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Research Article

Normed Ordered and *E*-Metric Spaces

Ahmed Al-Rawashdeh,[1] Wasfi Shatanawi,[2] and Muna Khandaqji[2]

[1] *Department of Mathematical Sciences, UAEU, Al-Ain 17551, UAE*
[2] *Department of Mathematics, Hashemite University, Zarqa 13115, Jordan*

Correspondence should be addressed to Ahmed Al-Rawashdeh, aalrawashdeh@uaeu.ac.ae

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In 2007, Haung and Zhang introduced the notion of cone metric spaces. In this paper, we define an ordered space E, and we discuss some properties and examples. Also, normed ordered space is introduced. We recall properties of $\mathbb{R}$, and we discuss their extension to E. We introduce the notion of E-metric spaces and characterize cone metric space. Afterwards, we get generalizations of notions of convergence and Cauchy theory. In particular, we get a fixed point theorem of a contractive mapping in E-metric spaces. Finally, by extending the notion of a contractive sequence in a real-valued metric space, we show that in E-metric spaces, a contractive sequence is Cauchy.

1. Introduction

Recall that the space of real numbers $\mathbb{R}$ is a normed space which having the usual ordering $\leq$, such that it is translation invariant, that is, for all x, y, and z in $\mathbb{R}$, $x \leq y$ implies that $x + z \leq y + z$. Also, for any space X, a metric d (real-valued metric) defines a metric space (X, d).

Recently, the concept of a cone metric space has been studied in [1–15], and others. Indeed, they proved some fixed-point theorems of generalized contractive mappings. In particular, Huang and Zhang in [7] introduced the following.

Definition 1.1 (see [7]). Let P be a subset of a normed space E with $\mathrm{Int}(P) \neq \emptyset$. Then P is called a cone if

(1) P is closed and $P \neq \{0\}$.

(2) $a, b \in \mathbb{R}^+,\ x, y \in P \Rightarrow ax + by \in P$.

(3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

A cone P induces a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. The notion $x < y$ means that $x \leq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{Int}(P)$. A cone P is called normal if there exists a number $k > 0$ such that for all $x, y \in E$, we have

$$0 \leq x \leq y \Longrightarrow \|x\| \leq k\|y\|. \tag{1.1}$$

The least positive number k, satisfying (1.1), is called the normal constant of P. Vandergraft in [16] (or see Example 2.1 of [10]) presented an example of a nonnormal cone, that is, a cone of a normed space E which does not satisfy (1.1).

Definition 1.2 (See [7]). Let X be a nonempty set, and let E be a normed space having a normal cone P with $\operatorname{Int}(P) \neq \emptyset$. Suppose, the mapping $d : X \times X \to E$ satisfies the following:

(1) $0 < d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$, for all $x, y \in X$,

(3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Remark 1.3. In [7] and in order to define a cone metric space, the authors started by a Banach space E (see [7, page 1469]). But in fact, and up to the results of the current paper, they just need E to be a normed space only.

More generally, Karapınar in [17] and Shatanawi in [18, 19] studied the couple fixed-point theorems in cone metric spaces. Also, Shatanawi in [20] studied common coincidence point in cone metric spaces.

In this paper, we start with different approach, and without starting by cones, we introduce the notion of normed ordered spaces in general (see Definition 2.1) which generalizes most of the properties of $\mathbb{R}$ (also the notion of ordered field is already defined, see [21, page 7]). We show that many results can be extended to a normed ordered space E; however, some properties cannot, as depending on crucial properties of $\mathbb{R}$. In the second part, and considering a nonempty set X, we replace the real-valued metric with an E-valued metric (denoted by d^E), and (X, d^E) is called E-metric space, we discuss some examples. We generalize results of real-valued metric spaces (see, e.g., [22]) concerning sequences to the case of E-metric spaces. Afterwards, we introduce the notions of convergence and Cauchy sequences in E-metric space. Considering E to be a normed ordered space and that X has an E-valued matric, we give a characterization of cone metric spaces in the sense of Huang and Zhang in [7]. Then we get results concerning convergence and Cauchy theory in the case of E-metric space. In particular, we get a fixed-point theorem of a contractive mapping. Finally, and in a similar way of a real-valued metric space, we introduce the notion of a contractive sequence in an E-metric space, and we prove that every contractive sequence is Cauchy.

