C_{(2,2)} + 2C_{(4,0)} + C_{(0,4)} + 2C_{(0,2)} + 2C_{(2,0)}, \\
X_2 C_{(1,1)} &= C_{(1,2)} + 2C_{(3,0)} + C_{(1,1)} + 2X_1, \\
X_2 C_{(2,0)} &= C_{(2,1)} + 2X_2; \qquad X_2^2 = C_{(0,2)} + 2C_{(2,0)} + 4.
\end{aligned}
\tag{C.5}
$$

The 3- and 4-term recursion relations are solved independently, giving us $C_{(0,b)}$, $C_{(a,0)}$, $C_{(1,b)}$, and $C_{(a,1)}$ for all a and b, for example, see Table 3.

C.2. Recursion Relations for S Functions of C_2

The generic relations for S functions are readily obtained from those of C functions by replacing C by S, and by making appropriate sign changes.

All C functions of C_2 are real valued. Here are a few examples of C_2 characters:

$$
\begin{aligned}
&\# = 0: \\
&\chi_{(0,0)} = C_{(0,0)} = 1, \\
&\chi_{(0,1)} = 1 + C_{(0,1)} = 1 + X_2, \\
&\chi_{(2,0)} = 2 + X_2 + C_{(2,0)}, \\
&\chi_{(0,2)} = 2 + X_2 + C_{(2,0)} + C_{(0,2)}, \\
&\chi_{(2,1)} = 3 + 3X_2 + 2C_{(2,0)} + C_{(0,2)} + C_{(2,1)}, \\
&\chi_{(0,3)} = 2 + 2X_2 + C_{(2,0)} + C_{(0,2)} + C_{(2,1)} + C_{(0,3)}, \\
&\chi_{(4,0)} = 3 + 2X_2 + 2C_{(2,0)} + C_{(0,2)} + C_{(2,1)} + C_{(4,0)}, \\
&\chi_{(2,2)} = 5 + 4X_2 + 4C_{(2,0)} + 3C_{(0,2)} + 2C_{(2,1)} + C_{(0,3)} + C_{(4,0)} + C_{(2,2)}, \\
&\# = 1: \\
&\chi_{(1,0)} = C_{(1,0)} = X_1, \\
&\chi_{(1,1)} = 2X_1 + C_{(1,1)},
\end{aligned}
$$

$$
\begin{aligned}
\chi_{(3,0)} &= 2X_1 + C_{(1,1)} + C_{(3,0)}, \\
\chi_{(1,2)} &= 3X_1 + 2C_{(1,1)} + C_{(3,0)} + C_{(1,2)}, \\
\chi_{(3,1)} &= 4X_1 + 3C_{(1,1)} + 2C_{(3,0)} + C_{(1,2)} + C_{(3,1)}, \\
\chi_{(1,3)} &= 4X_1 + 3C_{(1,1)} + 2C_{(3,0)} + 2C_{(1,2)} + C_{(3,1)} + C_{(1,3)}.
\end{aligned}
\tag{C.6}
$$

Using these characters and Table 3, we can calculate all irreducible S polynomials of degree up to four with respect to the variables X_1 and X_2, see Table 4. Note that $\chi_{(0,4)}$ yields the polynomial of order five.

D. Recursion Relations for G_2 Orbit Functions

All G_2 weights fall into the same congruence class # = 0. Thus there are no congruence classes to distinguish in G_2. The variables are the orbit functions of the two fundamental weights:

$$
X_1 := C_{(1,0)}(x,y), \qquad X_2 := C_{(0,1)}(x,y), \qquad S = S_{(1,1)}(x,y), \qquad S_{(a,b)}(X_1,X_2) = \chi_{(a,b)}(x,y).
\tag{D.1}
$$

D.1. Recursion Relations for C Functions of G_2

There are two generic recursion relations for C polynomials of G_2, each containing one product term and six C polynomials.

$$
\begin{gathered}
\text{For } a \geqslant 3,\ b \geqslant 4: \\
X_1C_{(a,b)} = C_{(a+1,b)} + C_{(a-1,b+3)} + C_{(a+2,b-3)} + C_{(a-2,b+3)} + C_{(a+1,b-3)} + C_{(a-1,b)}. \\
\text{For } a \geqslant 2,\ b \geqslant 3: \\
X_2C_{(a,b)} = C_{(a,b+1)} + C_{(a+1,b-1)} + C_{(a-1,b+2)} + C_{(a+1,b-2)} + C_{(a-1,b+1)} + C_{(a,b-1)}.
\end{gathered}
\tag{D.2}
$$

