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Research Article
Pseudovaluations on WFI Algebras

Young Bae Jun,[1] Min Su Kang,[2] and Eun Hwan Roh[3]

[1] *Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Republic of Korea*
[2] *Department of Mathematics, Hanyang University, Seoul 133-791, Republic of Korea*
[3] *Department of Mathematics Education, Chinju National University of Education, Chinju 660-756, Republic of Korea*

Correspondence should be addressed to Min Su Kang, sinchangmyun@hanmail.net

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Using Buşneag's model, the notion of pseudovaluations (valuations) on a WFI algebra is introduced, and a pseudometric is induced by a pseudovaluation on WFI algebras. Given a valuation with additional condition, we show that the binary operation in WFI algebras is uniformly continuous.

1. Introduction

In 1990, Wu [1] introduced the notion of fuzzy implication algebras (FI algebra, for short) and investigated several properties. In [2], Li and Zheng introduced the notion of distributive (regular, and commutative, resp.) FI algebras and investigated the relations between such FI algebras and MV algebras. In [3], Jun discussed several aspects of WFI algebras. He introduced the notion of associative (normal and medial, resp.) WFI algebras and investigated several properties. He gave conditions for a WFI algebra to be associative/medial, provided characterizations of associative/medial WFI algebras, and showed that every associative WFI algebra is a group in which every element is an involution. He also verified that the class of all medial WFI algebras is a variety. Jun et al. [4] introduced the concept of ideals of WFI algebras, and gave relations between a filter and an ideal. Moreover, they provided characterizations of an ideal, and established an extension property for an ideal. Buşneag [5] defined pseudovaluation on a Hilbert algebra and proved that every pseudovaluation induces a pseudometric on a Hilbert algebra. Also, Buşneag [6] provided several theorems on extensions of pseudovaluations. Buşneag [7] introduced the notions of pseudovaluations

(valuations) on residuated lattices, and proved some theorems of extension for these (using the model of Hilbert algebras ([6])).

In this paper, using Buşneag's model, we introduce the notion of pseudovaluations (valuations) on WFI algebras, and we induce a pseudometric by using a pseudovaluation on WFI algebras. Given a valuation with additional condition, we show that the binary operation in WFI algebras is uniformly continuous.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a WFI *algebra,* we mean an algebra $(X; \ominus, \theta) \in K(\tau)$ in which the following axioms hold:

(a1) $(\forall x \in X)\ (x \ominus x = \theta)$,

(a2) $(\forall x, y \in X)\ (x \ominus y = y \ominus x = \theta \ \Rightarrow\ x = y)$,

(a3) $(\forall x, y, z \in X)\ (x \ominus (y \ominus z) = y \ominus (x \ominus z))$,

(a4) $(\forall x, y, z \in X)\ ((x \ominus y) \ominus ((y \ominus z) \ominus (x \ominus z)) = \theta)$.

For the convenience of notation, we will write $[x, y_1, y_2, \ldots, y_n]$ for

$$(\cdots((x \ominus y_1) \ominus y_2) \ominus \cdots) \ominus y_n. \tag{2.1}$$

We define $[x, y]^0 = x$, and for $n > 0$, $[x, y]^n = [x, y, y, \ldots, y]$, where y occurs n-times.

Proposition 2.1 (see [3]). *In a WFI algebra X, the following are true:*

(b1) $x \ominus [x, y]^2 = \theta$,

(b2) $\theta \ominus x = \theta \ \Rightarrow\ x = \theta$,

(b3) $\theta \ominus x = x$,

(b4) $x \ominus y = \theta \Rightarrow (y \ominus z) \ominus (x \ominus z) = \theta,\ (z \ominus x) \ominus (z \ominus y) = \theta$,

(b5) $(x \ominus y) \ominus \theta = (x \ominus \theta) \ominus (y \ominus \theta)$,

(b6) $[x, y]^3 = x \ominus y$.

We define a relation "$\preceq$" on X by $x \preceq y$ if and only if $x \ominus y = \theta$. It is easy to verify that a WFI algebra is a partially ordered set with respect to $\preceq$. A nonempty subset S of a WFI algebra X is called a *subalgebra* of X if $x \ominus y \in S$ whenever $x, y \in S$. A nonempty subset F of a WFI algebra X is called a *filter* of X if it satisfies:

(c1) $\theta \in F$,

(c2) $(\forall x \in F)\ (\forall y \in X)\ (x \ominus y \in F \Rightarrow y \in F)$.

