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Research Article

Strong Unique Continuation for Solutions of a $p(x)$-Laplacian Problem

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We study the strong unique continuation property for solutions to the quasilinear elliptic equation $-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + V(x)|u|^{p(x)-2}u = 0$ in Ω where $V(x) \in L^{N/p(x)}(\Omega)$, Ω is a smooth bounded domain in $\mathbb{R}^N$, and $1 < p(x) < N$ for x in Ω.

1. Introduction and Preliminary Results

Let Ω be an open, connected subset in $\mathbb{R}^N$. Consider the Schrödinger Operator $H = -\Delta + V$. If $Hu = 0$, and if u vanishes of infinite order at one point $x_0 \in \Omega$ (see definitions in Section 3) imply that $u \equiv 0$ in Ω, then H has the Strong Unique Continuation Property (S.U.C.P). If, on the other hand, $Hu = 0$ in Ω, and $u = 0$ in Ω', an open subset of Ω, imply that $u \equiv 0$ in Ω, we say that H has the Weak Unique Continuation Property (W.U.C.P). In 1939 Carleman [1] showed that $H = -\Delta + V$ has the S.U.C.P whenever $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$. In order to prove this result he introduced a method, the so-called Carleman estimates, which has permeated almost all the subsequent works in the subject. For instance, Jerison and Kenig [2] showed that if $n > 2$, $p \geqslant N/2$ and $V \in L^p_{\mathrm{loc}}$, then H has the S.U.C.P.; Fabes et al. in [3] gave a positive answer for a radial potential V to Simon's conjecture, which stated that for a potential V in the Stummel-Kato class and $u \in H^1(\Omega)$ then H has the S.U.C.P. Other results were obtained by de Figueiredo and Gossez, but for Linear Elliptic Operators in the case $V \in L^{N/2}(\Omega)$, $N > 2$, [4]. Also, Loulit extended this property to $N = 2$ [5]. More recently, Hadi and Tsouli [6] proved Strong Unique Continuation Property for the p-Laplacian in the case $V \in L^{N/p}(\Omega)$, $p < N$ and p constant.

Equations involving variable exponent growth conditions have been intensively discussed in the last decade. A strong motivation in the study of such kind of problems is due to the fact that they can model with high accuracy various phenomena which arise from the study of elastic mechanics, electrorheological fluids, or image restoration; for information on modeling physical phenomena by equations involving $p(x)$-growth condition we refer to [7–12]. The understanding of such physical models was facilitated by the development of variable Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p(x)$ is a real-valued function. Variable exponent Lebesgue spaces appeared for the first time in literature as early as 1931 in an article by Orlicz [13]. The spaces $L^{p(x)}$ are special cases of Orlicz spaces L^{φ} originated by Nakano [14] and developed by Musielak and Orlicz [15, 16], where $f \in L^{\varphi}$ if and only if $\int \varphi(x, |f(x)|)dx < \infty$ for a suitable φ. For some interesting results on elliptic equation involving variable exponent growth conditions see [17–19]. We point out the presence of the $p(x)$-Laplace operator. This is a natural extension of the p-Laplace operator, with p positive constant. However, such generalizations are not trivial since the $p(x)$-Laplace operator possesses a more complicated structure than p-Laplace operator; for example, it is inhomogeneous.

In this paper we prove Strong Unique Continuation Property of the solutions of the quasilinear elliptic equation:

$$-\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + V(x)|u|^{p(x)-2}u = 0 \quad \text{in } \Omega, \tag{1.1}$$

where $1 < p(x) < N$, $V \in L^{N/p(x)}(\Omega)$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

Finally, we recall some definitions and basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^N$.

Set $C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1\}$. For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x), \qquad h^- = \inf_{x\in\Omega} h(x). \tag{1.2}$$

For $p \in C_+(\overline{\Omega})$, we introduce *the variable exponent Lebesgue space:*

$$L^{p(\cdot)}(\Omega) = \left\{u : u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}dx < \infty\right\}, \tag{1.3}$$

endowed with the so-called *Luxemburg norm:*

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0;\ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}, \tag{1.4}$$

which is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [20]. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents in $C_+(\overline{\Omega})$ such that $p_1 \le p_2$ in Ω, then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous [20, Theorem 2.8].

Let $L^{p'(\cdot)}(\Omega)$ be the conjugate space of $L^{p(\cdot)}(\Omega)$, obtained by conjugating the exponent pointwise, that is, $1/p(x) + 1/p'(x) = 1$, [20, Corollary 2.7]. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$ the following Hölder type inequality

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)} \tag{1.5}$$

is valid.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the $p(\cdot)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)}dx. \tag{1.6}$$

If (u_n), $u \in L^{p(\cdot)}(\Omega)$ then the following relations hold:

$$|u|_{p(\cdot)} < 1 \quad (= 1;\ > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 \quad (= 1;\ > 1), \tag{1.7}$$

$$|u|_{p(\cdot)} > 1 \Longrightarrow |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}, \tag{1.8}$$

$$|u|_{p(\cdot)} < 1 \Longrightarrow |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}, \tag{1.9}$$

$$|u_n - u|_{p(\cdot)} \longrightarrow 0 \Longleftrightarrow \rho_{p(\cdot)}(u_n - u) \longrightarrow 0, \tag{1.10}$$

since $p^+ < \infty$. For a proof of these facts see [20]. Spaces with $p^+ = \infty$ have been studied by Edmunds et al. [21].