Notice that ordered spaces need not be totally ordered. In fact, many properties of $\mathbb{R}$ are deduced as $\mathbb{R}$ being totally ordered. We say that E has the completeness property (CP) if every subset of an upper bound has a supremum in E and every subset of a lower bound has an infemum in E. If E is an ordered space, then all types of intervals can be defined. Indeed the unbounded intervals (left and right rays) are

$$(a, \infty) = \{x \in E;\ a < x\}, \qquad (-\infty, a) = \{x \in E;\ x < a\}. \tag{1.2}$$

2. Normed Ordered Spaces

In this section, we define normed ordered spaces, and we give different examples. Then we discuss generalizations of main results in the real space to normed ordered spaces. In particular, the Bolzano-Weierstrass theorem and intermediate value theorem are not true in general normed ordered spaces. The properties (CP) and totally ordered of $\mathbb{R}$, are in fact seems to be crucial.

Definition 2.1. An ordered space E is a vector space over the real numbers, with a partial order relation $\leq$ such that

(O1) for all x, y, and z in E, $x \leq y$ implies $x + z \leq y + z$ (translation invariant),

(O2) for all $\alpha \in \mathbb{R}^+$ and $x \in E$ with $x \geq 0$, $\alpha x \geq 0$.

Moreover, if E is equipped with a norm $\|\cdot\|$ such that

(O3) there exists a real number $k > 0$ and for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$, then E is called a normed ordered space.

By the translation invariant (O1), $x \leq y$ means $y - x \geq 0$. The strict inequality $x < y$ stands for $x \leq y$ and $x \neq y$.

Proposition 2.2. *Let E be an ordered space. Then for all x and y in E, we have*

(i) $x \geq 0$ *implies that* $-x \leq 0$,

(ii) *If* $x \geq 0$ *and* $-x \geq 0$, *then* $x = 0$,

(iii) *If* $x \geq 0$ *and* $y \geq 0$, *then* $x + y \geq 0$,

(iv) *for all* $\alpha \in \mathbb{R}^-$ *and* $x \geq 0$, $\alpha x \leq 0$,

(v) $x > y$ *and* $\alpha \in \mathbb{R}^+$ *implies that* $\alpha x > \alpha y$,

(vi) $x > y$ *and* $\alpha \in \mathbb{R}^-$ *implies that* $\alpha x < \alpha y$.

Proof. (i) As $x \geq 0$, by translation invariant, we have $-x + x \geq -x$, hence $-x \leq 0$. (ii) Assuming that $-x \geq 0$ implies by (i) that $x = -(-x) \leq 0$ so we get $x \leq 0 \leq x$, and as the relation is an antisymmetric, we have $x = 0$. (iii) As $x \geq 0$, we get $x + y \geq y \geq 0$, hence the result holds by transitivity. (iv) As $\alpha \in \mathbb{R}^-$, we have $-\alpha \in \mathbb{R}^+$ and by (O2) this implies that $-\alpha x \geq 0$. Using (i) $-(-\alpha x) \leq 0$, hence $\alpha x \leq 0$. (v) As $x > y$, then $x - y > 0$, and by (O2) we get $\alpha(x - y) > 0$, hence $\alpha x > \alpha y$. (vi) is similar. □

Now let us introduce the following examples of normed ordered spaces.

Example 2.3. (a) The set of real numbers $\mathbb{R}$ with usual ordering and the absolute value.

(b) The set $\mathbb{R}^n$ with the ordering defined by

$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \Longleftrightarrow x_i \leq y_i, \quad 1 \leq i \leq n. \tag{2.1}$$

This ordering is called the simplicial ordering of $\mathbb{R}^n$. Then $\mathbb{R}^n$ together with simplicial ordering and the Euclidean norm is a normed ordered space.

(c) The set of rational numbers $\mathbb{Q}$ as a vector space over itself together with absolute value and usual ordering.

(d) The complex space $\mathbb{C}$ together with the modulus, and the order defined by

$$z \leq w \Longleftrightarrow w - z \in \mathbb{R}, \quad w - z \geq 0 \tag{2.2}$$

is a normed ordered space.

(e) The space $C[0,1]$ of all continuous real-valued functions on $[0,1]$ together with the supremum norm,

$$\|f\| = \sup\{f(x);\ x \in [0,1]\} \tag{2.3}$$

and the pointwise ordering,

$$f \leq g \Longleftrightarrow f(x) \leq g(x); \quad \forall x \in [0,1] \tag{2.4}$$

is a normed ordered space.

