

Research Article

Some Properties of Multiple Generalized q -Genocchi Polynomials with Weight α and Weak Weight β

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The present paper deals with the various q -Genocchi numbers and polynomials. We define a new type of multiple generalized q -Genocchi numbers and polynomials with weight α and weak weight β by applying the method of p -adic q -integral. We will find a link between their numbers and polynomials with weight α and weak weight β . Also we will obtain the interesting properties of their numbers and polynomials with weight α and weak weight β . Moreover, we construct a Hurwitz-type zeta function which interpolates multiple generalized q -Genocchi polynomials with weight α and weak weight β and find some combinatorial relations.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and \mathbb{C}_p denote the ring of p -adic rational integers, the field of p -adic rational numbers, the complex number field, and the completion of the algebraic closure of \mathbb{Q}_p , respectively. Let \mathbb{N} be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-v_p(p)} = 1/p$ (see [1–21]). When one talks of q -extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < 1$.

Throughout this paper, we use the following notation:

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}. \quad (1.1)$$

Hence $\lim_{q \rightarrow 1} [x]_q = x$ for all $x \in \mathbb{Z}_p$ (see [1–14, 16, 18, 20, 21]).

We say that $g : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and we write $g \in \text{UD}(\mathbb{Z}_p)$ if the difference quotients $\Phi_g : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{C}_p$ such that

$$\Phi_g(x, y) = \frac{g(x) - g(y)}{x - y} \quad (1.2)$$

have a limit $g'(a)$ as $(x, y) \rightarrow (a, a)$.

Let d be a fixed integer, and let p be a fixed prime number. For any positive integer N , we set

$$\begin{aligned} X = X_d &= \varprojlim_N \left(\frac{\mathbb{Z}}{dp^N \mathbb{Z}} \right), & X_1 &= \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a, p) = 1}} (a + dp\mathbb{Z}_p), \\ a + dp^N \mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned} \quad (1.3)$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

For any positive integer N ,

$$\mu_q(a + dp^N \mathbb{Z}_p) = \frac{q^a}{[dp^N]_q} \quad (1.4)$$

is known to be a distribution on X .

For $g \in \text{UD}(\mathbb{Z}_p)$, Kim defined the q -deformed fermionic p -adic integral on \mathbb{Z}_p :

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x) (-q)^x. \quad (1.5)$$

(see [1–13]), and note that

$$\int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \int_X g(x) d\mu_{-q}(x). \quad (1.6)$$

We consider the case $q \in (-1, 0)$ corresponding to q -deformed fermionic certain and annihilation operators and the literature given there in [9, 13, 14].

In [9, 12, 14, 19], we introduced multiple generalized Genocchi number and polynomials. Let χ be a primitive Dirichlet character of conductor $f \in \mathbb{N}$. We assume that f

is odd. Then the multiple generalized Genocchi numbers, $G_{n,\chi}^{(r)}$, and the multiple generalized Genocchi polynomials, $G_{n,\chi}^{(r)}(x)$, associated with χ , are defined by

$$F_{\chi}^{(r)}(t) = \left(\frac{2t \sum_{a=0}^{f-1} \chi(a)(-1)^a e^{at}}{e^{ft} + 1} \right)^r = \sum_{n=0}^{\infty} G_{n,\chi}^{(r)} \frac{t^n}{n!},$$

$$F_{\chi}^{(r)}(t, x) = \left(\frac{2t \sum_{a=0}^{f-1} \chi(a)(-1)^a e^{at}}{e^{ft} + 1} e^{tx} \right)^r = \sum_{n=0}^{\infty} G_{n,\chi}^{(r)}(x) \frac{t^n}{n!}.$$
(1.7)

In the special case $x = 0$, $G_{n,\chi}^{(r)} = G_{n,\chi}^{(r)}(0)$ are called the n th multiple generalized Genocchi numbers attached to χ .

Now, having discussed the multiple generalized Genocchi numbers and polynomials, we were ready to multiple-generalize them to their q -analogues. In generalizing the generating functions of the Genocchi numbers and polynomials to their respective q -analogues; it is more useful than defining the generating function for the Genocchi numbers and polynomials (see [12]).