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}. \tag{1.11}$$

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a separable and reflexive Banach space. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous, where $p^*(x) = Np(x)/(N - p(x))$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \ge N$ [20, Theorems 3.9 and 3.3] (see also [22, Theorems 1.3 and 1.1]).

The bounded variable exponent p is said to be Log-Hölder continuous if there is a constant $C > 0$ such that

$$|p(x) - p(y)| \le \frac{C}{-\log(|x - y|)} \tag{1.12}$$

for all $x, y \in \mathbb{R}^N$, such that $|x - y| \le 1/2$.

A bounded exponent p is Log-Hölder continuous in Ω if and only if there exists a constant $C > 0$ such that

$$|B|^{p_B^- - p_B^+} \leq C \tag{1.13}$$

for every ball $B \subset \Omega$ [23, Lemma 4.1.6, page 101].

As a result of the Log-Hölder continuous condition we have

$$\begin{gathered} r^{-(p_B^+ - p_B^-)} \leq C, \\ C^{-1} r^{-p(y)} \leq r^{p(x)} \leq C r^{-p(y)}, \end{gathered} \tag{1.14}$$

for all $x, y \in B := B(x_0, r) \subset \Omega$ and the constant C depends only on the constant Log-Hölder continuous. It's well known that Smooth Functions are dense in Variable Exponent Sobolev Spaces if the exponent p satisfies the Log-Hölder condition [23, Proposition 11.2.3, page 346].

2. On Fefferman's Type Inequality

For every $u \in W_0^{1,p(\cdot)}(\Omega)$ the norm Poincaré inequality

$$|u|_{L^{p(\cdot)}(\Omega)} \leq c \ \mathrm{diam}(\Omega) |\nabla u|_{L^{p(\cdot)}}, \tag{2.1}$$

$c = C(N, \Omega, c \log(p))$ holds (we refer to [24] for notations and proofs). Nevertheless, the modular inequality

$$\int_\Omega |u|^{p(x)} dx \leq C \int_\Omega |\nabla u|^{p(x)} dx, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega) \tag{2.2}$$

not always holds (see [18, Theorem 3.1]). It is known that (2.2) holds if, for instance (i) $N > 1$, and the function $f(t) := p(x_o + tw)$ is monotone [18, Theorem 3.4] with $x_o + tw$ with an appropriate setting in Ω; (ii) if there exists a function $\xi \geq 0$ such that $\nabla p \cdot \nabla \xi \geq 0$, $\|\nabla \xi\| \neq 0$ [25, Theorem 1]; (iii) If there exists $a : \Omega \to \mathbb{R}^N$ bounded such that div $a(x) \geq a_0 > 0$ for all $x \in \overline{\Omega}$ and $a(x) \cdot \nabla p(x) = 0$ for all $x \in \Omega$, [26, Theorem 1]. To the best of our knowledge necessary and sufficient conditions in order to ensure that

$$\inf_{u \in W^{1,p(\cdot)}(\Omega)/\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}}{\int_\Omega |u|^{p(x)}} > 0 \tag{2.3}$$

have not been obtained yet, except in the case $N = 1$, [18, Theorem 3.2]. The following definition is in order.

Definition 2.1. We say that $p(\cdot)$ belongs to the Modular Poincaré Inequality Class, MPIC(Ω), if there exist necessary conditions to ensure that

$$\int_{\Omega} |u|^{p(x)} \leq C \int_{\Omega} |\nabla u|^{p(x)}, \quad \forall u \in W_0^{1,p(\cdot)}(\Omega), \tag{2.4}$$

$C = C(N, \Omega, c_{\log}(p)) > 0$ holds.

In [27] Fefferman proved the following inequality:

$$\int_{\mathbb{R}^N} |u(x)|^p \left|f(x)\right| dx \leq C \int_{\mathbb{R}^N} |\nabla u(x)|^p dx \quad \forall u \in C_0^{\infty}\left(\mathbb{R}^N\right). \tag{2.5}$$

in the case $p = 2$, assuming f in the Morrey's space $L^{r,N-2r}(\mathbb{R}^N)$, with $1 < r \leq N/2$. Later in [28] Schechter showed the same result taking f in the Stummel-Kato class $S(\mathbb{R}^N)$. Chiarenza and Frasca [29] generalized Fefferman's result proving (2.5) under the assumption $f \in L^{r,N-pr}(\mathbb{R}^N)$, with $1 < r < N/p$ and $1 < p < N$. Zamboni in [30] generalized Schecter's result proving (2.5) under the assumption $f \in \widetilde{M}_p(\mathbb{R}^N)$, with $1 < p < N$. We stress out that it is not possible to compare the assumptions $f \in L^{r,N-pr}(\mathbb{R}^N)$, the Morrey class, and $f \in S(\mathbb{R}^N)$, the Stumel-Kato class. All the mentioned results were obtained for fixed p. The theory for a variable exponent spaces is a growing area but Modular Fefferman-type inequalities are more scarce than Poincaré inequalities in variable exponent setting. In [31] Cuadro and López proved inequality (2.6) for variable exponent spaces. We use such inequality in order to prove S.U.C.P. We include the proof for the convenience of the reader.