(f) Let (X, μ) be any measure space. Then for all $1 \leq p \leq \infty$, the space $L^p(X)$ with the norm,

$$\|f\|_p = \left(\int_X |f^p| d\mu\right)^{1/p} \tag{2.5}$$

and the pointwise ordering is a normed order space.

Definition 2.4. Let E be an ordered space. Then an element $v \in E$ is called positive if $v \geq 0$, and it is called strict positive if $v > 0$. The set of all positive elements in E is denoted by E^+.

Remark 2.5. The following result is valid for any ordered space having a norm, and not necessarily satisfying the property (O3). So the space needs not to be a normed ordered space. Then according to Definition 2.1 with the characterization in Theorem 3.8, this is equivalent to say that a cone P is nonnormal (indeed this exists, see Example 2.1 of [10]).

Proposition 2.6. *Let E be an ordered space having a norm. Then E^+ is a closed set.*

Proof. We will prove that $E \setminus E^+$ is an open subset of E. Given that $a \in E \setminus E^+$. Define that $d = \inf\{\|x - a\|;\ x \in E^+\}$, then

$$B(a, d) = \{x \in E; \|x - a\| < d\} \tag{2.6}$$

is an open ball containing a. Now we claim that $B(a, d) \subseteq E \setminus E^+$: if not, then there exists $x_0 \in B(a, d)$, which means that $\|x_0 - a\| < d$, and $x_0 \in E^+$, $\|x_0 - a\| \geq d$, hence we get a contradiction. Then the claim is proved. □

The interior of E^+ is denoted by $\mathrm{Int}(E^+)$. In the following example, we show that $\mathrm{Int}(C[0,1]^+)$ consists of functions that never touch the x-axis.

Example 2.7. If $E = C[0,1]$, then $\operatorname{Int}(E^+) = \{f \in E^+;\ f(x) \neq 0;\ \forall x\}$.

Proof. Suppose that $f \in E^+$ and $f(x) \neq 0$, for all $x \in [0,1]$. Choose $\epsilon > 0$ such that $\epsilon < \inf\{f(x);\ \forall x \in [0,1]\}$. Now given any g in the open ball $B(f,\epsilon)$, and assume that $g(x_0) = 0$ for some $x_0 \in [0,1]$. Then

$$\epsilon < \inf\{|f(x)| < |f(x_0)| < \sup\{|f(x) - g(x)|;\ x \in [0,1]\} = \|f - g\| < \epsilon\}. \tag{2.7}$$

Therefore, $\{f \in E^+;\ f(x) \neq 0;\ \forall x\}$ is an open subset, so we get that $\{f \in E^+;\ f(x) \neq 0;\ \forall x\} \subseteq \operatorname{Int}(E^+)$. Conversely, if $g \in \operatorname{Int}(E^+)$, then there exists $\epsilon_0 > 0$ such that $B(g,\epsilon_0) \subseteq E^+$. If for some $x_0 \in [0,1]$, $g(x_0) = 0$, then define that $h(x) = g(x) - \epsilon_0/2$. Therefore, $h \in B(g,\epsilon_0)$ and $h \notin E^+$ as $h(x_0) = -\epsilon_0/2$ which is a contradiction, so $g(x) \neq 0$, for all $x \in [0,1]$. □

The following example shows that $\operatorname{Int}(E^+)$ can be empty.

Example 2.8. In $\mathbb{R}^2$, consider the subspace $E = \{(x,-x);\ \forall x \in \mathbb{R}\}$, with the induced norm and simplicial ordering as in Example 2.3(b). Then $E^+ = \{(0,0)\}$, hence $\operatorname{Int}(E^+) = \emptyset$.

For all x and y in E, by $x \ll y$ we mean $y - x \in \operatorname{Int}(E^+)$, so $x \gg 0$ means $x \in \operatorname{Int}(E^+)$. Then we have the following properties.

