

Research Article

Tripled Coincidence Point Theorems for Nonlinear Contractions in Partially Ordered Metric Spaces

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Tripled fixed points are extensions of the idea of coupled fixed points introduced in a recent paper by Berinde and Borcut, 2011. Here using a separate methodology we extend this result to a triple coincidence point theorem in partially ordered metric spaces. We have defined several concepts pertaining to our results. The main results have several corollaries and an illustrative example. The example shows that the extension proved here is actual and also the main theorem properly contains all its corollaries.

1. Introduction and Preliminaries

In recent times coupled fixed point theory has experienced a rapid growth in partially ordered metric spaces. The speciality of this line of research is that the problems herein utilize both order theoretic and analytic methods. References [1–19] are some instances of these works.

Definition 1.1 (see [14]). A function $g : R \rightarrow R$ is said to be monotone nondecreasing (or increasing) if $x \geq y$ implies $g(x) \geq g(y)$.

Definition 1.2 (see [14]). Let X be a nonempty set. Let $F : X \times X \rightarrow X$ be a mapping. An element (x, y) is called a coupled fixed point of F if

$$F(x, y) = x, \quad F(y, x) = y. \quad (1.1)$$

Recently, Berinde and Borcut [20] extended the idea of coupled fixed points to tripled fixed points. The definition is as follows.

Definition 1.3 (see [20]). Let X be a nonempty set. Let $F : X \times X \times X \rightarrow X$ be a mapping. An element (x, y, z) is called a tripled fixed point of F if

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z. \quad (1.2)$$

They also extended the mixed monotone property to functions with three arguments.

Definition 1.4 (see [20]). Let (X, \leq) be a partially ordered set and $F : X \times X \times X \rightarrow X$. The mapping F is said to have the mixed monotone property if for any $x, y, z \in X$

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \leq x_2 &\implies F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, \quad y_1 \leq y_2 &\implies F(x, y_1, z) \geq F(x, y_2, z), \\ z_1, z_2 \in X, \quad z_1 \leq z_2 &\implies F(x, y, z_1) \leq F(x, y, z_2). \end{aligned} \quad (1.3)$$

Our purpose here is to establish tripled coincidence point results in metric spaces with partial ordering. For that purpose we define mixed g -monotone property in the following. Mixed g -monotone property was already defined in the context of coupled fixed points [14]. Here in the spirit of Definition 1.4 we have made an extension of that.

Definition 1.5. Let (X, \leq) be a partially ordered set. Let $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow X$. The mapping F is said to have the mixed g -monotone property if for any $x, y, z \in X$.

$$\begin{aligned} x_1, x_2 \in X, \quad gx_1 \leq gx_2 &\implies F(x_1, y, z) \leq F(x_2, y, z), \\ y_1, y_2 \in X, \quad gy_1 \leq gy_2 &\implies F(x, y_1, z) \geq F(x, y_2, z), \\ z_1, z_2 \in X, \quad gz_1 \leq gz_2 &\implies F(x, y, z_1) \leq F(x, y, z_2). \end{aligned} \quad (1.4)$$

Coupled coincidence point was defined by Lakshmikantham and Ćirić [14]. We also extend the concept of coupled coincidence point to tripled coincidence point in the following.

Definition 1.6. Let X be a nonempty set. Let $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow X$. An element (x, y, z) is called a tripled coincidence point of g and F if

$$F(x, y, z) = gx, \quad F(y, x, y) = gy, \quad F(z, y, x) = gz. \quad (1.5)$$

We extend the concept of commuting mappings given by Lakshmikantham and Ćirić [14], in the following definition.

Definition 1.7. Let X be a nonempty set. Then one says that the mappings $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow X$ are commuting if for all $x, y, z \in X$

$$g(F(x, y, z)) = F(gx, gy, gz). \quad (1.6)$$

The following is the definition of compatible mappings which is an extension of the compatibility defined by Choudhury and Kundu in [8].