Theorem 2.2. *Let p be a Log-Hölder continuous exponent with $1 < p(x) < N$, and $p \in MPIC(\Omega)$. Let $V \in L^1_{\mathrm{loc}}(\Omega)$ with $0 < \varepsilon < V(x)$ almost everywhere. Then there exists a positive constant $C = C(N, \Omega, c_{\log}(p))$ such that*

$$\int_{\Omega} V(x) |u(x)|^{p(x)} dx \leq C \int_{\Omega} |\nabla u(x)|^{p(x)} dx \tag{2.6}$$

for any $u \in W_0^{1,p(x)}(\Omega)$.

Proof. Let $u \in W_0^{1,p(x)}(\Omega)$ supported in $B(x_0, r)$. Given that $V \in L^1_{\mathrm{loc}}(\Omega)$ the function

$$w(x) := \left(\int_{x_1^0}^{x_1} V(\xi_1, x_2, \ldots, x_n) d\xi_1, \ldots, \int_{x_N^0}^{x_N} V(x_1, \ldots, x_{N-1}, \xi_N) d\xi_N \right), \tag{2.7}$$

where $x_0 = (x_1^0, \ldots, x_N^0)$ and $x = (x_1, \ldots, x_N) \in B(x_0, r)$, is well defined. Notice that $\int_{x_i^0}^{x_i} V(x_1, \ldots, \xi_i, \ldots, x_n) d\xi_i \in C[x_i^0, x_i]$ for $i = 1, \ldots, N$ (Lemme VIII.2 [32]) so that $\operatorname{div} w(x) = NV(x)$. Moreover,

$$|V(x)|_{L^1(B(x_0,r))} \geq \int_{x_1^0}^{x_1} \cdots \int_{x_N^0}^{x_N} V(\xi) d\xi_n \cdots d\xi_1, \tag{2.8}$$

where $\xi = (\xi_1, \ldots, \xi_N)$. Therefore, $|w(x)| \leq \sqrt{N} |V(x)|_{L^1(B(x_0,r))}$.

A direct calculation leads to

$$\begin{aligned} \operatorname{div}\left(|u|^{p(x)} w(x)\right) = {} & |u(x)|^{p(x)} \operatorname{div} w(x) + p(x) |u|^{p(x)-2} u \nabla u \cdot w(x) \\ & + |u|^{p(x)} \log u \nabla p(x) \cdot w(x). \end{aligned} \tag{2.9}$$

Now the Divergence Theorem implies $\int_{B(x_0,r)} \operatorname{div}(|u|^{p(x)} w(x)) = 0$, and so

$$\begin{aligned} \int_{B(x_0,r)} |u(x)|^{p(x)} \operatorname{div} w(x) dx \leq {} & p^+ \int_{B(x_0,r)} |u(x)|^{p(x)-1} |\nabla u(x)| |w(x)| dx \\ & + \int_{B(x_0,r)} |u(x)|^{p(x)} \log |u(x)| \left| \nabla p(x) \right| |w(x)| dx. \end{aligned} \tag{2.10}$$

Set

$$\begin{aligned} I_1 &:= p^+ \int_{B(x_0,r)} |u(x)|^{p(x)-1} |\nabla u(x)| |w(x)| dx, \\ I_2 &:= \int_{B(x_0,r)} |u(x)|^{p(x)} \log |u(x)| \left| \nabla p(x) \right| |w(x)| dx. \end{aligned} \tag{2.11}$$

Now we estimate I_2 by distinguishing the case when $|u(x)| \leq 1$ and $|u(x)| > 1$. Notice that the relations

$$\sup_{0 \leq t \leq 1} t^\eta \left| \log t \right| < \infty \tag{2.12}$$

$$\sup_{t > 1} t^{-\eta} \log t < \infty \tag{2.13}$$

hold for $\eta > 0$.

Let $\Omega_1 =: \{x \in B_r : |u(x)| \leq 1\}$ and $\Omega_2 =: \{x \in B_r : |u(x)| > 1\}$, then for (2.12) and (2.13) we have

$$I_2 \leq C_1 \int_{\Omega_1} |w(x)| |u(x)|^{p(x)-\eta_1} dx + C_2 \int_{\Omega_2} |w(x)| |u(x)|^{p(x)+\eta_2} dx. \tag{2.14}$$

We can choose $k \in \mathbb{N}$ such that $p(x) - 1/k \geq p^-$. Since $u \in L^{p^-}(B(x_0, r))$ and in Ω_1, $|u(x)| \leq 1$ we have

$$|u(x)|^{p(x)-1/n} \leq |u(x)|^{p^-}, \tag{2.15}$$

for $n > k$. The Lebesgue Dominated Convergence Theorem implies

$$\lim_{n\to\infty} \int_{\Omega_1} |u(x)|^{p(x)-1/n} dx = \int_{\Omega_1} |u(x)|^{p(x)} dx. \tag{2.16}$$