Proposition 2.9. *Let E be an ordered space having a norm. Then*

(i) *The zero element is not in* $\operatorname{Int}(E^+)$*,*

(ii) *If* $x \gg 0$*, then* $-x \ll 0$*,*

(iii) *If* $x \gg 0$ *and* $\alpha \in \mathbb{R}^+$*, then* $\alpha x \gg 0$*,*

(iv) *If* $x \gg 0$ *and* $\alpha \in \mathbb{R}^-$*, then* $\alpha x \ll 0$*.*

Proof. (i) Assume that $0 \in \operatorname{Int}(E^+)$, then there is an ϵ-neighborhood $B(0,\epsilon)$ of 0 such that

$$B(0,\epsilon) = \{x \in E;\ \|x\| < \epsilon\} \subseteq E^+, \tag{2.8}$$

so this open neighborhood contains a nonzero x (if not then $B(0,\epsilon) = \{0\}$; hence, $\{0\}$ is an open subset which is a contradiction, as metric spaces are Hausdorff). Also, we have $-x \in B(0,\epsilon)$, hence by Proposition 2.2(ii), we get a contradiction. If $x \gg 0$, then $0 - (-x) = x \in \operatorname{Int}(E^+)$; hence, this proves (ii). To prove (iii) assume that $x \in \operatorname{Int}(E^+)$ and let $\alpha > 0$. There exists an ϵ-neighborhood $B(x,\epsilon)$ of x such that

$$B(x,\epsilon) = \{e \in E; \|e - x\| < \epsilon\} \subseteq E^+. \tag{2.9}$$

Consider that ball $B(\alpha x, \alpha\epsilon)$ of αx, we claim that it is contained in E^+. If $e \in B(\alpha x, \alpha\epsilon)$, then $\|e/\alpha - x\| < \epsilon$ which means that $e/\alpha \in B(x,\epsilon)$ and therefore $e/\alpha \in E^+$ so by (O2) , we get $e \geq 0$, which proves the claim. (iv) follows by (ii) and (iii), hence the proposition has been checked. □

Proposition 2.10. *Let E be an ordered space having a norm with* $\operatorname{Int}(E^+) \neq \emptyset$*. If* $\epsilon > 0$*, then there exists* $c \gg 0$*, such that* $\|c\| < \epsilon$*.*

Proof. Given that $\epsilon > 0$ and let $x \in \mathrm{Int}(E^+)$, by Proposition 2.9(i), we have $x \neq 0$ and by Proposition 2.9(iii), we have $x/\|x\| \gg 0$. Choose a real number $t \in (0, \epsilon)$ and put $c := t(x/\|x\|)$. Then $c \gg 0$ and $\|c\| = t < \epsilon$, hence the lemma is checked. □

Now let us consider a sequence $\{x_n\}_{n=1}^{\infty}$ in a normed ordered space E. We recall some properties that hold for $\mathbb{R}$ and try to extend it to E. As $\mathbb{R}$ is totally ordered, we know that x_n have a monotonic subsequence. This is not true in general even though x_n is assumed to be bounded sequence, the following example explains this.

Example 2.11. Consider that $E = \mathbb{R}^2$, with the simplicial ordering. Let γ be a bijection from $\mathbb{N}$ onto $\mathbb{Q} \cap (1, 2)$ and define the sequence $\{x_n\}_{n=1}^{\infty}$ by $x_n = (\gamma(n), 1/\gamma(n))$. Then x_n is a bounded sequence in E; for example, $(2, 1)$ is an upper bound, with no monotonic subsequence, as every two terms are not comparable. Also, it is a divergent sequence.

Remark 2.12. Recall that by the completeness property (CP), we know that $\mathbb{R}$ has the bounded monotone convergence theorem. Also, the spaces $\mathbb{R}^n$ and $C[0, 1]$ (Example 2.3(b), (e), resp.) have the bounded monotone convergence theorem.

The following example shows that the bounded monotone convergence theorem is not true in a normed ordered space E, in general. The (CP) does not hold in the normed ordered space $\mathbb{Q}$. Moreover, the Bolzano-Weierstrass theorem (B-W) is not true in general.

Example 2.13. Consider the normed ordered space in Example 2.3(c) and the sequence $x_n = (1+1/n)^n$. It is bounded and increasing sequence but has no limit in $\mathbb{Q}$. As every subsequence of x_n is convergent to e, so no subsequence is convergent in $\mathbb{Q}$, that is, (B-W) fails.

Now, let us consider a continuous function f defined on a normed ordered space E into a normed ordered space F. The following examples show that the maximum-minimum theorem (Max-Min) and the intermediate value theorem (IVT) in the case of real-valued functions do not hold for f.