Our aim in this paper is to define multiple generalized q -Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β . We investigate some properties which are related to multiple generalized q -Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β . We also derive the existence of a specific interpolation function which interpolate multiple generalized q -Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β at negative integers.

2. The Generating Functions of Multiple Generalized q -Genocchi Numbers and Polynomials with Weight α and Weak Weight β

Many mathematicians constructed various kinds of generating functions of the q -Genocchi numbers and polynomials by using p -adic q -Vokenborn integral. First we introduce multiple generalized q -Genocchi numbers and polynomials with weight α and weak weight β .

Let us define the generalized q -Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta)}(x)$ with weight α and weak weight β , respectively,

$$F_{\chi,q}^{(\alpha,\beta)}(t) = \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta)} \frac{t^n}{n!} = \int_X t\chi(x) e^{[x]_{q^\alpha} t} d\mu_{-q^\beta}(x),$$

$$F_{\chi,q}^{(\alpha,\beta)}(t, x) = \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta)}(x) \frac{t^n}{n!} = \int_X t\chi(y) e^{[x+y]_{q^\alpha} t} d\mu_{-q^\beta}(y).$$
(2.1)

By using the Taylor expansion of $e^{[x]_{q^\alpha} t}$, we have

$$\sum_{n=0}^{\infty} \int_X \chi(x) [x]_{q^\alpha}^n d\mu_{-q^\beta}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta)} \frac{t^{n-1}}{n!} = G_{0,\chi,q}^{(\alpha,\beta)} + \sum_{n=0}^{\infty} \frac{G_{n+1,\chi,q}^{(\alpha,\beta)}}{n+1} \frac{t^n}{n!}.$$
(2.2)

By comparing the coefficient of both sides of $t^n/n!$ in (2.2), we get

$$\frac{G_{n+1, \chi, q}^{(\alpha, \beta)}}{n+1} = \frac{[2]_{q^\beta}}{(1-q^\alpha)^n} \sum_{a=0}^{f-1} (-1)^a q^{\beta a} \chi(a) \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha a l} \frac{1}{1+q^{f(\alpha l + \beta)}}. \quad (2.3)$$

From (2.2) and (2.3), we can easily obtain that

$$\sum_{n=0}^{\infty} G_{n, \chi, q}^{(\alpha, \beta)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(t \int_X \chi(x) [x]_{q^\alpha}^n d\mu_{-q^\beta}(x) \right) \frac{t^n}{n!} = [2]_{q^\beta} t \sum_{l=0}^{\infty} (-1)^l q^{\beta l} \chi(l) e^{[l]_{q^\alpha} t}. \quad (2.4)$$

Therefore, we obtain

$$F_{X, q}^{(\alpha, \beta)}(t) = [2]_{q^\beta} t \sum_{l=0}^{\infty} (-1)^l q^{\beta l} \chi(l) e^{[l]_{q^\alpha} t} = \sum_{n=0}^{\infty} G_{n, \chi, q}^{(\alpha, \beta)} \frac{t^n}{n!}. \quad (2.5)$$

Similarly, we find the generating function of generalized q -Genocchi polynomials with weight α and weak weight β :

$$G_{0, \chi, q}^{(\alpha, \beta)}(x) = 0, \quad \frac{G_{n+1, \chi, q}^{(\alpha, \beta)}(x)}{n+1} = \int_X \chi(y) [x+y]_{q^\alpha}^n d\mu_{-q^\beta}(y) = [2]_{q^\beta} \sum_{l=0}^{\infty} (-1)^l q^{\beta l} \chi(l) [x+l]_{q^\alpha}^n. \quad (2.6)$$

From (2.6), we have

$$F_{X, q}^{(\alpha, \beta)}(t, x) = [2]_{q^\beta} t \sum_{l=0}^{\infty} (-1)^l q^{\beta l} \chi(l) e^{[x+l]_{q^\alpha} t} = \sum_{n=0}^{\infty} G_{n, \chi, q}^{(\alpha, \beta)}(x) \frac{t^n}{n!}. \quad (2.7)$$

Observe that $F_{X, q}^{(\alpha, \beta)}(t) = F_{X, q}^{(\alpha, \beta)}(t, 0)$. Hence we have $G_{n, \chi, q}^{(\alpha, \beta)} = G_{n, \chi, q}^{(\alpha, \beta)}(0)$. If $q \rightarrow 1$ into (2.7), then we easily obtain $F_X(t, x)$.