For Ω_2 we can choose k' such that $p(x) + 1/k' \leq (p(x))^* = Np(x)/(N - p(x))$. So

$$|u(x)|^{p(x)+1/n} \leq |u(x)|^{(p(x))^*}, \tag{2.17}$$

$n > k'$, and $x \in \Omega_2$. Since $u \in L^{(p(x))^*}(B(x_0, r))$ [23, Theorem 8.3.1] we may use the Lebesgue Theorem again to obtain

$$\lim_{n\to\infty} \int_{\Omega_2} |u(x)|^{p(x)+1/n} dx = \int_{\Omega_2} |u(x)|^{p(x)} dx. \tag{2.18}$$

Given that $p \in \mathrm{MPIC}(\Omega)$ we have

$$I_2 \leq C \int_{B(x_0,r)} |u|^{p(x)} dx \leq C \int_{B(x_0,r)} |\nabla u|^{p(x)} dx. \tag{2.19}$$

Now we estimate I_1 by using the modular Young's inequality [24, equation (3.2.21)]:

$$I_1 \leq p^+ C_1 \int_{B(x_0,r)} |w(x)|^{p(x)/(p(x)-1)} |u(x)|^{p(x)} + p^+ C_2 \int_{B(x_0,r)} |\nabla u(x)|^{p(x)}. \tag{2.20}$$

Again, since $p \in \mathrm{MPIC}(\Omega)$ we obtain

$$I_1 \leq C \int_{B(x_0,r)} |\nabla u|^{p(x)} dx. \tag{2.21}$$

Finally, recalling that $\operatorname{div} w(x) = NV(x)$ we get

$$N \int_{B(x_o,r)} V(x) |u(x)|^{p(x)} \leq C \int_{B(x_0,r)} |\nabla u(x)|^{p(x)} dx, \tag{2.22}$$

which leads to the claim of the theorem. □

3. Strong Unique Continuation

Consider the equation

$$Hu := \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + V(x)|u|^{p(x)-2}u = 0, \quad x \in \Omega, \tag{3.1}$$

$u \in W_{\mathrm{loc}}^{1\cdot p(x)}(\Omega)$, $1 < p(x) < N$, $V \in L^{N/p(x)}(\Omega)$.

A weak solution of (3.1) is the function $u \in W_{\mathrm{loc}}^{1\cdot p(x)}(\Omega)$ such that

$$\int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi \, dx + \int_{\Omega} V(x)|u|^{p(x)-2}u \cdot \varphi \, dx = 0, \tag{3.2}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

The main interest of this section is to prove a unique continuation result for solutions of (3.1) according to the following definition.

Definition 3.1. A function $u \in L_{\mathrm{loc}}^{p(x)}(\Omega)$ has a zero of infinite order in the $p(x)$-mean at a point $x_0 \in \Omega$ if, for each $k \in \mathbb{N}$,

$$\int_{|x-x_0|\leqslant R} |u|^{p(x)} dx = O\left(R^k\right). \tag{3.3}$$

Recall that $\Omega \subset \mathbb{R}^N$ is a bounded open set. We want to prove estimates' independency of p^+ for bounded solutions. For this purpose we assume throughout this section that $1 < p^- \le p^+ < \infty$ and p is Lipschitz continuous. In particular, p is Log-Hölder continuous. The new feature in the estimate is the choice of a test function which includes the variable exponent. This has both advantages and disadvantages: we need to assume that p is differentiable almost everywhere, but, on the other hand, we avoid terms involving p^+, which would be impossible to control later, see [24].

Before proving Theorem 3.5 which is the main result of this paper we require the following Lemmas.

Lemma 3.2. *Let p be a Log-Hölder continuous exponent with $1 < p(x) < N$, and $p \in MPIC(\Omega)$. Let $V \in L^{N/p(x)}$, $0 < \epsilon < V(x)$, almost everywhere and $u \in W_0^{1,p(x)}(\Omega)$. Then, for each $\epsilon_o > 0$, there exists K such that*

$$\int_{\Omega} V|u|^{p(x)} dx \le \epsilon_o \int_{\Omega} |\nabla u|^{p(x)} + K\int_{\Omega} |u|^{p(x)} dx. \tag{3.4}$$