A filter F of a WFI algebra X is said to be *closed* (see [3]) if F is also a subalgebra of X. A nonempty subset I of a WFI algebra X is called an *ideal* of X (see [4]) if it satisfies the condition (c1) and

(c3) $(\forall x, y \in X)\ (\forall z \in I)\ ((x \ominus z) \ominus y \in I \Rightarrow x \ominus y \in I)$.

Proposition 2.2 (see [3]). *Let F be a filter of a WFI algebra X. Then F is closed if and only if $x \ominus \theta \in F$ for all $x \in F$.*

Proposition 2.3 (see [3]). *In a finite WFI algebra, every filter is closed.*

Note that every ideal of a WFI algebra is a closed filter (see [4, Theorem 4.3]). For a WFI algebra X, the set

$$\mathcal{S}(X) := \{x \in X \mid x \preceq \theta\} \tag{2.2}$$

is called the *simulative part* of X.

3. WFI Algebras with Pseudovaluations

In what follows, let X denote a WFI algebra unless otherwise specified.

Definition 3.1. A mapping $f : X \to \mathbb{R}$ is called a *pesudovaluation* on X if it satisfies the following two conditions:

(i) $f(\theta) = 0$,

(ii) $(\forall x, y \in X)\ (f(x \ominus y) + f(x) \geq f(y))$.

A pseudovaluation f on X satisfying the following condition:

$$(\forall x \in X)\ \left(x \neq \theta \Longrightarrow f(x) \neq 0\right) \tag{3.1}$$

is called a *valuation* on X.

Obviously, a mapping

$$f : X \longrightarrow \mathbb{R}, \quad x \longmapsto 0 \tag{3.2}$$

is a pseudovaluation on X, which is called the *trivial pseudovaluation.*

Example 3.2. Let $f : X \to \mathbb{R}$ be a mapping defined by

$$f(x) = \begin{cases} 0 & \text{if } x = \theta, \\ k & \text{if } x \in X \setminus \{\theta\}, \end{cases} \tag{3.3}$$

where k is a positive real number. Then, f is a pseudovaluation on X. Moreover, it is a valuation on X.

Example 3.3. Let $\mathbb{Z}$ be the set of integers. Then, $(\mathbb{Z}; \ominus, \theta)$ is a WFI algebra, where $\theta = 0$ and $x \ominus y = y - x$ for all $x, y \in \mathbb{Z}$ (see [8]). Let $f : \mathbb{Z} \to \mathbb{R}$ be a mapping defined by

$$f(x) = \begin{cases} 0 & \text{if } x = \theta, \\ ax + b & \text{otherwise}, \end{cases} \tag{3.4}$$

for all $x \in \mathbb{Z}$, where a and b are real numbers with $a \neq 0$ and $b \geq 0$. Then, f is a pseudovaluation on $\mathbb{Z}$.

Example 3.4. Let $X = \{\theta, a, b\}$ be a set with the following Cayley table:

$$\begin{array}{c|ccc} \ominus & \theta & a & b \\ \hline \theta & \theta & a & b \\ a & \theta & \theta & b \\ b & b & b & \theta \end{array}. \tag{3.5}$$

Then, $(X;\ominus,\theta)$ is a WFI algebra (see [3]). Define a mapping $f : X \to \mathbb{R}$ by $f(\theta) = 0, f(a) = 2$ and $f(b) = 9$. Then, f is a pseudovaluation on X. Also, it is a valuation on X.

Proposition 3.5. *Every pseudovaluation f on X satisfies the following conditions:*

(1) $(\forall x, y \in X)$ $(x \preceq y \Rightarrow f(x) \geq f(y))$,

(2) $(\forall x, y, z \in X)$ $(f(x \ominus z) \leq f(x \ominus y) + f(y \ominus z))$,

(3) $(\forall x, y \in X)$ $(f(x \ominus y) + f(y \ominus x) \geq 0)$.