First, we define the multiple generalized q -Genocchi numbers $G_{n, \chi, q}^{(\alpha, \beta, r)}$ with weight α and weak weight β :

$$\begin{aligned} F_{X, q}^{(\alpha, \beta, r)}(t) &= [2]_{q^\beta}^r t^r \sum_{k_1, \dots, k_r=0}^{\infty} (-1)^{\sum_{i=1}^r k_i} q^{\beta \sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i) \right) e^{[\sum_{i=1}^r k_i]_{q^\alpha} t} \\ &= t^r \underbrace{\int_X \cdots \int_X}_{r\text{-times}} \chi(x_1) \cdots \chi(x_r) e^{[x_1 + \cdots + x_r]_{q^\alpha} t} d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_r) \\ &= \sum_{n=0}^{\infty} G_{n, \chi, q}^{(\alpha, \beta, r)} \frac{t^n}{n!}. \end{aligned} \quad (2.8)$$

Then we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \underbrace{\int_X \cdots \int_X}_{r\text{-times}} \chi(x_1) \cdots \chi(x_r) [x_1 + \cdots + x_r]_{q^\alpha}^n d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_r) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)} \frac{t^{n-r}}{n!} = \sum_{n=0}^{r-1} G_{n,\chi,q}^{(\alpha,\beta,r)} \frac{t^{n-r}}{n!} + \sum_{n=0}^{\infty} \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}}{\binom{n+r}{r} r!} \frac{t^n}{n!}, \end{aligned} \tag{2.9}$$

where $\binom{n+r}{r} = (n+r)!/n!r!$.

By comparing the coefficients on the both sides of (2.9), we obtain the following theorem.

Theorem 2.1. *Let $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$ and $n \in \mathbb{Z}_+$. Then one has*

$$\begin{aligned} G_{0,\chi,q}^{(\alpha,\beta,r)} &= G_{1,\chi,q}^{(\alpha,\beta,r)} = \cdots = G_{r-1,\chi,q}^{(\alpha,\beta,r)} = 0, \\ \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}}{\binom{n+r}{r} r!} &= \underbrace{\int_X \cdots \int_X}_{r\text{-times}} \chi(x_1) \cdots \chi(x_r) [x_1 + \cdots + x_r]_{q^\alpha}^n d\mu_{-q^\beta}(x_1) \cdots d\mu_{-q^\beta}(x_r) \\ &= \frac{[2]_{q^\beta}^r}{(1 - q^\alpha)^n} \sum_{a_1, \dots, a_r=0}^{f-1} \sum_{l=0}^n \binom{n}{l} \left(\prod_{i=1}^r \chi(a_i) \right) \frac{(-1)^{l+\sum_{i=1}^l a_i} q^{(\alpha+l\beta) \sum_{i=1}^l a_i}}{(1 + q^{f(\alpha+l\beta)})^r} \\ &= [2]_{q^\beta}^r \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_r=0}^{f-1} \binom{m+r-1}{m} (-1)^{\sum_{i=1}^r a_i+m} q^{\beta(\sum_{i=1}^r a_i+fm)} \times \left(\prod_{i=1}^r \chi(a_i) \right) \left[\sum_{i=1}^r a_i + fm \right]_{q^\alpha}^n. \end{aligned} \tag{2.10}$$

From now on, we define the multiple generalized q -Genocchi polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β .

$$\begin{aligned} F_{\chi,q}^{(\alpha,\beta,r)}(t, x) &= [2]_{q^\beta}^r t^r \sum_{k_1, \dots, k_r=0}^{\infty} (-1)^{\sum_{i=1}^r k_i} q^{\beta \sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i) \right) e^{[\sum_{i=1}^r k_i+x]_{q^\alpha} t} \\ &= t^r \underbrace{\int_X \cdots \int_X}_{r\text{-times}} \chi(y_1) \cdots \chi(y_r) e^{[x+y_1+\cdots+y_r]_{q^\alpha} t} d\mu_{-q^\beta}(y_1) \cdots d\mu_{-q^\beta}(y_r) \\ &= \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.11}$$