Proof. Let $\epsilon_o > 0$ be given. We have

$$\begin{aligned}\int_\Omega V(x)|u|^{p(x)} &= \int_{\{x:V(x)>t\}} V(x)|u|^{p(x)} + \int_{\{x:V(x)\le t\}} V(x)|u|^{p(x)} \\ &\le \int_{\{x:V(x)>t\}} V(x)|u|^{p(x)} + t\int_\Omega |u|^{p(x)} \\ &\le C\int_{\{x:V(x)>t\}} |\nabla u|^{p(x)} + t\int_\Omega |u|^{p(x)},\end{aligned} \tag{3.5}$$

where the last inequality follows from Theorem 2.2. Now, notice that the measure $\lambda(E) = \int_E |\nabla u|^{p(x)}$ is absolutely continuous with respect to the Lebesgue measure μ. It follows that for $\epsilon_1 := (\epsilon_o/C)\int_\Omega |\nabla u|^{p(x)} > 0$ there exists $\delta > 0$ such that $\int_E |\nabla u|^{p(x)} d\mu < \epsilon_1$ whenever $\mu(E) < \delta$. Moreover, by Chebyshev's type inequality,

$$\mu(\{x : V(x) > t\}) \le t^{-N/p^+}\int_\Omega V(x)^{N/p(x)}. \tag{3.6}$$

So taking t sufficiently big, we get the desired inequality. □

Lemma 3.3. *Let $p : \Omega \to (1, N)$ be an exponent with $1 < p^- \le p^+ < \infty$ and such that $p \in MPIC(\Omega)$ is Lipschitz continuous. Let u be solution of* (3.1) *in Ω, and B_r and B_{2r} two concentric balls contained in Ω. Then*

$$\int_{B_r} |\nabla u|^{p(x)} \le \frac{C}{r^{p(x_0)}}\int_{B_{2r}} |u|^{p(x)}, \tag{3.7}$$

where the constant C does not depend on r, $x_0 \in B_{2r}$ and $V \in L^{N/p(x)}$.

Proof. Take $\eta \in C_0^\infty(\Omega)$, with $\sup p\eta \subset B_{2r}$, $0 \le \eta \le 1$ such that $\eta(x) = 1$ for any $x \in B_r$ and $|\nabla\eta| \le C/r$. We want to use as test function $\varphi = \eta^{p(x)}u$. To this end we show first that $\varphi \in W_0^{1,p(x)}(\Omega)$; it is clear that $\varphi \in L^{p(x)}(\Omega)$ since u is solution of (3.1). Furthermore, since $0 \leqslant \eta \leqslant 1$ then $|\eta\log\eta| \leqslant a$ for some constant a, so

$$\begin{aligned}|\nabla\varphi| &\le \left|\nabla u\eta^{p(x)}\right| + \left|up(x)\eta^{p(x)-1}\nabla\eta\right| + \left|u\eta^{p(x)}\log\eta\nabla p(x)\right| \\ &\le \left|\nabla u\eta^{p(x)}\right| + \left|up(x)\eta^{p(x)-1}\nabla\eta\right| + \left|u\eta^{p(x)-1}\nabla p(x)\eta\log\eta\right| \\ &\le \left|\nabla u\eta^{p(x)}\right| + \left|up(x)\eta^{p(x)-1}\nabla\eta\right| + \left|u\eta^{p(x)-1}\nabla p(x)\right|a \\ &\le \left|\nabla u\eta^{p(x)}\right| + \left|u\eta^{p(x)-1}\right|\left(\left|\nabla\eta\right|p(x) + a\left|\nabla p(x)\right|\right) \\ &\le |\nabla u| + |u|(Cp^+ + aL) \\ &\le |\nabla u| + C_p|u|.\end{aligned} \tag{3.8}$$

Hence, $|\nabla\varphi|^{p(x)} \leqslant 2^{p^+-1}|\nabla u|^{p(x)} + C_p^{p^+}2^{p^+-1}|u|^{p(x)}$. Therefore, $|\nabla\varphi| \in L^{p(x)}(\Omega)$.
Now we can use $\varphi = \eta^{p(x)}u$ as a test function to obtain

$$\begin{aligned}
0 &= \int_{B_{2r}} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx + \int_{B_{2r}} V|u|^{p(x)-2}u\varphi\, dx \\
&= \int_{B_{2r}} |\nabla u|^{p(x)}\eta^{p(x)}dx + \int_{B_{2r}} |\nabla u|^{p(x)-2}u\nabla u \cdot \left(p(x)\eta^{p(x)-1}\nabla\eta + \eta^{p(x)}\log\eta\nabla p(x)\right)dx \\
&\quad + \int_{B_{2r}} V\eta^{p(x)}|u|^{p(x)}dx.
\end{aligned} \tag{3.9}$$

Let

$$\begin{gathered}
I_1 := \int_{B_{2r}} |\nabla u|^{p(x)-2}u\nabla u \cdot \left(p(x)\eta^{p(x)-1}\nabla\eta + \eta^{p(x)}\log\eta\nabla p(x)\right)dx, \\
I_2 := \int_{B_{2r}} V\eta^{p(x)}|u|^{p(x)}dx.
\end{gathered} \tag{3.10}$$