Proof. (1) Let $x, y \in X$ be such that $x \preceq y$. Then, $x \ominus y = \theta$, and so

$$f(y) \leq f(x \ominus y) + f(x) = f(\theta) + f(x) = 0 + f(x) = f(x). \tag{3.6}$$

(2) Using (a4), we have $x \ominus y \preceq (y \ominus z) \ominus (x \ominus z)$ for all $x, y, z \in X$. It follows from (1) and Definition 3.1(ii) that

$$f(x \ominus y) \geq f((y \ominus z) \ominus (x \ominus z)) \geq f(x \ominus z) - f(y \ominus z), \tag{3.7}$$

so that $f(x \ominus z) \leq f(x \ominus y) + f(y \ominus z)$ for all $x, y, z \in X$.

(3) Let $x, y \in X$. Using Definition 3.1(ii), we have $f(x \ominus y) + f(x) \geq f(y)$ and $f(y \ominus x) + f(y) \geq f(x)$; that is, $f(x \ominus y) \geq f(y) - f(x)$ and $f(y \ominus x) \geq f(x) - f(y)$. It follows that $f(x \ominus y) + f(y \ominus x) \geq 0$. □

Corollary 3.6. *Let $f : X \to \mathbb{R}$ be a pseudovaluation on X. Then, $f(x) \geq 0$ for all $x \in \mathcal{S}(X)$.*

Proof. Since $x \preceq \theta$ for all $x \in \mathcal{S}(X)$, we have $f(x) \geq f(\theta) = 0$ by Proposition 3.5(1) and Definition 3.1(i). □

The following example shows that the converse of Corollary 3.6 may not be true.

Example 3.7. Let X be a WFI algebra which is considered in Example 3.4. Let $g : X \to \mathbb{R}$ be a mapping defined by

$$g = \begin{pmatrix} \theta & a & b \\ 0 & -3 & 2 \end{pmatrix}. \tag{3.8}$$

Then, $\mathcal{S}(X) = \{\theta, b\}$, $g(\theta) = 0$ and $g(b) = 2 \geq 0$. But g is not a pseudovaluation on X, since

$$g(a \ominus \theta) + g(a) = g(\theta) + g(a) = -3 \ngeq 0 = g(\theta). \tag{3.9}$$

Let $f : X \to \mathbb{R}$ be a pseudovaluation on X. If $x_1 \ominus x = \theta$, that is, $x_1 \preceq x$, for all $x, x_1 \in X$, then $f(x) \leq f(x_1)$ by Proposition 3.5(1). If $x_2 \ominus (x_1 \ominus x) = \theta$ for all $x, x_1, x_2 \in X$, then $x_2 \preceq x_1 \ominus x$, and so, $f(x_2) \geq f(x_1 \ominus x) \geq f(x) - f(x_1)$ by Proposition 3.5(1) and Definition 3.1(ii). Hence, $f(x) \leq f(x_1) + f(x_2)$. Now, if $x_3 \ominus (x_2 \ominus (x_1 \ominus x)) = \theta$ for all $x, x_1, x_2, x_3 \in X$, then $x_3 \preceq x_2 \ominus (x_1 \ominus x)$. It follows from Proposition 3.5(1) and Definition 3.1(ii) that

$$f(x_3) \geq f(x_2 \ominus (x_1 \ominus x)) \geq f(x_1 \ominus x) - f(x_2) \geq f(x) - f(x_1) - f(x_2), \tag{3.10}$$

so that $f(x) \leq f(x_1) + f(x_2) + f(x_3)$. Continuing this process, we have the following proposition.

Proposition 3.8. *Let $f : X \to \mathbb{R}$ be a pseudovaluation on X. For any elements $x, x_1, x_2, \ldots, x_n$ of X, if $x_n \ominus (\cdots \ominus (x_2 \ominus (x_1 \ominus x)) \cdots) = \theta$, then $f(x) \leq \sum_{k=1}^{n} f(x_k)$.*

Theorem 3.9. *Let F be a filter of X, and let $f_F : X \to \mathbb{R}$ be a mapping defined by*

$$f_F(x) = \begin{cases} 0 & \text{if } x \in F, \\ k & \text{if } x \notin F, \end{cases} \tag{3.11}$$

where k is a positive real number. Then, f_F is a pseudovaluation on X. In particular, f_F is a valuation on X if and only if $F = \{\theta\}$.