Example 2.14. Consider that the continuous function $f : \mathbb{R} \to \mathbb{R}^2$, which is defined by $f(x) = (x, -x)$, and consider that $M = [0, 1]$. Then M is a compact subset of $\mathbb{R}$ and $f(M)$ is the line segment from $(0, 0)$ to $(1, -1)$. Indeed, $\sup f(M) = (1, 0)$ and $\inf f(M) = (0, -1)$ and both do not belong to $f(M)$.

More precisely, using known theorems about normed spaces and compactness (see, e.g., [22]), one may easily deduce the following.

Proposition 2.15. *Let E be a normed ordered space having (CP), let X be a metric space and $f : X \to E$ be any continuous function. If M is any compact subset of X, then $f(M)$ attains its* sup *and* inf *in E.*

Example 2.16 (IVT is invalid). Consider the function $g : \mathbb{R} \to \mathbb{R}^2$, which is defined by

$$g(x) = \begin{cases} \left(x, -\sqrt{1-x^2}\right), & -1 \leq x \leq 1, \\ (x, 0), & x \geq 1 \quad \text{or} \quad x \leq -1. \end{cases} \tag{2.10}$$

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation mapping by the angle $\pi/4$, that is, for any $z = (x, y) \in \mathbb{R}^2$, $h(z) = e^{i(\pi/4)}z$. Set $f := h \circ g$, so f is a continuous function defined on real numbers with values in $\mathbb{R}^2$. Now

$$f(-1) = (-1, -1) \leq (0, 0) \leq (1, 1) = f(1), \tag{2.11}$$

but no $c \in (-1, 1)$ with $f(c) = (0, 0)$.

3. *E*-Metric Spaces

Recall that in order to define a metric on a set X, it is necessary to have an ordered space E only. In this section, we define E-metric spaces, and we give main examples. We characterize cone metric spaces as in [7], by using the notion of E-metric space of a normed ordered space E. Then we generalize main theorems in real-valued metric spaces such as Cauchy, convergence theories, and contractive sequences (see, e.g., [21, 22]) to the case of E-metric spaces. Moreover, we get a fixed point theorem of a contractive mapping.

Definition 3.1. Let X be any nonempty set and let E be any ordered space, over the real scalars. An ordered E-metric on X is an E-valued function $d^E : X \times X \to E$ such that for all x, y, and z in X, we have

(i) $d^E(x, y) > 0$ and $d^E(x, y) = 0$ if and only if $x = y$,

(ii) $\mathrm{d}^E(x, y) = d^E(y, x)$,

(iii) $d^E(x, y) \leq d^E(x, z) + d^E(z, y)$.

Then the pair (X, d^E) is called E-metric space.

Now, consider an ordered space E with a norm and consider an E-metric space (X, d^E), let p be a point in X and $c \in \mathrm{Int}(E^+)$. Then the open ball in X centered at p of radius c is

$$B(p, c) = \left\{x \in X; d^E(x, p) < c\right\}. \tag{3.1}$$

Example 3.2. (a) Fixing $X = E = \mathbb{R}$, with the usual distance, is reduced to the usual metric space of real numbers, having the open intervals as open balls.

(b) Consider that $X = \mathbb{R}^2$ and $E = \mathbb{R}^2$ with the simplicial ordering. Define the function $d^{\mathbb{R}^2}$ from $\mathbb{R}^2 \times \mathbb{R}^2$ into $\mathbb{R}^2$ by

$$d^{\mathbb{R}^2}((x_1, y_1), (x_2, y_2)) = (|x_2 - x_1|, |y_2 - y_1|). \tag{3.2}$$

Then $d^{\mathbb{R}^2}$ is a metric, and hence $(\mathbb{R}^2, d^{\mathbb{R}^2})$ is an $\mathbb{R}^2$-metric space. Moreover, the open balls are realized as the open rectangles in $\mathbb{R}^2$ space.