Then we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \underbrace{\int_X \cdots \int_X}_{r\text{-times}} \chi(y_1) \cdots \chi(y_r) [x + y_1 + \cdots + y_r]_{q^\alpha}^n d\mu_{-q^\beta}(y_1) \cdots d\mu_{-q^\beta}(y_r) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x) \frac{t^{n-r}}{n!} = \sum_{n=0}^{r-1} G_{n,\chi,q}^{(\alpha,\beta,r)}(x) \frac{t^{n-r}}{n!} + \sum_{n=0}^{\infty} \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x) t^n}{\binom{n+r}{r} r! n!}, \end{aligned} \quad (2.12)$$

where $\binom{n+r}{r} = (n+r)!/n!r!$.

By comparing the coefficients on the both sides of (2.12), we have the following theorem.

Theorem 2.2. *Let $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$ and $n \in \mathbb{Z}_+$. Then one has*

$$\begin{aligned} G_{0,\chi,q}^{(\alpha,\beta,r)}(x) &= G_{1,\chi,q}^{(\alpha,\beta,r)}(x) = \cdots = G_{r-1,\chi,q}^{(\alpha,\beta,r)}(x) = 0, \\ \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!} &= \underbrace{\int_X \cdots \int_X}_{r\text{-times}} \chi(y_1) \cdots \chi(y_r) [x + y_1 + \cdots + y_r]_{q^\alpha}^n d\mu_{-q^\beta}(y_1) \cdots d\mu_{-q^\beta}(y_r) \\ &= \frac{[2]_{q^\beta}^r}{(1 - q^\alpha)^n} \sum_{a_1, \dots, a_r=0}^{f-1} \sum_{l=0}^n \binom{n}{l} \left(\prod_{i=1}^r \chi(a_i) \right) \frac{(-1)^{l + \sum_{i=1}^r a_i} q^{alx + (al+\beta) \sum_{i=1}^r a_i}}{(1 + q^{f(al+\beta)})^r} \\ &= [2]_{q^\beta}^r \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_r=0}^{f-1} \binom{m+r-1}{m} (-1)^{\sum_{i=1}^r a_i + m} q^{\beta(\sum_{i=1}^r a_i + fm)} \\ &\quad \times \left(\prod_{i=1}^r \chi(a_i) \right) \left[\sum_{i=1}^r a_i + fm + x \right]_{q^\alpha}^n. \end{aligned} \quad (2.13)$$

In (2.11), we simply identify that

$$\begin{aligned} \lim_{q \rightarrow 1} F_{\chi,q}^{(\alpha,\beta,r)}(t, x) &= 2^r t^r \sum_{k_1, \dots, k_r=0}^{\infty} (-1)^{\sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i) \right) e^{(\sum_{i=1}^r k_i + x)t} \\ &= \left(\frac{2t \sum_{a=0}^{f-1} (-1)^a \chi(a) e^{at}}{1 + e^{ft}} \right)^r e^{tx} = F_X^{(r)}(t, x). \end{aligned} \quad (2.14)$$

So far, we have studied the generating functions of the multiple generalized q -Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β .

3. Modified Multiple Generalized q -Genocchi Polynomials with Weight α and Weak Weight β

In this section, we will investigate about modified multiple generalized q -Genocchi numbers and polynomials with weight α and weak weight β . Also, we will find their relations in multiple generalized q -Genocchi numbers and polynomials with weight α and weak weight β .

Firstly, we modify generating functions of $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$. We access some relations connected to these numbers and polynomials with weight α and weak weight β . For this reason, we assign generating function of modified multiple generalized q -Genocchi numbers and polynomials with weight α and weak weight β which are implied by $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$. We give relations between these numbers and polynomials with weight α and weak weight β .