Specializing the first of the generic relations to either $a \in \{0,1,2\}$ or $b \in \{0,1,2,3\}$, we have

$$
\begin{aligned}
X_1C_{(2,b)} &= C_{(3,b)} + C_{(1,b+3)} + C_{(4,b-3)} + 2C_{(0,b+3)} + C_{(3,b-3)} + C_{(1,b)}, \\
X_1C_{(1,b)} &= C_{(2,b)} + 2C_{(0,b+3)} + C_{(3,b-3)} + C_{(2,b-3)} + C_{(1,b)} + 2C_{(0,b)}, \\
X_1C_{(0,b)} &= C_{(1,b)} + C_{(2,b-3)} + C_{(1,b-3)}, \\
X_1C_{(a,3)} &= C_{(a+1,3)} + C_{(a-1,6)} + 2C_{(a+2,0)} + C_{(a-2,6)} + 2C_{(a+1,0)} + C_{(a-1,3)}, \\
X_1C_{(a,2)} &= C_{(a+1,2)} + C_{(a-1,5)} + C_{(a+1,1)} + C_{(a-2,5)} + C_{(a,1)} + C_{(a-1,2)}, \\
X_1C_{(a,1)} &= C_{(a+1,1)} + C_{(a-1,4)} + C_{(a-2,4)} + C_{(a-1,1)} + C_{(a,2)} + C_{(a-1,2)},
\end{aligned}
$$

$$
\begin{aligned}
X_1C_{(a,0)} &= C_{(a+1,0)} + C_{(a-1,3)} + C_{(a-2,3)} + C_{(a-1,0)},\\
X_1C_{(2,3)} &= C_{(3,3)} + C_{(1,6)} + 2C_{(4,0)} + 2C_{(0,6)} + 2C_{(3,0)} + C_{(1,3)},\\
X_1C_{(2,2)} &= C_{(3,2)} + C_{(1,5)} + C_{(0,5)} + C_{(1,2)} + C_{(2,1)} + C_{(3,1)},\\
X_1C_{(2,1)} &= C_{(3,1)} + C_{(1,4)} + 2C_{(0,4)} + C_{(1,1)} + C_{(2,2)} + C_{(1,2)},\\
X_1C_{(2,0)} &= C_{(3,0)} + C_{(1,3)} + 2C_{(0,3)} + X_1,\\
X_1C_{(1,3)} &= C_{(2,3)} + 2C_{(0,6)} + 2C_{(3,0)} + 2C_{(2,0)} + C_{(1,3)} + 2C_{(0,3)},\\
X_1C_{(1,2)} &= C_{(2,2)} + 2C_{(0,5)} + C_{(1,2)} + 2C_{(0,2)} + C_{(2,1)} + C_{(1,1)},\\
X_1C_{(1,1)} &= C_{(2,1)} + 2C_{(0,4)} + C_{(1,2)} + 2C_{(0,2)} + C_{(1,1)} + 2X_2,\\
X_1C_{(0,3)} &= C_{(1,3)} + 2C_{(2,0)} + 2X_1,\\
X_1C_{(0,2)} &= C_{(1,2)} + C_{(1,1)} + 2X_2,\\
X_1X_1 &= C_{(2,0)} + 2C_{(0,3)} + 2X_1 + 6,\\
X_1X_2 &= C_{(1,1)} + 2C_{(0,2)} + 2X_2.
\end{aligned}
\tag{D.3}
$$

Specializing the second of the generic relations to either $a \in \{0,1\}$ or $b \in \{0,1,2\}$, we have

$$
\begin{aligned}
X_2C_{(1,b)} &= C_{(1,b+1)} + C_{(2,b-1)} + 2C_{(0,b+2)} + C_{(2,b-2)} + 2C_{(0,b+1)} + C_{(1,b-1)},\\
X_2C_{(0,b)} &= C_{(0,b+1)} + C_{(1,b-1)} + C_{(1,b-2)} + C_{(0,b-1)},\\
X_2C_{(a,1)} &= C_{(a,2)} + 2C_{(a+1,0)} + C_{(a-1,3)} + C_{(a-1,2)} + 2C_{(a,0)} + C_{(a,1)},\\
X_2C_{(a,0)} &= C_{(a,1)} + C_{(a-1,2)} + C_{(a-1,1)},\\
X_2C_{(1,2)} &= C_{(1,3)} + C_{(2,1)} + 2C_{(0,4)} + 2C_{(0,3)} + C_{(1,1)} + 2C_{(2,0)},\\
X_2C_{(1,1)} &= C_{(1,2)} + 2C_{(2,0)} + 2C_{(0,3)} + 2C_{(0,2)} + C_{(1,1)} + 2X_1,\\
X_2C_{(0,2)} &= C_{(0,3)} + C_{(1,1)} + 2X_1 + X_2,\\
X_2X_2 &= C_{(0,2)} + 2X_1 + 2X_2 + 6.
\end{aligned}
\tag{D.4}
$$

Remark D.1. It can be seen from Table 5 that order of $C_{(a,b)}$ polynomial sometimes exceeds $a+b$.