Definition 1.8 (see [8]). Let (X, d) be a metric space. The mappings g and F , where $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow X$ are said to be compatible if

$$\begin{aligned}\lim_{n \rightarrow \infty} d(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n)) &= 0, \\ \lim_{n \rightarrow \infty} d(gF(y_n, x_n, y_n), F(gy_n, gx_n, gy_n)) &= 0, \\ \lim_{n \rightarrow \infty} d(gF(z_n, y_n, x_n), F(gz_n, gy_n, gx_n)) &= 0,\end{aligned}\tag{1.7}$$

whenever $\{x_n\}, \{y_n\}, \{z_n\}$ are sequences in X such that

$$\begin{aligned}\lim_{n \rightarrow \infty} F(x_n, y_n, z_n) &= gx_n = x, \\ \lim_{n \rightarrow \infty} F(y_n, x_n, y_n) &= gy_n = y, \\ \lim_{n \rightarrow \infty} F(z_n, y_n, x_n) &= gz_n = z.\end{aligned}\tag{1.8}$$

2. Main Results

Theorem 2.1. Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ are such that, g is monotone increasing, F has the mixed g -monotone property and

$$d(F(x, y, z), F(u, v, w)) \leq \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\})\tag{2.1}$$

for all $x, y, z \in X$ for which $gx \leq gu$, $gy \geq gv$ and $gz \leq gw$, where $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is such that $\psi(t)$ is monotone, $\psi(t) < t$ and $\lim_{r \rightarrow t^+} \psi(r) < t$ for all $t > 0$. Suppose $F(X \times X \times X) \subseteq g(X)$, g is continuous, and $\{g, F\}$ is a compatible pair. Suppose either

- (a) F is continuous or
- (b) X has the following properties:
 - (i) if a nondecreasing sequence $\{\alpha_n\} \rightarrow \alpha$, then $\alpha_n \leq \alpha$ for all n ,
 - (ii) if a nonincreasing sequence $\{\beta_n\} \rightarrow \beta$, then $\beta_n \geq \beta$ for all n .

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = gx, \quad F(y, x, y) = gy, \quad F(z, y, x) = gz,\tag{2.2}$$

that is, g and F have a tripled coincidence point.

Proof. By a condition of the theorem, there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$. Since $F(X \times X \times X) \subseteq g(X)$, we can choose $x_1, y_1, z_1 \in X$ such that

$$gx_1 = F(x_0, y_0, z_0), \quad gy_1 = F(y_0, x_0, y_0), \quad gz_1 = F(z_0, y_0, x_0). \quad (2.3)$$

Continuing this process, we can construct sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n, z_n), \quad gy_{n+1} = F(y_n, x_n, y_n), \quad gz_{n+1} = F(z_n, y_n, x_n). \quad (2.4)$$

Next we will show that, for $n \geq 0$,

$$gx_n \leq gx_{n+1}, \quad gy_n \geq gy_{n+1}, \quad gz_n \leq gz_{n+1}. \quad (2.5)$$

Since, $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, by (2.3), we get

$$gx_0 \leq gx_1, \quad gy_0 \geq gy_1, \quad gz_0 \leq gz_1, \quad (2.6)$$

that is, (2.5) holds for $n = 0$.

We presume that (2.5) holds for some $n = m > 0$. As F has the mixed g -monotone property and $gx_m \leq gx_{m+1}$, $gy_m \geq gy_{m+1}$ and $gz_m \leq gz_{m+1}$, we obtain

$$\begin{aligned} gx_{m+1} &= F(x_m, y_m, z_m) \\ &\leq F(x_{m+1}, y_m, z_m) \\ &\leq F(x_{m+1}, y_m, z_{m+1}) \\ &\leq F(x_{m+1}, y_{m+1}, z_{m+1}) = gx_{m+2}, \end{aligned} \quad (2.7)$$