We modify (2.11) as follows:

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t, x) = F_{\chi,q}^{(\alpha,\beta,r)}(q^{-\alpha x}t, x), \quad (3.1)$$

where $F_{\chi,q}^{(\alpha,\beta,r)}(t, x)$ is defined in (2.11).

From the above we know that

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t, x) = \sum_{n=0}^{\infty} q^{-(n+r)\alpha x} G_{n,\chi,q}^{(\alpha,\beta,r)}(x) \frac{t^n}{n!}. \quad (3.2)$$

After some elementary calculations, we attain

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t, x) = q^{-\alpha r x} e^{(q^{-\alpha x} [x]_{q^\alpha} t)} F_{\chi,q}^{(\alpha,\beta,r)}(t), \quad (3.3)$$

where $F_{\chi,q}^{(\alpha,\beta,r)}(t)$ is defined in (2.8).

From the above, we can assign the modified multiple generalized q -Genocchi polynomials $\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β as follows:

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t, x) = \sum_{n=0}^{\infty} \varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x) \frac{t^n}{n!}. \quad (3.4)$$

Then we have

$$\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x) = q^{-(n+r)\alpha x} G_{n,\chi,q}^{(\alpha,\beta,r)}(x). \quad (3.5)$$

Theorem 3.1. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has

$$\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x) = q^{-(n+r)\alpha x} \sum_{i=0}^n \binom{n}{i} q^{\alpha i x} [x]_{q^\alpha}^{n-i} G_{i,\chi,q}^{(\alpha,\beta,r)}. \quad (3.6)$$

Corollary 3.2. For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, by using (3.7), one easily obtains

$$\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x) = q^{-(n+r)\alpha x} \sum_{m=0}^{\infty} \sum_{j=0}^n \sum_{l=0}^{n-j} \binom{n}{j,l,n-j-l} \binom{n-j+m-1}{m} (-1)^l q^{\alpha\{(j+l)x+m\}} G_{j,\chi,q}^{(\alpha,\beta,r)}. \quad (3.7)$$

Secandly, by using generating function of the multiple generalized q -Genocchi polynomials with weight α and weak weight β , which is defined by (2.11), we obtain the following identities.

By using (2.13), we find that

$$\begin{aligned} \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!} &= [2]_{q^\beta}^r \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_r=0}^{f-1} \binom{m+r-1}{m} (-1)^{\sum_{i=1}^r a_i+m} \\ &\quad \times q^{\beta(\sum_{i=1}^r a_i+fm)} \left(\prod_{i=1}^r \chi(a_i) \right) \left[\sum_{i=1}^r a_i + fm + x \right]_{q^\alpha}^n \\ &= [2]_{q^\beta}^r \sum_{a_1, \dots, a_r=0}^{f-1} \sum_{l=0}^n \sum_{a=0}^l \binom{n}{a,l-a,n-l} (-1)^{a+\sum_{i=1}^r a_i} q^{\{\alpha(a+n-l)+\beta\} \sum_{i=1}^r a_i} \\ &\quad \times \left(\prod_{i=1}^r \chi(a_i) \right) \frac{[x]_{q^\alpha}^{n-l}}{(1-q^\alpha)^l (1+q^{\{\alpha(a+n-l)+\beta\}})^r}. \end{aligned} \quad (3.8)$$

Thus we have the following theorem.

Theorem 3.3. Let $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $r \in \mathbb{N}$. Then one has

$$\begin{aligned} \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!} &= [2]_{q^\beta}^r \sum_{a_1, \dots, a_r=0}^{f-1} \sum_{l=0}^n \sum_{a=0}^l \binom{n}{a,l-a,n-l} (-1)^{a+\sum_{i=1}^r a_i} q^{\{\alpha(a+n-l)+\beta\} \sum_{i=1}^r a_i} \\ &\quad \times \left(\prod_{i=1}^r \chi(a_i) \right) \frac{[x]_{q^\alpha}^{n-l}}{(1-q^\alpha)^l (1+q^{\{\alpha(a+n-l)+\beta\}})^r}. \end{aligned} \quad (3.9)$$