Table 3: The irreducible C polynomials of C_2 of degree up to 4.

	C polynomials
# = 0	
$C_{(0,1)}$	X_2
$C_{(2,0)}$	$X_1^2 - 2X_2 - 4$
$C_{(2,1)}$	$X_1^2X_2 - 2X_2^2 - 6X_2$
$C_{(4,0)}$	$4 - 4X_1^2 + X_1^4 + 8X_2 - 4X_1^2X_2 + 2X_2^2$
$C_{(0,2)}$	$4 - 2X_1^2 + 4X_2 + X_2^2$
$C_{(0,3)}$	$9X_2 - 3X_1^2X_2 + 6X_2^2 + X_2^3$
$C_{(2,2)}$	$-8 + 10X_1^2 - 2X_1^4 - 20X_2 + 8X_1^2X_2 - 12X_2^2 + X_1^2X_2^2 - 2X_2^3$
$C_{(0,4)}$	$4 - 8X_1^2 + 2X_1^4 + 16X_2 - 8X_1^2X_2 + 20X_2^2 - 4X_1^2X_2^2 + 8X_2^3 + X_2^4$
# = 1	
$C_{(1,0)}$	X_1
$C_{(1,1)}$	$X_1X_2 - 2X_1$
$C_{(3,0)}$	$X_1^3 - 3X_1X_2 - 3X_1$
$C_{(3,1)}$	$2X_1 - 4X_1X_2 + X_1^3X_2 - 3X_1X_2^2$
$C_{(1,2)}$	$6X_1 - 2X_1^3 + 3X_1X_2 + X_1X_2^2$
$C_{(1,3)}$	$-6X_1 + 2X_1^3 + 6X_1X_2 - 3X_1^3X_2 + 5X_1X_2^2 + X_1X_2^3$

Table 4: The irreducible S polynomials of C_2 of degree up to 4.

	S polynomials
# = 0	
$S_{(0,1)}$	$1 + X_2$
$S_{(2,0)}$	$-2 + X_1^2 - X_2$
$S_{(0,2)}$	$2 - X_1^2 + 3X_2 + X_2^2$
$S_{(2,1)}$	$-1 - 3X_2 + X_1^2X_2 - X_2^2$
$S_{(0,3)}$	$2 - X_1^2 + 7X_2 - 2X_1^2X_2 + 5X_2^2 + X_2^3$
$S_{(4,0)}$	$3 - 4X_1^2 + X_1^4 + 4X_2 - 3X_1^2X_2 + X_2^2$
$S_{(2,2)}$	$-3 + 4X_1^2 - X_1^4 - 7X_2 + 3X_1^2X_2 - 5X_2^2 + X_1^2X_2^2 - X_2^3$
# = 1	
$S_{(1,0)}$	X_1
$S_{(1,1)}$	X_1X_2
$S_{(3,0)}$	$-3X_1 + X_1^3 - 2X_1X_2$
$S_{(1,2)}$	$2X_1 - X_1^3 + 2X_1X_2 + X_1X_2^2$
$S_{(3,1)}$	$-4X_1X_2 + X_1^3X_2 - 2X_1X_2^2$
$S_{(1,3)}$	$5X_1X_2 - 2X_1^3X_2 + 4X_1X_2^2 + X_1X_2^3$

D.2. Recursion Relations for S Functions of G_2

Generic recursion relations for S polynomials differ very little from those for C polynomials.

For $a \geqslant 3,\ b \geqslant 4$:

$$X_1 S_{(a,b)} = S_{(a+1,b)} + S_{(a-1,b+3)} + S_{(a+2,b-3)} + S_{(a-2,b+3)} + S_{(a+1,b-3)} + S_{(a-1,b)}.$$

For $a \geqslant 2,\ b \geqslant 3$:

$$X_2 S_{(a,b)} = S_{(a,b+1)} + S_{(a+1,b-1)} + S_{(a-1,b+2)} + S_{(a+1,b-2)} + S_{(a-1,b+1)} + S_{(a,b-1)}. \tag{D.5}$$

The S polynomials need not be calculated independently. They can be read off the tables [27] as the characters of G_2 representations, see Table 6.