Proof. Straightforward. □

We say f_F is a pseudovaluation induced by a filter F.

Theorem 3.10. *If a mapping $f : X \to \mathbb{R}$ is a pseudovaluation on X, then the set*

$$F_f := \{x \in X \mid f(x) \leq 0\} \tag{3.12}$$

is a filter of X.

Proof. Obviously, $\theta \in F_f$. Let $x, y \in X$ be such that $x \in F_f$ and $x \ominus y \in F_f$. Then, $f(x) \leq 0$ and $f(x \ominus y) \leq 0$. It follows from Definition 3.1(ii) that $f(y) \leq f(x \ominus y) + f(x) \leq 0$ so that $y \in F_f$. Hence, F_f is a filter of X. □

We say F_f is a filter induced by a pseudovaluation f on X.

Corollary 3.11. *If a mapping $f : X \to \mathbb{R}$ is a pseudovaluation on a finite WFI algebra X, then the set*

$$F_f := \{x \in X \mid f(x) \leq 0\} \tag{3.13}$$

is a closed filter of X.

Proof. It follows from Proposition 2.3 and Theorem 3.10. □

Remark 3.12. A filter induced by a pseudovaluation on X may not be closed. Indeed, in Example 3.3, if we take $a = 1$ and $b = 0$, then $f : \mathbb{Z} \to \mathbb{R}$, $x \mapsto x$, is a pseudovaluation on $\mathbb{Z}$. Then, $F_f = \{\theta\} \cup \{k \in \mathbb{Z} \mid k < \theta\}$ which is a filter but not a subalgebra of $\mathbb{Z}$, since $(-3) \ominus (-1) = -1 - (-3) = 2 \notin F_f$. Hence, F_f is not a closed filter of $\mathbb{Z}$.

Theorem 3.13. *For any pseudovaluation $f : X \to \mathbb{R}$, if F is a filter of X, then $F_{f_F} = F$.*

Proof. We have $F_{f_F} = \{x \in X \mid f_F(x) \leq 0\} = \{x \in X \mid x \in F\} = F$. □

The following example shows that the converse of Theorem 3.10 may not be true; that is, there exist a WFI algebra X and a mapping $f : X \to \mathbb{R}$ such that

(1) f is not a pseudovaluation on X,

(2) $F_f := \{x \in X \mid f(x) \leq 0\}$ is a filter of X.

Example 3.14. Let $X = \{\theta, 1, 2, a, b\}$ be a set with the following Cayley table:

$\ominus$	θ	1	2	a	b
θ	θ	1	2	a	b
1	θ	θ	2	a	b
2	θ	θ	θ	a	a
a	a	a	b	θ	2
b	a	a	a	θ	θ

(3.14)

Then $(X; \ominus, \theta)$ is a WFI algebra. Let $f : X \to \mathbb{R}$ be a mapping defined by

$$f = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0 & -4 & 3 & -2 & 5 \end{pmatrix}. \tag{3.15}$$

Then, $F_f = \{\theta, 1, a\}$ is a filter of X. But f is not a pseudovaluation on X, since

$$f(a \ominus b) + f(a) = 1 \ngeq 5 = f(b). \tag{3.16}$$

Definition 3.15. A pseudovaluation (or, valuation) f on X is said to be *positive* if $f(x) \geq 0$ for all $x \in X$.

The pseudovaluation f on X which is given in Example 3.4 is positive.

Theorem 3.16. *If a pseudovaluation f on X is positive, then the set*

$$F_f^{=} := \{x \in X \mid f(x) = 0\} \tag{3.17}$$

is a filter of X.

Proof. Clearly, $\theta \in F_f^=$. Let $x, y \in X$ be such that $x \in F_f^=$ and $x \ominus y \in F_f^=$. Then, $f(x) = 0$ and $f(x \ominus y) = 0$. Since f is positive, it follows from Definition 3.1(ii) that

$$0 \leq f(y) \leq f(x \ominus y) + f(x) = 0, \tag{3.18}$$

so that $f(y) = 0$, that is, $y \in F_f^=$. Hence, $F_f^=$ is a filter of X. □

The following example shows that two distinct pseudovaluations induce the same filter.