By using (2.13), we have

$$\begin{aligned} F_{\chi,q}^{(\alpha,\beta,r)}(t,x) &= [2]_{q^\beta}^r t^r \sum_{n=0}^{\infty} \sum_{l=0}^n \binom{n}{l} \frac{(-1)^l q^{\alpha l x}}{(1-q)^n} \sum_{a_1, \dots, a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} \\ &\quad \times q^{(\alpha l + \beta)(\sum_{i=1}^r a_i)} \left(\prod_{i=1}^r \chi(a_i) \right) \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-q^{\{\alpha l + \beta\}})^m \frac{t^n}{n!}. \end{aligned} \quad (3.10)$$

Thus we have

$$\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} [2]_{q^\beta}^r t^n \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} (1 - q^\alpha)^{-n} \sum_{a_1, \dots, a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} \times q^{(\alpha l + \beta)(\sum_{i=1}^r a_i)} \left(\prod_{i=1}^r \chi(a_i) \right) \left(1 + q^{f(\alpha l + \beta)} \right)^{-r} \frac{t^n}{n!}. \tag{3.11}$$

By comparing the coefficients of both sides of $(n + r)!/t^{n+r}$ in the above, we arrive at the following theorem.

Theorem 3.4. *Let $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$, $r \in \mathbb{N}$. Then one has*

$$\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!} = [2]_{q^\beta}^r \sum_{l=0}^n \binom{n}{l} (-1)^l q^{\alpha l x} (1 - q^\alpha)^{-n} \sum_{a_1, \dots, a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} \times q^{(\alpha l + \beta)(\sum_{i=1}^r a_i)} \left(\prod_{i=1}^r \chi(a_i) \right) \left(1 + q^{f(\alpha l + \beta)} \right)^{-r}. \tag{3.12}$$

From (2.12), we easily know that

$$\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} [2]_{q^\beta}^r \binom{n+r}{r} r! \sum_{k_1, \dots, k_r=0}^{\infty} (-1)^{\sum_{i=1}^r k_i} q^{\beta \sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i) \right) \times \left[x + \sum_{i=1}^r k_i \right]_{q^\alpha}^n \frac{t^{n+r}}{(n+r)!}. \tag{3.13}$$

From the above, we get the following theorem.

Theorem 3.5. *Let $r \in \mathbb{N}$, $k \in \mathbb{Z}_+$. Then one has*

$$G_{0,\chi,q}^{(\alpha,\beta,r)}(x) = G_{1,\chi,q}^{(\alpha,\beta,r)}(x) = \dots = G_{r-1,\chi,q}^{(\alpha,\beta,r)}(x) = 0, \tag{3.14}$$

$$G_{l+r,\chi,q}^{(\alpha,\beta,r)}(x) = [2]_{q^\beta}^r \binom{l+r}{r} r! \sum_{k_1, \dots, k_r=0}^{\infty} (-1)^{\sum_{i=1}^r k_i} q^{\beta \sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i) \right) \left[x + \sum_{i=1}^r k_i \right]_{q^\alpha}^l.$$

From (2.13), we have

$$\begin{aligned}
& \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,s)}(x) \frac{t^n}{n!} \\
&= [2]_{q^\beta}^{r+s} t^{r+s} \sum_{a_1, \dots, a_r=0}^{f-1} \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^{\sum_{i=1}^r a_i+m} q^{\beta(\sum_{i=1}^r a_i+fm)} \\
&\quad \times \left(\prod_{i=1}^r \chi(a_i) \right) e^{[\sum_{i=1}^r a_i+fm+x]_{q^\alpha} t} \sum_{b_1, \dots, b_s=0}^{f-1} \sum_{k=0}^{\infty} \binom{k+s-1}{k} (-1)^{\sum_{i=1}^s b_i+k} \\
&\quad \times q^{\beta(\sum_{i=1}^s b_i+fk)} \left(\prod_{i=1}^s \chi(b_i) \right) e^{[\sum_{i=1}^s b_i+fk+x]_{q^\alpha} t}.
\end{aligned} \tag{3.15}$$