Here are all G_2-characters $\chi_{(a,b)}$ with $a + b \leqslant 3$:

$$\begin{aligned}
\chi_{(1,0)} &= 1 + C_{(1,0)} = 1 + X_1,\\
\chi_{(0,1)} &= 2 + C_{(1,0)} + C_{(0,1)} = 2 + X_1 + X_2,\\
\chi_{(2,0)} &= 3 + 2X_1 + X_2 + C_{(2,0)},\\
\chi_{(1,1)} &= 4 + 4X_1 + 2X_2 + 2C_{(2,0)} + C_{(1,1)},\\
\chi_{(3,0)} &= 5 + 4X_1 + 3X_2 + 2C_{(2,0)} + C_{(3,0)},\\
\chi_{(0,2)} &= 5 + 3X_1 + 3X_2 + 2C_{(2,0)} + C_{(1,1)} + C_{(3,0)} + C_{(0,2)},\\
\chi_{(2,1)} &= 9 + 8X_1 + 6X_2 + 5C_{(2,0)} + 3C_{(1,1)} + 2C_{(3,0)} + C_{(0,2)} + C_{(2,1)},\\
\chi_{(1,2)} &= 10 + 10X_1 + 7X_2 + 7C_{(2,0)} + 5C_{(1,1)} + 3C_{(3,0)} + 3C_{(3,0)} + 2C_{(0,2)} + 2C_{(2,1)} + C_{(4,0)} + C_{(1,2)},\\
\chi_{(0,3)} &= 9 + 7X_1 + 7X_2 + 5C_{(2,0)} + 4C_{(1,1)} + 4C_{(3,0)} + 3C_{(0,2)} + 2C_{(1,1)} + C_{(4,0)} + C_{(1,2)},\\
&\quad + C_{(3,1)} + C_{(0,3)}.
\end{aligned} \tag{D.6}$$

E. Recursion Relations for Lie Algebras of Rank 3

E.1. Recursion Relations for C-Functions of A_3

There are 4 congruence classes of A_3 defined by

$$\#(a,b,c) = a + 2b + 3c \bmod 4. \tag{E.1}$$

The variables of the A_3 polynomials are chosen to be

$$\begin{gathered}
X_1 := C_{(1,0,0)}(x_1,x_2,x_3), \qquad X_2 := C_{(0,1,0)}(x_1,x_2,x_3),\\
X_3 := C_{(0,0,1)}(x_1,x_2,x_3) \text{ and } S_{(a,b,c)}(X_1,X_2,X_3) = \chi_{(a,b,c)}(x,y,z).
\end{gathered} \tag{E.2}$$

Table 5: The irreducible $C_{(a,b)}$ polynomials of G_2.

	C polynomials
$C_{(1,0)}$	X_1
$C_{(0,1)}$	X_2
$C_{(0,2)}$	$X_2^2 - 2X_1 - 2X_2 - 6$
$C_{(1,1)}$	$X_1X_2 - 2X_2^2 + 4X_1 + 2X_2 + 12$
$C_{(0,3)}$	$X_2^3 - 3X_1X_2 - 6X_1 - 9X_2 - 12$
$C_{(2,0)}$	$X_1^2 - 2X_2^3 + 6X_1X_2 + 10X_1 + 18X_2 + 18$
$C_{(1,2)}$	$X_1X_2 - 2X_1^2 - 3X_1X_2 + 2X_2^2 - 10X_1 - 4X_2 - 12$
$C_{(2,1)}$	$X_1^2X_2 - 2X_2^4 + 5X_1X_2^2 + 2X_1^2 + 12X_1X_2 + 18X_2^2 + 6X_1 + 20X_2$
$C_{(1,3)}$	$X_1X_2^3 - 3X_1^2X_2 - 8X_1^2 + 4X_2^3 - 21X_1X_2 - 34X_1 - 36X_2 - 36$
$C_{(3,0)}$	$X_1^3 - 3X_1X_2^3 + 9X_1^2X_2 + 18X_1^2 - 6X_2^3 + 45X_1X_2 + 63X_1 + 54X_2 + 60$
$C_{(2,2)}$	$X_1^2X_2^2 - 2X_2^5 - 2X_1^3 + 10X_1X_2^3 - 14X_1^2X_2 + 2X_2^4 + 6X_1X_2^2 - 30X_1^2 + 30X_2^3 - 84X_1X_2 - 108X_1 - 124X_2 - 108$
$C_{(3,1)}$	$X_1^3X_2 - 3X_1X_2^4 + 8X_1^2X_2^2 + 2X_2^5 + 2X_1^3 - 10X_1X_2^3 + 31X_1^2X_2 - 6X_2^4 + 34X_1X_2^2 + 28X_1^2 - 30X_2^3 + 135X_1X_2 + 36X_2^2 + 102X_1 + 164X_2 + 108$
$C_{(2,3)}$	$X_1^2X_2^3 - 2X_2^6 - 3X_1^3X_2 + 12X_1X_2^4 - 18X_1^2X_2^2 - 6X_1^3 + 21X_1X_2^3 - 72X_1^2X_2 + 36X_2^4 - 108X_1X_2^2 - 64X_1^2 + 38X_2^3 - 303X_{1X_2} - 162X_2^2 - 196X_1 - 342X_2 - 180$
$C_{(3,2)}$	$X_1^3X_2^2 - 3X_1X_2^5 - 2X_1^4 + 15X_1^2X_2^3 - 20X_1^3X_2 + 5X_1X_2^4 + 4X_1^2X_2^2 - 6X_2^5 - 44X_1^3 + 75X_1X_2^3 - 175X_1^2X_2 + 10X_2^4 - 13X_1X_2^2 - 238X_1^2 + 90X_2^3 - 480X_1X_2 - 34X_2^2 - 484X_1 - 418X_2 - 336$