(c) Recall Example 2.3(e), let $X = C[0, 1]$, and consider the normed space $E = C[0, 1]$. Define the function $d^E : X \times X \to E$ by $d^E(f, g) = |f - g|$. Then (X, d^E) is an E-metric space. Moreover, fix $g \in C[0, 1]$ and let $\epsilon(x)$ be a function with all its values are strictly positive; that is, $\epsilon(x)$ never touch the x-axis, so by Example 2.7 we have $\epsilon(x) \in \mathrm{Int}(E^+)$. Then the open ball $B(g, \epsilon)$ is a fiber that consists of all functions f with values

$$g(x) - \epsilon(x) < f(x) < g(x) + \epsilon(x), \quad \forall x \in [0,1]. \tag{3.3}$$

Definition 3.3. Let E be a normed ordered space, and let (X, d^E) be an E-metric space. Then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is called convergent to a point $x_0 \in X$, if for all $c \in \mathrm{Int}(E^+)$, there exists a positive integer N such that $d^E(x_n, x_0) < c$, for all $n > N$.

If a sequence $\{x_n\}_{n=1}^{\infty}$ converges to a point $x_0 \in X$, then we write

$$\lim_{n \to \infty} x_n = x_0, \quad \text{or simply } x_n \longrightarrow x_0. \tag{3.4}$$

Definition 3.4. Let E be a normed ordered space, and let (X, d^E) be an E-metric space. Then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is called Cauchy, if for all $c \in \mathrm{Int}(E^+)$, there exists a positive integer N such that $d^E(x_n, x_m) < c$, for all $n, m > N$.

Proposition 3.5. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a E-metric space (X, d^E), where E is a normed ordered space. If x_n is convergent, then it is a Cauchy sequence.*

Proof. Assume that for some $x_0 \in X$, $x_n \to x_0$. Let $c \in \mathrm{Int}(E^+)$. Then there exists a positive integer N such that $d^E(x_n, x_0) < c/2$, for all $n > N$. Therefore for all $n, m > N$, $d^E(x_n, x_m) \leq d^E(x_n, x_0) + d^E(x_m, x_0) < c$. □

Proposition 3.6. *Let E be a normed ordered space with $\mathrm{Int}(E^+) \neq \emptyset$ and let (X, d^E) be an E-metric space. Then any convergent sequence $\{x_n\}_{n=1}^{n}$ in X has a unique limit.*

Proof. Assume that $x_n \to x_1$ and $x_n \to x_2$ and let $c \gg 0$. Then there exist positive integers N_1 and N_2 such that

$$\begin{aligned} d^E(x_n, x_1) &< \frac{c}{2}, \quad \forall n > N_1, \\ d^E(x_n, x_2) &< \frac{c}{2}, \quad \forall n > N_2. \end{aligned} \tag{3.5}$$

Choose that $N = \max\{N_1, N_2\} + 1$, then we get the following:

$$0 \leq d^E(x_1, x_2) \leq d^E(x_1, x_N) + d^E(x_N, x_2) \leq \frac{c}{2} + \frac{c}{2} = c. \tag{3.6}$$

As E is a normed ordered space, there exists $k > 0$ such that $\|d^E(x_1, x_2)\| \leq k\|c\|$. As c an arbitrary, then $\|c\| \to 0$, which gives $\|d^E(x_1, x_2)\| = 0$, then $d^E(x_1, x_2) = 0$, hence $x_1 = x_2$. □

Definition 3.7. Let E be a normed ordered space, and let (X, d^E) be an E-metric space. Then (X, d^E) is called complete E-metric space if every Cauchy sequence in X is convergent.

Now, using the concepts of E-metric spaces, let us have the following characterization of cone metric spaces (Definition 1.2) in the sense of Huang and Zhang as in [7].

Theorem 3.8. *Let X be any nonempty set and E be a space over the real scalars. Then the following are equivalent.*

(a) *The pair (X, d^E) is an E-metric space, where E is a normed ordered space, with* $\mathrm{Int}(E^+) \neq \emptyset$,

(b) *The pair (X, d) is a cone metric space.*

Proof. First (a) implies (b). Consider that $P = E^+$, then by Proposition 2.6, P is closed, and as $\mathrm{Int}(E^+) \neq \emptyset$, we have $P \neq \{0\}$. If $a, b \geq 0$ real numbers and $x, y \in P$, then, by (O2) of Definition 2.1, we get $ax, by \geq 0$, then, by Proposition 2.2(iii), we have $ax + by \in P$. Now if $x \in P$ and $-x \in P$, then by Proposition 2.2(ii), we get $x = 0$, so P is a cone in the sense of Definition 1.1. As E is a normed space, then by definition, P is a normal cone. Then considering that $d = d^E$, we get that (X, d) is a cone metric space. □