We can estimate I_1 by

$$\begin{aligned}
I_1 &\leq \int_{B_{2r}} |\nabla u|^{p(x)-2}|u||\nabla u|\left(\left|p(x)\eta^{p(x)-1}\nabla\eta\right| + \left|\eta^{p(x)}\log\eta\nabla p(x)\right|\right)dx \\
&\leq \int_{B_{2r}} \left(|\nabla u|\left|\eta\right|\right)^{p(x)-1}\left(|u|\left|\nabla\eta\right|\right)\left(\left|p(x)\right| + a\left|\nabla p(x)\right|\right)dx \\
&\leq \int_{B_{2r}} \left(|\nabla u|\left|\eta\right|\right)^{p(x)-1}\left(|u|\left|\nabla\eta\right|\right)\left(p^+ + aL\right)dx \\
&\leq C_p\int_{B_{2r}} \left(|\nabla u|\left|\eta\right|\right)^{p(x)-1}\left(|u|\left|\nabla\eta\right|\right)dx \\
&\leq C_p\int_{B_{2r}} \epsilon\left(|\nabla u|\left|\eta\right|\right)^{p(x)} + \left(\frac{1}{\epsilon}\right)^{p(x)-1}\left(|u|\left|\nabla\eta\right|\right)^{p(x)}dx,
\end{aligned} \tag{3.11}$$

where the Young-type inequality

$$fg \leq \epsilon f^{p(x)/(p(x)-1)} + \left(\frac{1}{\epsilon}\right)^{p(x)-1}g^{p(x)} \tag{3.12}$$

was used in the last inequality. Moreover,

$$\begin{aligned}
I_1 &\leq C_p\epsilon\int_{B_{2r}} |\nabla u|^{p(x)}\eta^{p(x)}dx + C_p\int_{B_{2r}} \left(\frac{1}{\epsilon}\right)^{p(x)-1}|u|^{p(x)}\left|\nabla\eta\right|^{p(x)}dx \\
&\leq C_p\epsilon\int_{B_{2r}} |\nabla u|^{p(x)}\eta^{p(x)}dx + C_p\left(\frac{1}{\epsilon}\right)^{p^+-1}\int_{B_{2r}} |u|^{p(x)}\left|\nabla\eta\right|^{p(x)}dx.
\end{aligned} \tag{3.13}$$

We estimate I_2:

$$\begin{aligned} I_2 &= \int_{B_{2r}} V\eta^{p(x)}|u|^{p(x)}dx \\ &= \int_{B_{2r}} V\left|\eta u\right|^{p(x)}dx \\ &\leq \epsilon\int_{B_{2r}} \left|\nabla(u\eta)\right|^{p(x)}dx + C_\epsilon\int_{B_{2r}}\left|\eta u\right|^{p(x)}dx, \end{aligned} \tag{3.14}$$

where Lemma (3.2) was used in the last inequality.

Now using the estimates for I_1 and I_2, we have

$$\begin{aligned} \int_{B_{2r}} |\nabla u|^{p(x)}\eta^{p(x)}dx &\leq C_p\epsilon\int_{B_{2r}}|\nabla u|^{p(x)}\eta^{p(x)}dx + C_p\left(\frac{1}{\epsilon}\right)^{p^+-1}\int_{B_{2r}}|u|^{p(x)}\left|\nabla\eta\right|^{p(x)}dx\,dx \\ &\quad + \epsilon\int_{B_{2r}}\left|\nabla(u\eta)\right|^{p(x)} + C_\epsilon\int_{B_{2r}}\left|\eta u\right|^{p(x)}dx \\ &\leq \epsilon(C_1+C_2)\int_{B_{2r}}|\nabla u|^{p(x)}\eta^{p(x)}dx \\ &\quad + \frac{\epsilon}{\epsilon^{p^+}}\left(C_1+C_2\epsilon^{p^+}\right)\int_{B_{2r}}|u|^{p(x)}\left|\nabla\eta\right|^{p(x)}dx + C_\epsilon\int_{B_{2r}}\eta^{p(x)}|u|^{p(x)}dx \end{aligned} \tag{3.15}$$

for $0<\epsilon\leq 1$. By choosing $\epsilon \leq \min\{1, 1/2(C_1+C_2)\}$, we have

$$\begin{aligned} \int_{B_{2r}} |\nabla u|^{p(x)}\eta^{p(x)}dx &\leq \frac{1}{\epsilon^{p^+}}\int_{B_{2r}}|u|^{p(x)}\left|\nabla\eta\right|^{p(x)}dx + 2C_\epsilon\int_{B_{2r}}|u|^{p(x)}\eta^{p(x)}dx \\ &\leq \frac{1}{\epsilon^{p^+}}\int_{B_{2r}}|u|^{p(x)}\left(\frac{C}{r}\right)^{p(x)}dx + 2C_\epsilon\int_{B_{2r}}|u|^{p(x)}\left(\frac{C}{r}\right)^{p(x)}dx. \end{aligned} \tag{3.16}$$

Since $p(x)$ is Log-Hölder $r^{-p(x)} \leq Cr^{-p(x_0)}$ for all $x_0 \in B_{2r}$, then

$$\begin{aligned} \int_{B_{2r}} |\nabla u|^{p(x)}\eta^{p(x)}dx &\leq \frac{1}{\epsilon^{p^+}}\int_{B_{2r}}|u|^{p(x)}\left(\frac{C}{r}\right)^{p(x_0)}dx + 2C_\epsilon\int_{B_{2r}}|u|^{p(x)}\left(\frac{C}{r}\right)^{p(x_0)}dx \\ &\leq \left(\frac{1}{\epsilon^{p^+}}+C_\epsilon\right)\frac{C^{p(x_0)}}{r^{p(x_0)}}\int_{B_{2r}}|u|^{p(x)}dx \\ &\leq \frac{C}{r^{p(x_0)}}\int_{B_{2r}}|u|^{p(x)}dx. \end{aligned} \tag{3.17}$$