Table 6: The irreducible $S_{(a,b)}$ polynomials of G_2 with $a + b \leqslant 5$.

	S polynomials
$S_{(0,1)}$	$X_2 + 1$
$S_{(1,0)}$	$X_1 + X_2 + 2$
$S_{(0,2)}$	$X_2^2 - X_1 - 3$
$S_{(1,1)}$	$X_1X_2 + 2X_1 + 2X_2 + 4$
$S_{(2,0)}$	$X_1^2 - X_2^3 + 4X_1X_2 + 7X_1 + 10X_2 + 11$
$S_{(1,2)}$	$X_1X_2^2 - X_1^2 + X_2^2 - 4X_1 - 3$
$S_{(2,1)}$	$X_1^2X_2 - X_2^4 + 3X_1X_2^2 + 2X_1^2 - X_2^3 + 10X_1X_2 + 9X_2^2 + 9X_1 + 19X_2 + 10$
$S_{(2,2)}$	$X_1^2X_2^2 - X_2^5 - X_1^3 + 4X_1X_2^3 - 3X_1^2X_2 + 6X_1X_2^2 - 9X_1^2 + 12X_2^3 - 18X_1X_2 + 9X_2^2 - 27X_1 - 27X_2 - 27$

For C functions the generic recursion relations are the following ones, where we assume $a, b, c \geqslant 2$:

$$
\begin{aligned}
X_1C_{(a,b,c)} &= C_{(a+1,b,c)} + C_{(a-1,b+1,c)} + C_{(a,b-1,c+1)} + C_{(a,b,c-1)},\\
X_2C_{(a,b,c)} &= C_{(a,b+1,c)} + C_{(a+1,b-1,c+1)} + C_{(a-1,b,c+1)} + C_{(a+1,b,c-1)} + C_{(a-1,b+1,c-1)} + C_{(a,b-1,c)},\\
X_3C_{(a,b,c)} &= C_{(a,b,c+1)} + C_{(a,b+1,c-1)} + C_{(a+1,b-1,c)} + C_{(a-1,b,c)}.
\end{aligned}
\tag{E.3}
$$

Note that the first and the third relations are easily obtained from each other by interchanging the first and third component of all dominant weights. Thus $X_1 \leftrightarrow X_3$ and $(a, b, c) \leftrightarrow (c, b, a)$.

Table 7: The irreducible C polynomials and S polynomials of A_3.