$$\begin{aligned} gy_{m+1} &= F(y_m, x_m, y_m) \\ &\geq F(y_m, x_m, y_{m+1}) \\ &\geq F(y_{m+1}, x_m, y_{m+1}) \\ &\geq F(y_{m+1}, x_{m+1}, y_{m+1}) = gy_{m+2}, \end{aligned} \quad (2.8)$$

$$\begin{aligned} gz_{m+1} &= F(z_m, y_m, x_m) \leq F(z_{m+1}, y_m, x_m) \\ &\leq F(z_{m+1}, y_{m+1}, x_m) \\ &\leq F(z_{m+1}, y_{m+1}, x_{m+1}) = gz_{m+2}. \end{aligned} \quad (2.9)$$

Thus, (2.5) holds for $n = m + 1$. Then, by induction, we conclude that (2.5) holds for $n \geq 1$.

If for some $n \in \mathbb{N}$,

$$gx_n = gx_{n+1}, \quad gy_n = gy_{n+1}, \quad gz_n = gz_{n+1}, \quad (2.10)$$

then, by (2.4), (x_n, y_n, z_n) is a tripled coincidence point of g and F . Therefore we assume, for any $n \in \mathbb{N}$,

$$(gx_n, gy_n, gz_n) \neq (gx_{n+1}, gy_{n+1}, gz_{n+1}). \quad (2.11)$$

Set $\delta_n = \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\}$.

Then

$$\delta_n > 0 \quad \forall n \geq 0. \quad (2.12)$$

Then, by (2.1), (2.4) and (2.5), we have

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\ &\leq \psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\}), \\ d(gy_n, gy_{n+1}) &= d(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n)) \\ &\leq \psi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}), \\ d(gz_n, gz_{n+1}) &= d(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_n, y_n, x_n)) \\ &\leq \psi(\max\{d(gz_{n-1}, gz_n), d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}). \end{aligned} \quad (2.13)$$

Thus, from (2.13) we obtain that

$$\begin{aligned} \delta_n &= \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\} \\ &\leq \psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\}). \end{aligned} \quad (2.14)$$

It then follows from (2.12) and a property ψ , that for all $n \geq 1$,

$$\delta_n \leq \psi(\delta_{n-1}) < \delta_{n-1}. \quad (2.15)$$

Thus, $\{\delta_n\}$ is a monotone decreasing sequence of nonnegative real numbers. So, there exist a $\delta \geq 0$ such that

$$\lim_{n \rightarrow \infty} \delta_n = \delta. \quad (2.16)$$

Suppose $\delta > 0$. Letting $n \rightarrow \infty$ in (2.14), using (2.15), (2.16), and a property of ψ , we get

$$\delta \leq \psi(\delta) < \delta, \quad (2.17)$$

which is a contradiction. Thus $\delta = 0$, or

$$\lim_{n \rightarrow \infty} \delta_n = 0, \quad (2.18)$$

or

$$\begin{aligned}\lim_{n \rightarrow \infty} d(gx_{n+1}, gx_n) &= 0, \\ \lim_{n \rightarrow \infty} d(gy_{n+1}, gy_n) &= 0, \\ \lim_{n \rightarrow \infty} d(gz_{n+1}, gz_n) &= 0.\end{aligned}\tag{2.19}$$

Now, we will prove that $\{gx_n\}$, $\{gy_n\}$, and $\{gz_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$, $\{gy_n\}$, and $\{gz_n\}$ is not a Cauchy sequence. So, there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$ of $\{gx_n\}$, $\{gy_{n(k)}\}$ of $\{gy_n\}$, and $\{gz_{n(k)}\}$ of $\{gz_n\}$ with $n(k) > m(k) \geq k$ such that

$$\alpha_k = \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\} \geq \varepsilon.\tag{2.20}$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.20). Then, for all $k \geq 0$,

$$\max\{d(gx_{n(k)-1}, gx_{m(k)}), d(gy_{n(k)-1}, gy_{m(k)}), d(gz_{n(k)-1}, gz_{m(k)})\} < \varepsilon.\tag{2.21}$$