Example 3.17. Consider a WFI algebra $X = \{\theta, 1, 2, a, b\}$ which is given in Example 3.14. Let g and h be mappings from X to $\mathbb{R}$ defined by

$$g = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0 & 0 & 4 & 3 & 5 \end{pmatrix}, \qquad h = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0 & 0 & 4 & 2 & 3 \end{pmatrix}. \tag{3.19}$$

Then, g and h are pseudovaluations on X, and $F_g = \{\theta, 1\} = F_h$.

For a mapping $f : X \to \mathbb{R}$, define a mapping $d_f : X \times X \to \mathbb{R}$ by $d_f(x, y) = f(x \ominus y) + f(y \ominus x)$ for all $(x, y) \in X \times X$. Note that $d_f(x, y) \geq 0$ for all $(x, y) \in X \times X$.

Theorem 3.18. *If $f : X \to \mathbb{R}$ is a pseudovaluation on X, then d_f is a pseudometric on X, and so (X, d_f) is a pseudometric space.*

We say d_f is called the *pseudometric induced by pseudovaluation f.*

Proof. Let $x, y, z \in X$. Then, $d_f(x, y) = f(x \ominus y) + f(y \ominus x) \geq 0$ by Proposition 3.5(3), and obviously, $d_f(x, y) = d_f(y, x)$ and $d_f(x, x) = 0$. Now,

$$\begin{aligned} d_f(x, y) + d_f(y, z) &= [f(x \ominus y) + f(y \ominus x)] + [f(y \ominus z) + f(z \ominus y)] \\ &= [f(x \ominus y) + f(y \ominus z)] + [f(z \ominus y) + f(y \ominus x)] \\ &\geq f(x \ominus z) + f(z \ominus x) = d_f(x, z). \end{aligned} \tag{3.20}$$

Therefore, (X, d_f) is a pseudometric space. □

Proposition 3.19. *Every pseudometric d_f induced by pseudovaluation f satisfies the following inequalities:*

(1) $d_f(x, y) \geq d_f(x \ominus a, y \ominus a)$,

(2) $d_f(x, y) \geq d_f(a \ominus x, a \ominus y)$,

(3) $d_f(x \ominus y, a \ominus b) \leq d_f(x \ominus y, a \ominus y) + d_f(a \ominus y, a \ominus b)$,

for all $x, y, a, b \in X$.

Proof. (1) Let $x, y, a \in X$. Since $(x\ominus y)\ominus((y\ominus a)\ominus(x\ominus a)) = \theta$ and $(y\ominus x)\ominus((x\ominus a)\ominus(y\ominus a)) = \theta$, it follows from Proposition 3.5(1) that $f(x \ominus y) \geq f((y \ominus a) \ominus (x \ominus a))$ and $f(y \ominus x) \geq f((x \ominus a) \ominus (y \ominus a))$ so that

$$\begin{aligned} d_f(x,y) &= f(x \ominus y) + f(y \ominus x) \\ &\geq f((y \ominus a) \ominus (x \ominus a)) + f((x \ominus a) \ominus (y \ominus a)) \\ &= d_f(x \ominus a,\ y \ominus a). \end{aligned} \tag{3.21}$$

(2) It is similar to the proof of (1).

(3) Using Proposition 3.5(2), we have

$$\begin{aligned} &f((x \ominus y) \ominus (a \ominus b)) \leq f((x \ominus y) \ominus (a \ominus y)) + f((a \ominus y) \ominus (a \ominus b)), \\ &f((a \ominus b) \ominus (x \ominus y)) \leq f((a \ominus b) \ominus (a \ominus y)) + f((a \ominus y) \ominus (x \ominus y)), \end{aligned} \tag{3.22}$$

for all $x, y, a, b \in X$. Hence,

$$\begin{aligned} d_f(x \ominus y, a \ominus b) &= f((x \ominus y) \ominus (a \ominus b)) + f((a \ominus b) \ominus (x \ominus y)) \\ &\leq [f((x \ominus y) \ominus (a \ominus y)) + f((a \ominus y) \ominus (a \ominus b))] \\ &\quad + [f((a \ominus b) \ominus (a \ominus y)) + f((a \ominus y) \ominus (x \ominus y))] \\ &= [f((x \ominus y) \ominus (a \ominus y)) + f((a \ominus y) \ominus (x \ominus y))] \\ &\quad + [f((a \ominus b) \ominus (a \ominus y)) + f((a \ominus y) \ominus (a \ominus b))] \\ &= d_f(x \ominus y, a \ominus y) + d_f(a \ominus y, a \ominus b) \end{aligned} \tag{3.23}$$

for all $x, y, a, b \in X$. □

Theorem 3.20. *Let $f : X \to \mathbb{R}$ be a pseudovaluation on X such that $F_f = \{x \in X \mid f(x) \leq 0\}$ is a closed filter of X. If d_f is a metric on X, then f is a valuation on X.*