Conversely, assume that (X, d) is a cone metric space. Then we have a cone P in E, and the order with respect to P ($x \leq y \Leftrightarrow y - x \in P$) defines a partial ordering on E. To check the translation invariant (O1): given any x, y and $z \in E$, with $x \leq y$. Then by the definition of the order, $y - x \in P$, therefore, $y + z - (x + z) \in P$, hence $x + z \leq y + z$. To prove (O2): by (ii) of Definition 1.1, we know that $0 \in P$, and taking $y = 0$, we get $ax - 0 \in P$, which means $ax \geq 0$. (O3) is direct, hence E is a normed ordered space. By the definition of cone metric space, $\mathrm{Int}(P) \neq \emptyset$. Finally, the cone metric d on X is chosen to be the ordered E-metric as in Definition 3.1; hence, the theorem has been checked.

Now, using the above characterization, we have the following results, including a fixed-point theorem of a contractive mapping. Indeed, Proposition 2.10 is used in the proof of Lemmas 1 and 4 of Haung and Zhang in [7].

Corollary 3.9. *Let E be a normed ordered space with* $\mathrm{Int}(E^+) \neq \emptyset$, *and let (X, d^E) be an E-metric space. Then a sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to $x \in X$ if and only if $d^E(x_n, x) \to 0$ in E.*

Proof. Direct by using Theorem 3.8 together with Lemma 1 in [7]. □

Corollary 3.10. *Let E be a normed ordered space with* $\mathrm{Int}(E^+) \neq \emptyset$, *and let (X, d^E) be an E-metric space. Then a sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy if and only if $d^E(x_n, x_m) \to 0$ in E as $n \to \infty$ and $m \to \infty$.*

Proof. The proof is directed by using Theorem 3.8 together with Lemma 4 in [7]. □

Finally, we have the following fixed-point theorem in normed ordered spaces.

Theorem 3.11. *Let E be a normed ordered space with* $\mathrm{Int}(E^+) \neq \emptyset$, *and let (X, d^E) be a complete E-metric space. If a function $f : X \to X$ satisfies the following contractive condition:*

$$d^E(f(x), f(y)) \leq k d^E(x, y), \quad \text{for some } k \in (0, 1), \tag{3.7}$$

then f has a unique fixed point in X.

Proof. It follows directly by using Theorem 3.8 and Theorem 1 in [7]. □

Finally, and extending the notion in real-valued metric spaces, let us introduce the notion of a contractive sequence in E-metric space, then proving that it is indeed a sufficient to be a Cauchy sequence.

Definition 3.12. Let E be an ordered space, and let (X, d^E) be an E-metric space. Then a sequence $\{x_n\}_{n=1}^{\infty}$ in X is called contractive if there exists a real number $l \in (0,1)$, such that $d^E(x_{n+2}, x_{n+1}) \leq l d^E(x_{n+1}, x_n)$.

Theorem 3.13. *Let E be a normed ordered space with $\operatorname{Int}(E^+) \neq \emptyset$, and let (X, d^E) be an E-metric space. Then every contractive sequence in X is Cauchy.*

Proof. Suppose that x_n is a contractive sequence in X. Then for some real number $l \in (0,1)$, we have

$$d^E(x_{n+2}, x_{n+1}) \leq l d^E(x_{n+1}, x_n) \leq \cdots \leq l^n d^E(x_2, x_1). \tag{3.8}$$

Therefore assuming that $m > n$, we get the following:

$$\begin{aligned} d^E(x_m, x_n) &\leq d^E(x_m, x_{m-1}) + d^E(x_{m-1}, x_{m-2}) + \cdots + d^E(x_{n+1}, x_n) \\ &\leq \left(l^{m-2} + l^{m-3} + \cdots + l^{n-1}\right) d^E(x_2, x_1) \\ &\leq \frac{l^{n-1}}{1-1} d^E(x_2, x_1). \end{aligned} \tag{3.9}$$

Therefore, for some $k \in (0,1)$,

$$\left\| d^E(x_m, x_n) \right\| \leq k \frac{l^{n-1}}{1-l} \left\| d^E(x_2, x_1) \right\|, \tag{3.10}$$

which implies that $\|d^E(x_m, x_n)\| \to 0$ and then $d^E(x_m, x_n) \to 0$ in E, hence by Corollary 3.10, x_n is a Cauchy sequence. □

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