	C polynomials		S polynomials
# = 0		# = 0	
$C_{(1,0,1)}$	$-4+X_1X_3$	$S_{(1,0,1)}$	$-1+X_1X_3$
$C_{(0,2,0)}$	$2-2X_1X_3+X_2^2$	$S_{(0,2,0)}$	$X_2^2-X_1X_3$
$C_{(0,1,2)}$	$4-X_1X_3-2X_2^2+X_2X_3^2$	$S_{(0,1,2)}$	$1-X_2^2-X_1X_3+X_2X_3^2$
$C_{(2,1,0)}$	$4-X_1X_3+X_1^2X_2-2X_2^2$	$S_{(2,1,0)}$	$1+X_1^2X_2-X_2^2-X_1X_3$
# = 1		# = 1	
$C_{(1,0,0)}$	X_1	$S_{(1,0,0)}$	X_1
$C_{(0,1,1)}$	$-3X_1+X_2X_3$	$S_{(0,1,1)}$	$-X_1+X_2X_3$
$C_{(0,0,3)}$	$3X_1-3X_2X_3+X_3^3$	$S_{(0,0,3)}$	$X_1-2X_2X_3+X_3^3$
$C_{(2,0,1)}$	$-X_1-2X_2X_3+X_1^2X_3$	$S_{(2,0,1)}$	$-X_1+X_1^2X_3-X_2X_3$
$C_{(1,2,0)}$	$5X_1-X_2X_3-2X_1^2X_3+X_1X_2^2$	$S_{(1,2,0)}$	$X_1+X_1X_2^2-X_1^2X_3-X_2X_3$
# = 2		# = 2	
$C_{(0,1,0)}$	X_2	$S_{(0,1,0)}$	X_2
$C_{(0,0,2)}$	$-2X_2+X_3^2$	$S_{(0,0,2)}$	$-X_2+X_3^2$
$C_{(2,0,0)}$	$-2X_2+X_1^2$	$S_{(2,0,0)}$	$X_1^2-X_2$
$C_{(1,1,1)}$	$4X_2-3X_3^2-3X_1^2+X_1X_2X_3$	$S_{(1,1,1)}$	$-X_1^2-X_3^2+X_1X_2X_3$
# = 3		# = 3	
$C_{(0,0,1)}$	X_3	$S_{(0,0,1)}$	X_3
$C_{(1,1,0)}$	$-3X_3+X_1X_2$	$S_{(1,1,0)}$	$X_1X_2-X_3$
$C_{(3,0,0)}$	$3X_3-3X_1X_2+X_1^3$	$S_{(3,0,0)}$	$X_1^3-2X_1X_2+X_3$
$C_{(1,0,2)}$	$-X_3-2X_1X_2+X_1X_3^2$	$S_{(1,0,2)}$	$-X_1X_2-X_3+X_1X_3^2$
$C_{(0,2,1)}$	$5X_3-X_1X_2-2X_1X_3^2+X_2^2X_3$	$S_{(0,2,1)}$	$-X_1X_2-X_3-X_1X_3^2+X_2^2X_3$

The special recursion relations are obtained from the same products, where some of the components a, b, c of the generic dominant weight take special values 1 and 0. The explicit form of these relations is available in [30] and here we skip them in order to save the space, instead of this we adduce all their solutions of form of Table 7.

E.2. S Polynomials of A_3

Generic recursion relations are decompositions of the following products, where we assume that $a, b, c > 1$:

$$\begin{aligned}
X_1S_{(a,b,c)} &= S_{(a+1,b,c)} + S_{(a-1,b+1,c)} + S_{(a,b-1,c+1)} + S_{(a,b,c-1)},\\
X_2S_{(a,b,c)} &= S_{(a,b+1,c)} + S_{(a+1,b-1,c+1)} + S_{(a-1,b,c+1)} + S_{(a+1,b,c-1)} + S_{(a-1,b+1,c-1)} + S_{(a,b-1,c)},\\
X_3S_{(a,b,c)} &= S_{(a,b,c+1)} + S_{(a,b+1,c-1)} + S_{(a+1,b-1,c)} + S_{(a-1,b,c)}.
\end{aligned}\tag{E.4}$$

To calculate S polynomials explicitly (see Table 7) we use the A_3 characters. The lowest ones from the congruence classes # = 0, # = 1, # = 2 and # = 3 are listed below:

= 0:

$$\chi_{(0,0,0)} = C_{(0,0,0)} = 1,$$

$$\chi_{(1,0,1)} = 3 + C_{(1,0,1)},$$

$$\chi_{(0,2,0)} = 2 + C_{(1,0,1)} + C_{(0,2,0)},$$

$$\chi_{(0,1,2)} = 3 + 2C_{(1,0,1)} + C_{(0,2,0)} + C_{(0,1,2)},$$

$$\chi_{(2,1,0)} = 3 + 2C_{(1,0,1)} + C_{(0,2,0)} + C_{(2,1,0)},$$

$\# = 1$:

$$\chi_{(1,0,0)} = C_{(1,0,0)} = X_1,$$

$$\chi_{(0,1,1)} = 2X_1 + C_{(0,1,1)},$$

$$\chi_{(2,0,1)} = 3X_1 + C_{(0,1,1)} + C_{(2,0,1)},$$

$$\chi_{(0,0,3)} = X_1 + C_{(0,1,1)} + C_{(0,0,3)},$$

$$\chi_{(1,2,0)} = 3X_1 + 2C_{(0,1,1)} + C_{(2,0,1)} + C_{(1,2,0)},$$

$\# = 2$:

$$\chi_{(0,1,0)} = C_{(0,1,0)} = X_2,$$

$$\chi_{(0,0,2)} = X_2 + C_{(0,0,2)},$$

$$\chi_{(2,0,0)} = X_2 + C_{(2,0,0)},$$

$$\chi_{(1,1,1)} = 4X_2 + 2C_{(0,0,2)} + 2C_{(2,0,0)} + C_{(1,1,1)},$$

$\# = 3$:

$$\chi_{(0,0,1)} = C_{(0,0,1)} = X_3,$$

$$\chi_{(1,1,0)} = 2X_3 + C_{(1,1,0)},$$

$$\chi_{(1,0,2)} = 3X_3 + C_{(1,1,0)} + C_{(1,0,2)},$$

$$\chi_{(3,0,0)} = X_3 + C_{(1,1,0)} + C_{(3,0,0)},$$

$$\chi_{(0,2,1)} = 3X_3 + 2C_{(1,1,0)} + C_{(1,0,2)} + C_{(0,2,1)}.$$

E.3. Recursion Relations for C and S Polynomials of B_3 and C_3

The two cases differ in many important respects in spite of the isomorphism of their Weyl groups.

We write the generic relations for the C polynomials of the Lie algebras B_3 and C_3 (resp., of the simple Lie group $O(7)$ and $Sp(6)$). The generic relations for the S-polynomials are obtained by replacing the C symbol by S.

The variables are denoted by the same symbols X_1, X_2, X_3 for all algebras of rank 3, namely, $X_j := C_{\omega_j}$, $j = 1, 2, 3$. There are two congruence classes of (a_1, a_2, a_3) for either of the two algebras.

We have

$$\#(B_3) = a_3 \bmod 2, \qquad \#(C_3) = a_1 + a_3 \bmod 2. \tag{E.5}$$

Table 8: C polynomials of B_3 split into two congruence classes # = 0 and # = 1.

	C polynomials
# = 0	
$C_{(1,0,0)}$	X_1
$C_{(0,1,0)}$	X_2
$C_{(2,0,0)}$	$-6-2X_2+X_1^2$
$C_{(0,0,2)}$	$-8-4X_1-2X_2+X_3^2$
$C_{(1,1,0)}$	$24+8X_1+6X_2-3X_3^2+X_1X_2$
$C_{(1,0,2)}$	$-8X_1-2X_2-4X_1^2-2X_1X_2+X_1X_3^2$
$C_{(3,0,0)}$	$-24-15X_1-6X_2+X_3^2-3X_1X_2+X_1^3$
$C_{(0,2,0)}$	$12+16X_1+8X_2+4X_1^2+4X_1X_2-2X_1X_3^2+X_2^2$
$C_{(0,1,2)}$	$-48-20X_1-20X_2+6X_3^2-6X_1X_2-2X_2^2+X_2X_3^2$
$C_{(2,1,0)}$	$8X_1-6X_2+4X_1^2+2X_1X_2-X_1X_3^2-2X_2^2+X_1^2X_2$
# = 1	
$C_{(0,0,1)}$	X_3
$C_{(1,0,1)}$	$-3X_3+X_1X_3$
$C_{(0,1,1)}$	$3X_3-2X_1X_3+X_2X_3$
$C_{(0,0,3)}$	$-9X_3-3X_1X_3-3X_2X_3+X_3^3$
$C_{(2,0,1)}$	$-3X_3-X_1X_3-2X_2X_3+X_1^2X_3$
$C_{(1,1,1)}$	$30X_3+12X_1X_3+8X_2X_3-3X_3^3-2X_1^2X_3+X_1X_2X_3$

For B_3, we have the generic recursion relations

$$\begin{aligned}
X_1C_{(a,b,c)} &= C_{(a+1,b,c)}+C_{(a-1,b+1,c)}+C_{(a,b-1,c+2)}+C_{(a,b+1,c-2)}+C_{(a+1,b-1,c)}\\
&\quad+C_{(a-1,b,c)}, \quad \text{for } a,b\geqslant 2,\ c\geqslant 3,\\
X_2C_{(a,b,c)} &= C_{(a,b+1,c)}+C_{(a+1,b-1,c+2)}+C_{(a-1,b,c+2)}+C_{(a+1,b+1,c-2)}+C_{(a-1,b+2,c-2)}\\
&\quad+C_{(a+2,b-1,c)}+C_{(a+1,b-2,c+2)}+C_{(a-2,b+1,c)}+C_{(a-1,b-1,c+2)}+C_{(a+1,b,c-2)}\\
&\quad+C_{(a-1,b+1,c-2)}+C_{(a,b-1,c)}, \quad a\geqslant 2,\ b,c\geqslant 3,\\
X_3C_{(a,b,c)} &= C_{(a,b,c+1)}+C_{(a,b+1,c-1)}+C_{(a+1,b-1,c+1)}+C_{(a-1,b,c+1)}-C_{(a+1,b,c-1)}\\
&\quad+C_{(a-1,b+1,c-1)}+C_{(a,b-1,c+1)}+C_{(a,b,c-1)}, \quad a,b,c\geqslant 2.
\end{aligned} \tag{E.6}$$