Proof. Suppose that f is not a valuation on X. Then, there exists $x \in X$ such that $x \neq \theta$ and $f(x) = 0$. Thus $\theta, x \in F_f$ and so $x \ominus \theta \in F_f$, since F_f is a closed filter of X. It follows that $f(x \ominus \theta) \leq 0$ so that

$$0 = f(\theta) \leq f(x \ominus \theta) + f(x) = f(x \ominus \theta) \leq 0. \tag{3.24}$$

Hence, $f(x \ominus \theta) = 0$, and thus $d_f(x,\theta) = f(x \ominus \theta) + f(\theta \ominus x) = f(x \ominus \theta) + f(x) = 0$. Thus, $x = \theta$ since d_f is a metric on X. This is a contradiction. Therefore, f is a valuation on X. □

Consider the pseudovaluation f on $\mathbb{Z}$ which is described in Example 3.3. If $a = -1$, then

$$f(x) = \begin{cases} 0 & \text{if } x = \theta, \\ -x + b & \text{otherwise}, \end{cases} \tag{3.25}$$

for all $x \in \mathbb{Z}$, and $F_f = \{x \in \mathbb{Z} \mid b \leq x\} \cup \{\theta\}$ which is not a closed filter of $\mathbb{Z}$. Since f is a pseudovaluation on $\mathbb{Z}$, we know that $(\mathbb{Z}, d_f)$ is a pseudometric space by Theorem 3.18. If $x \neq y$ in $\mathbb{Z}$, then

$$\begin{aligned} d_f(x, y) &= f(x \ominus y) + f(y \ominus x) = f(y - x) + f(x - y) \\ &= -y + x + b - x + y + b = 2b \neq 0. \end{aligned} \tag{3.26}$$

Hence, $(\mathbb{Z}, d_f)$ is a metric space. But $f(b) = 0$, and so, f is not a valuation on $\mathbb{Z}$. This shows that Theorem 3.20 may not be true when F_f is not a closed filter of X.

Theorem 3.21. *For a mapping $f : X \to \mathbb{R}$, if d_f is a pseudometric on X, then $(X \times X, d_f^*)$ is a pseudometric space, where*

$$d_f^*((x, y), (a, b)) = \max\{d_f(x, a), d_f(y, b)\} \tag{3.27}$$

for all $(x, y), (a, b) \in X \times X$.

Proof. Suppose d_f is a pseudometric on X. For any $(x, y), (a, b) \in X \times X$, we have

$$\begin{aligned} d_f^*((x, y), (x, y)) &= \max\{d_f(x, x), d_f(y, y)\} = 0, \\ d_f^*((x, y), (a, b)) &= \max\{d_f(x, a), d_f(y, b)\} \\ &= \max\{d_f(a, x), d_f(b, y)\} \\ &= d_f^*((a, b), (x, y)). \end{aligned} \tag{3.28}$$

Now, let $(x, y), (a, b), (u, v) \in X \times X$. Then,

$$\begin{aligned} d_f^*((x, y), (u, v)) + d_f^*((u, v), (a, b)) &= \max\{d_f(x, u), d_f(y, v)\} + \max\{d_f(u, a), d_f(v, b)\} \\ &\geq \max\{d_f(x, u) + d_f(u, a), d_f(y, v) + d_f(v, b)\} \\ &\geq \max\{d_f(x, a), d_f(y, b)\} \\ &= d_f^*((x, y), (a, b)). \end{aligned} \tag{3.29}$$

Therefore, $(X \times X, d_f^*)$ is a pseudometric space. □

Corollary 3.22. *If $f : X \to \mathbb{R}$ is a pseudovaluation on X, then $(X \times X, d_f^*)$ is a pseudometric space.*

It is natural to ask that if $f : X \to \mathbb{R}$ is a valuation on X, then is (X, d_f) a metric space. But, we see that it is incorrect in the following example.