By using Cauchy product in (3.15), we obtain

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{n-j,\chi,q}^{(\alpha,\beta,s)}(x) \frac{t^n}{n!} \\
&= [2]_{q^\beta}^{r+s} t^{r+s} \sum_{n=0}^{\infty} \sum_{j=0}^n \sum_{a_1, \dots, a_r=0}^{f-1} \sum_{b_1, \dots, b_s=0}^{f-1} \binom{j+r-1}{j} \binom{n-j+s-1}{n-j} \\
&\quad \times (-1)^{\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+n} q^{\beta(\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+fn)} \left(\prod_{i=1}^r \chi(a_i) \right) \left(\prod_{i=1}^s \chi(b_i) \right) \\
&\quad \times e^{[\sum_{i=1}^r a_i+fj+x]_{q^\alpha} t} e^{[\sum_{i=1}^s b_i+f(n-j)+x]_{q^\alpha} t}.
\end{aligned} \tag{3.16}$$

From (3.16), we have

$$\begin{aligned}
& \sum_{m=0}^{\infty} \left(\sum_{j=0}^m \binom{m}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{m-j,\chi,q}^{(\alpha,\beta,s)}(x) \right) \frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} [2]_{q^\beta}^{r+s} t^{r+s} \sum_{n=0}^{\infty} \sum_{j=0}^n \sum_{a_1, \dots, a_r=0}^{f-1} \sum_{b_1, \dots, b_s=0}^{f-1} \binom{j+r-1}{j} \binom{n-j+s-1}{n-j} \\
&\quad \times (-1)^{\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+n} q^{\beta(\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+fn)} \left(\prod_{i=1}^r \chi(a_i) \right) \left(\prod_{i=1}^s \chi(b_i) \right) \\
&\quad \times \left(\left[\sum_{i=1}^r a_i+fj+x \right]_{q^\alpha} + \left[\sum_{i=1}^s b_i+f(n-j)+x \right]_{q^\alpha} \right)^m \frac{t^m}{m!}.
\end{aligned} \tag{3.17}$$

By comparing the coefficients of both sides of $t^{m+r+s}/(m+r+s)!$ in (3.17), we have the following theorem.

Theorem 3.6. Let $r \in \mathbb{N}$ and $s \in \mathbb{Z}_+$. Then one has

$$\begin{aligned} & \frac{\sum_{j=0}^{l+r+s} \binom{l+r+s}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{l+r+s-j,\chi,q}^{(\alpha,\beta,s)}(x)}{\binom{l+r+s}{l} (r+s)!} \\ &= [2]_{q^\beta}^{r+s} \sum_{n=0}^{\infty} \sum_{j=0}^n \sum_{a_1,\dots,a_r=0}^{f-1} \sum_{b_1,\dots,b_s=0}^{f-1} \binom{j+r-1}{j} \binom{n-j+s-1}{n-j} \\ & \times (-1)^{\sum_{i=1}^r a_i + \sum_{i=1}^s b_i + n} q^{\beta(\sum_{i=1}^r a_i + \sum_{i=1}^s b_i + fn)} \left(\prod_{i=1}^r \chi(a_i) \right) \left(\prod_{i=1}^s \chi(b_i) \right) \\ & \times \left(\left[\sum_{i=1}^r a_i + fj + x \right]_{q^\alpha} + \left[\sum_{i=1}^s b_i + f(n-j) + x \right]_{q^\alpha} \right)^l. \end{aligned} \tag{3.18}$$

Corollary 3.7. In (3.18) setting $s = 1$, one has

$$\begin{aligned} & \frac{\sum_{j=0}^{l+r+1} \binom{l+r+1}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{l+r+1-j,\chi,q}^{(\alpha,\beta,1)}(x)}{\binom{l+r+1}{l} (r+1)!} \\ &= [2]_{q^\beta}^{r+1} \sum_{n=0}^{\infty} \sum_{j=0}^n \sum_{a_1,\dots,a_r=0}^{f-1} \sum_{b_1=0}^{f-1} \binom{j+r-1}{j} (-1)^{\sum_{i=1}^r a_i + b_1 + n} q^{\beta(\sum_{i=1}^r a_i + b_1 + fn)} \\ & \times \left(\chi(b_1) \prod_{i=1}^r \chi(a_i) \right) \left(\left[\sum_{i=1}^r a_i + fj + x \right]_{q^\alpha} + [b_1 + f(n-j) + x]_{q^\alpha} \right)^l. \end{aligned} \tag{3.19}$$

By using (2.13) we have the following theorem.