By using (2.20) and (2.21) we have for $k \geq 0$,

$$\begin{aligned}\varepsilon &\leq \alpha_k = \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\} \\ &\leq \max\{d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}), d(gy_{n(k)}, gy_{n(k)-1}) \\ &\quad + d(gy_{n(k)-1}, gy_{m(k)}), d(gz_{n(k)}, gz_{n(k)-1}) + d(gz_{n(k)-1}, gz_{m(k)})\} \\ &\leq \max\{d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1}), d(gz_{n(k)}, gz_{n(k)-1})\} + \varepsilon \\ &\leq \delta_{n(k)-1} + \varepsilon.\end{aligned}\tag{2.22}$$

Letting $k \rightarrow \infty$ in (2.22), and using (2.19), we get

$$\lim_{k \rightarrow \infty} \alpha_k = \lim_{k \rightarrow \infty} \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\} = \varepsilon.\tag{2.23}$$

Let, for $k \geq 0$,

$$\beta_k = \max\{d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), d(gz_{n(k)+1}, gz_{m(k)+1})\}.\tag{2.24}$$

Again, for all $k \geq 0$,

$$\begin{aligned}
 \alpha_k &= \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\} \\
 &\leq \max\{d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)}), \\
 &\quad (gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)}), \\
 &\quad d(gz_{n(k)}, gz_{n(k)+1}) + d(gz_{n(k)+1}, gz_{m(k)+1}) + d(gz_{m(k)+1}, gz_{m(k)})\} \\
 &\leq \max\{d(gx_{n(k)}, gx_{n(k)+1}), d(gy_{n(k)}, gy_{n(k)+1}), d(gz_{n(k)}, gz_{n(k)+1})\} \\
 &\quad + \max\{d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), d(gz_{n(k)+1}, gz_{m(k)+1})\} \\
 &\quad + \max\{d(gx_{m(k)}, gx_{m(k)+1}), d(gy_{m(k)}, gy_{m(k)+1}), d(gz_{m(k)}, gz_{m(k)+1})\} \\
 &\leq \delta_{n(k)+1} + \beta_k + \delta_{m(k)+1}.
 \end{aligned} \tag{2.25}$$

Analogously we have for $k \geq 0$,

$$\begin{aligned}
 \beta_k &= \max\{d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), d(gz_{n(k)+1}, gz_{m(k)+1})\} \\
 &\leq \max\{d(gx_{n(k)+1}, gx_{n(k)}) + d(gx_{n(k)}, gx_{m(k)}) + d(gx_{m(k)}, gx_{m(k)+1}), \\
 &\quad d(gy_{n(k)+1}, gy_{n(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gy_{m(k)}, gy_{m(k)+1}), \\
 &\quad d(gz_{n(k)+1}, gz_{n(k)}) + d(gz_{n(k)}, gz_{m(k)}) + d(gz_{m(k)}, gz_{m(k)+1})\} \\
 &\leq \max\{d(gx_{n(k)}, gx_{n(k)+1}), d(gy_{n(k)}, gy_{n(k)+1}), d(gz_{n(k)}, gz_{n(k)+1})\} \\
 &\quad + \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\} \\
 &\quad + \max\{d(gx_{m(k)}, gx_{m(k)+1}), d(gy_{m(k)}, gy_{m(k)+1}), d(gz_{m(k)}, gz_{m(k)+1})\} \\
 &\leq \delta_{n(k)+1} + \alpha_k + \delta_{m(k)+1}.
 \end{aligned} \tag{2.26}$$

Letting $k \rightarrow \infty$ in (2.25) and (2.26), we get that

$$\begin{aligned}
 &\lim_{k \rightarrow \infty} \max\{d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), d(gz_{n(k)+1}, gz_{m(k)+1})\} \\
 &= \lim_{k \rightarrow \infty} \beta_k = \varepsilon = \lim_{k \rightarrow \infty} \alpha_k.
 \end{aligned} \tag{2.27}$$