Example 3.23. For a WFI algebra $(\mathbb{Z}; \ominus, \theta)$, a mapping $f : \mathbb{Z} \to \mathbb{R}$ defined by $f(x) = (1/2)x$ for all $x \in \mathbb{Z}$ is a valuation on $\mathbb{Z}$. Then, d_f is a pseudometric on $\mathbb{Z}$. Note that $d_f(-2, 3) = f(-2 \ominus 3) + f(3 \ominus (-2)) = 0$, but $-2 \neq 3$. Hence, (X, d_f) is not a metric space.

Theorem 3.24. *If $f : X \to \mathbb{R}$ is a positive valuation on X, then (X, d_f) is a metric space.*

Proof. Suppose that f is a positive valuation on X. Then, (X, d_f) is a pseudometric space by Theorem 3.18. Let $x, y \in X$ be such that $d_f(x, y) = 0$. Then, $0 = d_f(x, y) = f(x \ominus y) + f(y \ominus x)$, and so $f(x \ominus y) = 0$ and $f(y \ominus x) = 0$, since f is positive. Also, since f is a valuation on X, it follows that $x \ominus y = \theta$ and $y \ominus x = \theta$ so from (a2) that $x = y$. Therefore, (X, d_f) is a metric space. □

Corollary 3.25. *If $f : X \to \mathbb{R}$ is a valuation on X such that $F_f = \{\theta\}$, then (X, d_f) is a metric space.*

Theorem 3.26. *If $f : X \to \mathbb{R}$ is a positive valuation on X, then $(X \times X, d_f^*)$ is a metric space.*

Proof. Note from Corollary 3.22 that $(X \times X, d_f^*)$ is a pseudometric space. Let $(x, y), (a, b) \in X \times X$ be such that $d_f^*((x, y), (a, b)) = 0$. Then,

$$0 = d_f^*\big((x, y), (a, b)\big) = \max\{d_f(x, a), d_f(y, b)\}, \tag{3.30}$$

and so $d_f(x, a) = 0 = d_f(y, b)$, since $d_f(x, y) \geq 0$ for all $(x, y) \in X \times X$. Hence,

$$\begin{aligned} 0 &= d_f(x, a) = f(x \ominus a) + f(a \ominus x), \\ 0 &= d_f(y, b) = f(y \ominus b) + f(b \ominus y). \end{aligned} \tag{3.31}$$

Since f is positive, it follows that $f(x \ominus a) = 0 = f(a \ominus x)$ and $f(y \ominus b) = 0 = f(b \ominus y)$ so that $x \ominus a = \theta = a \ominus x$ and $y \ominus b = \theta = b \ominus y$. Using (a2), we have $a = x$ and $b = y$, and so $(x, y) = (a, b)$. Therefore, $(X \times X, d_f^*)$ is a metric space. □

Theorem 3.27. *If f is a positive valuation on X, then the operation $\ominus : X \times X \to X$ is uniformly continuous. (Suppose that (X, d) and (Y, ρ) are metric spaces and $f : X \to Y$. We say that f is uniformly continuous provided that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any points x_1 and x_2 in X, if $d(x_1, x_2) < \delta$, then $\rho(f(x_1), f(x_2)) < \varepsilon$.)*

Proof. For any $\varepsilon > 0$, if $d_f^*((x, y), (a, b)) < \varepsilon/2$, then $d_f(x, a) < \varepsilon/2$, and $d_f(y, b) < \varepsilon/2$. Using Proposition 3.19, we have

$$\begin{aligned} d_f(x \ominus y, a \ominus b) &\leq d_f(x \ominus y, a \ominus y) + d_f(a \ominus y, a \ominus b) \\ &\leq d_f(x, a) + d_f(y, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{3.32}$$

Therefore, the operation $\ominus : X \times X \to X$ is uniformly continuous. □

Corollary 3.28. *If f is a valuation on X such that $F_f = \{\theta\}$, then the operation $\ominus : X \times X \to X$ is uniformly continuous.*

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