Therefore,

$$\int_{B_r} |\nabla u|^{p(x)} dx \leq \frac{C}{r^{p(x_0)}} \int_{B_{2r}} |u|^{p(x)} dx. \tag{3.18}$$

□

Lemma 3.4. *Let $u \in W^{1,1}(B(x_0, r))$, where $B(x_0, r)$ is the ball of radius $r > 0$ in $\mathbb{R}^N$ and $E = \{x \in B(x_0, r) : u(x) = 0\}$. Then there exists a constant $\beta > 0$ depending only on N, such that*

$$\int_D |u(x)| dx \leq \beta \frac{r^N}{|E|} |D|^{1/N} \int_{B(x_0,r)} |\nabla u(x)| dx \tag{3.19}$$

for all $B(x_0, r)$, u as above, and all mensurable sets $D \subset B(x_0, r)$.

Proof. See [33, Lemma 3.4, page 54]. □

Now we are ready to prove the main result in this paper.

Theorem 3.5. *Let Ω be a bounded domain in $\mathbb{R}^N$, $p : \Omega \longrightarrow (1, N)$ an exponent with $1 < p^- \leq p^+ < \infty$ and such that $p \in MPIC(\Omega)$ is Lipschitz continuous, and $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega)$ a solution of (3.1). If u vanishes on set $E \subset \Omega$ of positive measure, then u has a zero of infinite order in the $p(x)$-mean.*

Proof. We know that almost every point of E is a point of density, let x_0 be such a point, that is,

$$\frac{\left|E^C \cap B(x_0, r)\right|}{|B(x_0, r)|} \longrightarrow 0, \qquad \frac{|E \cap B(x_0, r)|}{|B(x_0, r)|} \longrightarrow 1 \tag{3.20}$$

as $r \longrightarrow 0$.

Let $B_r := B(x_0, r)$. So for a given $\epsilon > 0$, there exists $r_0 = r_0(\epsilon)$ such that for $r \leq r_0$,

$$\frac{\left|E^C \cap B_r\right|}{|B_r|} < \epsilon, \qquad \frac{|E \cap B_r|}{|B_r|} > 1 - \epsilon, \tag{3.21}$$

where E^C denote the complement of E in Ω. Taking r_0 smaller if necessary, we may assume that $B_{r_0} \subset \Omega$. Since $u = 0$ on E, and using Lemma 3.4 we have

$$\begin{aligned}
\int_{B_r} |u(x)|^{p(x)} dx &= \int_{B_r \cap E^C} |u(x)|^{p(x)} dx + \int_{B_r \cap E} |u(x)|^{p(x)} dx \\
&= \int_{B_r \cap E^C} |u(x)|^{p(x)} dx \\
&\leq \beta \frac{r^N}{|B_r \cap E|} \left|B_r \cap E^C\right|^{1/N} \int_{B_r} \left|\nabla\left(|u(x)|^{p(x)}\right)\right| dx \\
&\leq C\beta \frac{r^N}{r^{N-1}} \frac{\epsilon^{1/N}}{1-\epsilon} \int_{B_r} \left|\nabla\left(|u(x)|^{p(x)}\right)\right| dx \\
&= Cr \frac{\epsilon^{1/N}}{1-\epsilon} \int_{B_r} \left|\nabla\left(|u(x)|^{p(x)}\right)\right| dx.
\end{aligned} \tag{3.22}$$

But $|\nabla(|u(x)|^{p(x)})| \leq |p(x)||| u(x)|^{p(x)-1}||\nabla u(x)| + |\nabla p(x)||| u(x)|^{p(x)} \log |u(x)||$. Hence,

$$\int_{B_r} |u(x)|^{p(x)} dx \leq Cr\frac{\epsilon^{1/N}}{1-\epsilon}\left[\int_{B_r} p^+|u(x)|^{p(x)\,-1}|\nabla u(x)|dx + \int_{B_r} L|u(x)|^{p(x)}\left|\log|u(x)|\right|dx\right]. \tag{3.23}$$

Let

$$\begin{aligned} I_1 &:= \int_{B_r} |u(x)|^{p(x)-1}|\nabla u(x)|dx, \\ I_2 &:= \int_{B_r} \left||u(x)|^{p(x)} \log|u(x)|\right|dx. \end{aligned} \tag{3.24}$$

I_1 can be estimated using the Young type inequality with $\epsilon = 1/r$:

$$\int_{B_r} |u(x)|^{p(x)-1}|\nabla u(x)|dx \leqslant \int_{B_r} \frac{1}{r}|u(x)|^{p(x)}dx + \int_{B_r} r^{p(x)-1}|\nabla u(x)|^{p(x)}dx. \tag{3.25}$$

Now we estimate I_2 by distinguishing the case when $|u(x)| \leq 1$ and $|u(x)| > 1$, using the relations (2.2) and (2.13).