Since $n(k) > m(k)$, for $k \geq 0$, we have

$$\begin{aligned}
 gx_{n(k)} &\geq gx_{m(k)}, & gy_{n(k)} &\leq gy_{m(k)}, \\
 & & gz_{n(k)} &\geq gz_{m(k)}.
 \end{aligned} \tag{2.28}$$

Then from (2.1), (2.4), and (2.28), we have for $k \geq 0$,

$$\begin{aligned} d(gx_{n(k)+1}, gx_{m(k)+1}) &= d(F(x_{n(k)}, y_{n(k)}, z_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)})) \\ &\leq \psi(\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\}), \\ d(gy_{n(k)+1}, gy_{m(k)+1}) &= d(F(y_{n(k)}, x_{n(k)}, y_{n(k)}), F(y_{m(k)}, x_{m(k)}, y_{m(k)})) \\ &\leq \psi(\max\{d(gy_{n(k)}, gy_{m(k)}), d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\}), \\ d(gz_{n(k)+1}, gz_{m(k)+1}) &= d(F(z_{n(k)}, y_{n(k)}, x_{n(k)}), F(z_{m(k)}, y_{m(k)}, x_{m(k)})) \\ &\leq \psi(\max\{d(gz_{n(k)}, gz_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gx_{n(k)}, gx_{m(k)})\}). \end{aligned} \quad (2.29)$$

From (2.29) for $k \geq 0$, we get

$$\beta_k \leq \psi(\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\}) = \psi(\alpha_k). \quad (2.30)$$

Letting $k \rightarrow \infty$ in (2.30), using (2.20), (2.27), and a property of ψ , we get

$$\varepsilon \leq \psi(\varepsilon) < \varepsilon, \quad (2.31)$$

which is a contradiction. This shows that $\{gx_n\}$, $\{gy_n\}$, and $\{gz_n\}$ are Cauchy sequences.

Since X is complete, there exist $x, y, z \in X$ such that

$$\lim_{n \rightarrow \infty} gx_n = x, \quad \lim_{n \rightarrow \infty} gy_n = y, \quad \lim_{n \rightarrow \infty} gz_n = z. \quad (2.32)$$

From (2.4) and (2.32), using the continuity of g , we have

$$gx = \lim_{n \rightarrow \infty} g(gx_{n+1}) = \lim_{n \rightarrow \infty} g(F(x_n, y_n, z_n)), \quad (2.33)$$

$$gy = \lim_{n \rightarrow \infty} g(gy_{n+1}) = \lim_{n \rightarrow \infty} g(F(y_n, x_n, y_n)), \quad (2.34)$$

$$gz = \lim_{n \rightarrow \infty} g(gz_{n+1}) = \lim_{n \rightarrow \infty} g(F(z_n, y_n, x_n)). \quad (2.35)$$

Now we will show that $gx = F(x, y, z)$, $gy = F(y, x, y)$, and $gz = F(z, y, x)$.

Since g and F are compatible, in addition with (2.33), (2.34), and (2.35), respectively imply

$$\lim_{n \rightarrow \infty} d(g(F(x_n, y_n, z_n)), F(g(x_n), g(y_n), g(z_n))) = 0, \quad (2.36)$$

$$\lim_{n \rightarrow \infty} d(g(F(y_n, x_n, y_n)), F(g(y_n), g(x_n), g(y_n))) = 0, \quad (2.37)$$

$$\lim_{n \rightarrow \infty} d(g(F(z_n, y_n, x_n)), F(g(z_n), g(y_n), g(x_n))) = 0. \quad (2.38)$$

Suppose now the assumption (a) holds, that is, F is continuous.

For all $n \geq 0$, we have

$$d(gx, F(gx_n, gy_n, gz_n)) \leq d(gx, g(F(x_n, y_n, z_n))) + d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)). \quad (2.39)$$

Taking the limit as $n \rightarrow \infty$, using (2.32), (2.33), (2.36), and the facts that g and F are continuous, we have $d(gx, F(x, y, z)) = 0$.