Theorem 3.8. Distribution theorem is as follows:

$$\begin{aligned} G_{n+r,\chi,q}^{(\alpha,\beta,r)} &= \frac{[f]_{q^\alpha}^n}{[f]_{-q^\beta}^r} \sum_{a_1,\dots,a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} q^{\beta \sum_{i=1}^r a_i} \left(\prod_{i=1}^r \chi(a_i) \right) G_{n+r,q^f}^{(\alpha,\beta,r)} \left(\frac{a_1 + \dots + a_r}{f} \right), \\ G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x) &= \frac{[f]_{q^\alpha}^n}{[f]_{-q^\beta}^r} \sum_{a_1,\dots,a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} q^{\beta \sum_{i=1}^r a_i} \left(\prod_{i=1}^r \chi(a_i) \right) G_{n+r,q^f}^{(\alpha,\beta,r)} \left(\frac{x + a_1 + \dots + a_r}{f} \right). \end{aligned} \tag{3.20}$$

4. Interpolation Function of Multiple Generalized q -Genocchi Polynomials with Weight α and Weak Weight β

In this section, we see interpolation function of multiple generalized q -Genocchi polynomials with weak weight α and find some relations.

Let us define interpolation function of the $G_{k+r,q}^{(\alpha,\beta,r)}(x)$ as follows.

Definition 4.1. Let $q, s \in \mathbb{C}$ with $|q| < 1$ and $0 < x \leq 1$. Then one defines

$$\zeta_{\chi,q}^{(\alpha,\beta,r)}(s, x) = [2]_{q^\beta}^r \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-1)^{\sum_{i=1}^r k_i} q^{\beta \sum_{i=1}^r k_i} (\prod_{i=1}^r \chi(k_i))}{[x + \sum_{i=1}^r k_i]_{q^\alpha}^s}. \quad (4.1)$$

We call $\zeta_q^{(\alpha,\beta,r)}(s, x)$ the multiple generalized Hurwitz type q -zeta function.

In (4.1), setting $r = 1$, we have

$$\zeta_{\chi,q}^{(\alpha,\beta,1)}(s, x) = [2]_{q^\beta} \sum_{l=0}^{\infty} \frac{(-1)^l q^{\beta l} \chi(l)}{[x + l]_{q^\alpha}^s} = \zeta_{\chi,q}^{(\alpha,\beta)}(s, x). \quad (4.2)$$

Remark 4.2. It holds that

$$\lim_{q \rightarrow 1} \zeta_{\chi,q}^{(\alpha,\beta,r)}(s, x) = 2^r \sum_{k_1, \dots, k_r=0}^{\infty} \frac{(-1)^{\sum_{i=1}^r k_i} (\prod_{i=1}^r \chi(k_i))}{(x + \sum_{i=1}^r k_i)^s}. \quad (4.3)$$

Substituting $s = -n, n \in \mathbb{Z}_+$ into (4.1), then we have,

$$\zeta_{\chi,q}^{(\alpha,\beta,r)}(-n, x) = [2]_{q^\beta}^r \sum_{k_1, \dots, k_r=0}^{\infty} (-1)^{\sum_{i=1}^r k_i} q^{\beta \sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i) \right) \left[x + \sum_{i=1}^r k_i \right]_{q^\alpha}^{-n}. \quad (4.4)$$

Setting (3.14) into the above, we easily get the following theorem.

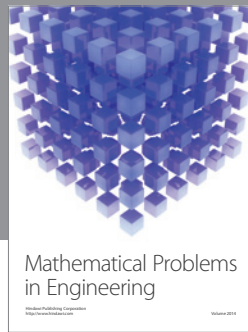
Theorem 4.3. Let $r \in \mathbb{N}, n \in \mathbb{Z}_+$. Then one has

$$\zeta_{\chi,q}^{(\alpha,\beta,r)}(-n, x) = \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!}. \quad (4.5)$$

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