Let $\Omega_1 =: \{x \in B_r : |u(x)| \leq 1\}$ and $\Omega_2 =: \{x \in B_r : |u(x)| > 1\}$, then

$$I_2 \leq C_1\int_{\Omega_1} |u(x)|^{p(x)-\eta_1}dx + C_2\int_{\Omega_2} |u(x)|^{p(x)+\eta_2}dx. \tag{3.26}$$

We can choose $k \in \mathbb{N}$ such that $p(x) - 1/k \geq p^-$. Since $u \in L^{p^-}(B(x_0, r))$ and in $\Omega_1, |u(x)| \leq 1$ we have

$$|u(x)|^{p(x)-1/n} \leq |u(x)|^{p^-}, \tag{3.27}$$

for $n > k$. The Lebesgue Dominated Convergence Theorem implies

$$\lim_{n\to\infty}\int_{\Omega_1} |u(x)|^{p(x)-1/n}dx = \int_{\Omega_1} |u(x)|^{p(x)}dx. \tag{3.28}$$

For Ω_2 we can choose k' such that $p(x) + 1/k' \leq (p(x))^* = Np(x)/(N - p(x))$. So

$$|u(x)|^{p(x)+1/n} \leq |u(x)|^{(p(x))^*}, \tag{3.29}$$

for $n > k'$, and $x \in \Omega_2$. Since $u \in L^{(p(x))^*}(B(x_0, r))$ [23, Theorem 8.3.1] we may use the Lebesgue Theorem again to obtain

$$\lim_{n\to\infty}\int_{\Omega_2} |u(x)|^{p(x)+1/n}dx = \int_{\Omega_2} |u(x)|^{p(x)}dx. \tag{3.30}$$

Therefore,

$$I_2 \leq C \int_{B(x_0,r)} |u|^{p(x)} dx. \tag{3.31}$$

Now, using estimates for I_1 and I_2, and noticing that for $0 < r < 1$ we have $r^{p(x)} < r^{p^-}$, we get

$$\begin{aligned}\int_{B_r} |u(x)|^{p(x)} dx \leq \frac{\epsilon^{1/N}}{1-\epsilon} \Bigg\{ & Cp^+ \int_{B_r} |u(x)|^{p(x)} dx \\ & + Cp^+ r^{p^-} \int_{B_r} |\nabla u(x)|^{p(x)} dx + C_p \int_{B_r} |u(x)|^{p(x)} dx \Bigg\}\end{aligned} \tag{3.32}$$

and by Lemma (3.3) we have

$$\begin{aligned}\int_{B_r} |u(x)|^{p(x)} dx &\leq \frac{\epsilon^{1/N}}{1-\epsilon} \Bigg\{ Cp^+ \int_{B_{2r}} |u(x)|^{p(x)} dx + Cp^+ r^{p^-} r^{-p(x_0)} \int_{B_{2r}} |u(x)|^{p(x)} dx \\ &\qquad + C_p \int_{B_{2r}} |u(x)|^{p(x)} dx \Bigg\} \\ &\leq C \frac{\epsilon^{1/N}}{1-\epsilon} \int_{B_{2r}} |u(x)|^{p(x)} dx,\end{aligned} \tag{3.33}$$

where C is independent of ε and of r as $r \to 0$. Note that $r^{p^- - p(x_0)} < C$ where C is the Log-Hölder constant. From this point the argument in the proof is standard, see, for instance, in [4] the proof of Lemma 1, page 344-345 from equation (10) to the end of the proof, or the proof of Theorem 2.1 [6], from inequality (2.18) to (2.23), page 216; we include this last part of the proof for the sake of completeness. Set $f(r) := \int_{B_r} |u(x)|^{p(x)} dx$. Let us fix $n \in \mathbb{N}$ and choose $\epsilon > 0$ such that $(C\epsilon^{1/N})/(1-\epsilon) \leq 2^{-n}$. Now, observe that r_0 depends on n, hence by the last inequality we deduce

$$f(r) \leq 2^{-n} f(2r), \quad \text{for } r \leq r_0. \tag{3.34}$$

Iterating (3.34), we get

$$f(\rho) \leq 2^{-kn} f\left(2^k \rho\right), \quad \text{if } 2^{k-1}\rho \leq r_0. \tag{3.35}$$

Thus, given that $0 < r < r_0(n)$ and choosing $k \in \mathbb{N}$ such that

$$2^{-k} r_0 \leq r \leq 2^{-(k-1)} r_0. \tag{3.36}$$

From (3.35), we conclude that

$$f(r) \leq 2^{-kn} f\left(2^k r\right) \leq 2^{-kn} f(2r_0), \tag{3.37}$$

and since $2^{-k} \leq r/r_0$, we get

$$f(r) \leq \left(\frac{r}{r_0}\right)^n f(2r_0), \tag{3.38}$$

which shows that x_0 is a zero infinite order in $p(x)$-mean. □

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