For C_3, we have the generic recursion relations

$$\begin{aligned}
X_1C_{(a,b,c)} &= C_{(a+1,b,c)}+C_{(a-1,b+1,c)}+C_{(a,b-1,c+1)}+C_{(a,b+1,c-1)}\\
&\quad+C_{(a+1,b-1,c)}+C_{(a-1,b,c)}, \quad a,b,c\geqslant 2,\\
X_2C_{(a,b,c)} &= C_{(a,b+1,c)}+C_{(a+1,b-1,c)}+C_{(a-1,b,c+1)}+C_{(a+1,b+1,c-1)}+C_{(a-1,b+2,c-1)}\\
&\quad+C_{(a+2,b-1,c)}+C_{(a+1,b-2,c+1)}+C_{(a-2,b+1,c)}+C_{(a+1,b,c-1)}+C_{(a-1,b-1,c+1)}\\
&\quad+C_{(a-1,b+1,c-1)}+C_{(a,b-1,c)}, \quad a,b\geqslant 3,\ c\geqslant 2,
\end{aligned}$$

Table 9: C polynomials of C_3 split into two congruence classes # = 0 and # = 1.

	C polynomials
# = 0	
$C_{(0,1,0)}$	X_2
$C_{(2,0,0)}$	$-6-2X_2+X_1^2$
$C_{(1,0,1)}$	$-2X_2+X_1X_3$
$C_{(0,2,0)}$	$12+8X_2-4X_1^2-2X_1X_3+X_2^2$
$C_{(2,1,0)}$	$-6X_2-X_1X_3-2X_2^2+X_1^2X_2$
$C_{(0,0,2)}$	$-8-8X_2+4X_1^2+4X_1X_3-2X_2^2+X_3^2$
$C_{(1,1,1)}$	$12X_2-4X_1X_3+4X_2^2-2X_1^2X_2-3X_3^2+X_1X_2X_3$
$C_{(0,3,0)}$	$9X_2+3X_1X_3+6X_2^2-3X_1^2X_2+3X_3^2-3X_1X_2X_3+X_2^3$
$C_{(0,1,2)}$	$-18X_2+3X_1X_3-12X_2^2+6X_1^2X_2+3X_3^2+3X_1X_2X_3-2X_2^3+X_2X_3^2$
# = 1	
$C_{(1,0,0)}$	X_1
$C_{(0,0,1)}$	X_3
$C_{(1,1,0)}$	$-4X_1-3X_3+X_1X_2$
$C_{(3,0,0)}$	$-3X_1+3X_3-3X_1X_2+X_1^3$
$C_{(0,1,1)}$	$4X_1+6X_3-2X_1X_2+X_2X_3$
$C_{(2,0,1)}$	$-9X_3-2X_2X_3+X_1^2X_3$
$C_{(1,2,0)}$	$12X_1-3X_3+9X_1X_2-4X_1^3-X_2X_3-2X_1^2X_3+X_1X_2^2$
$C_{(1,0,2)}$	$-12X_1-6X_3-6X_1X_2+4X_1^3-X_2X_3+4X_1^2X_3-2X_1X_2^2+X_1X_3^2$
$C_{(0,2,1)}$	$27X_3+12X_2X_3-6X_1^2X_3-2X_1X_3^2+X_2^2X_3$
$C_{(0,0,3)}$	$-27X_3-18X_2X_3+9X_1^2X_3+6X_1X_3^2-3X_2^2X_3+X_3^3$

$$\begin{aligned}X_3C_{(a,b,c)} &= C_{(a,b,c+1)}+C_{(a,b+2,c-1)}+C_{(a+2,b-2,c+1)}+C_{(a-2,b,c+1)}+C_{(a+2,b,c-1)}\\&\quad+C_{(a-2,b+2,c-1)}+C_{(a,b-2,c+1)}+C_{(a,b,c-1)},\quad a,b\geqslant 3,\ c\geqslant 2.\end{aligned}\tag{E.7}$$

Additional recursion relations for both cases are available in explicit form in [32] and here we present only their solutions in form of Tables 8 and 9.

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