Similarly, by using (2.32), (2.34), and (2.37) and (2.32), (2.35), and (2.38), respectively, and also the facts that g and F are continuous, we have $d(gy, F(y, x, y)) = 0$ and $d(gz, F(z, y, x)) = 0$.

Thus we have proved that g and F have a tripled coincidence point.

Suppose that the assumption (b) holds. Since $\{gx_n\}$, $\{gz_n\}$ are nondecreasing and $gx_n \rightarrow x$ with $gz_n \rightarrow z$ and also $\{gy_n\}$ is nonincreasing with $gy_n \rightarrow y$, by assumption (b) we have for all n

$$gx_n \leq x, \quad gy_n \geq y, \quad gz_n \leq z. \quad (2.40)$$

By virtue of monotone increasing property of g we have

$$ggx_n \leq gx, \quad ggy_n \geq gy, \quad ggz_n \leq gz. \quad (2.41)$$

Now using (2.4) we have

$$\begin{aligned} d(gx, F(x, y, z)) &\leq d(gx, g(gx_{n+1})) + d(g(gx_{n+1}), F(x, y, z)) \\ &\leq d(gx, g(gx_{n+1})) + d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)) \\ &\quad + d(F(gx_n, gy_n, gz_n), F(x, y, z)) \\ &\leq d(gx, g(gx_{n+1})) + d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)) \\ &\quad + \psi(\max\{d(ggx_n, gx), d(ggy_n, gy), d(ggz_n, gz)\}), \quad (\text{by (2.1), (2.41)}). \end{aligned} \quad (2.42)$$

Taking the limit as $n \rightarrow \infty$ in the above inequality, using (2.33), (2.36), and (2.41) we have

$$d(gx, F(x, y, z)) \leq \lim_{n \rightarrow \infty} \psi(\max\{d(ggx_n, gx), d(ggy_n, gy), d(ggz_n, gz)\}). \quad (2.43)$$

By (2.33), (2.34), (2.35), and the property of ψ , we have

$$d(gx, F(x, y, z)) \leq \psi(0) = 0, \quad (2.44)$$

that is

$$gx = F(x, y, z). \quad (2.45)$$

In a similar manner using (2.33), (2.34), (2.35), and (2.36), (2.37), (2.38), respectively, we obtain

$$\begin{aligned} gy &= F(y, x, y), \\ gz &= F(z, y, x). \end{aligned} \tag{2.46}$$

Thus, we proved that g and F have a tripled coincidence point.

This completes the proof of the theorem. \square

Corollary 2.2. *Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed g -monotone property and*

$$d(F(x, y, z), F(u, v, w)) \leq \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) \tag{2.47}$$

for any $x, y, z \in X$ for which $gx \leq gu$, $gy \geq gv$ and $gz \leq gw$, where $\psi : [0, +\infty) \rightarrow [0, +\infty)$ be such that $\psi(t)$ is monotone, $\psi(t) < t$ and $\lim_{r \rightarrow t^+} \psi(r) < t$ for all $t > 0$. Suppose $F(X \times X \times X) \subseteq g(X)$, g is continuous, and F and g are commuting. Suppose either

- (a) F is continuous, or
- (b) X has the following property:

- (i) if a nondecreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $\{y_n\} \rightarrow y$, then $y_n \geq y$ for all n .

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, y_0)$, and $gz_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = gx, \quad F(y, x, y) = gy, \quad F(z, y, x) = gz, \tag{2.48}$$

that is, F and g have a tripled coincidence point.

Proof. Since a commuting pair is also a compatible pair, the result of the Corollary 2.2 follows from Theorem 2.1. \square

Later, by an example, we will show that the Corollary 2.2 is properly contained in Theorem 2.1.

Corollary 2.3. *Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \rightarrow X$ be such that F has the mixed monotone property and*

$$d(F(x, y, z), F(u, v, w)) \leq \psi(\max\{d(x, u), d(y, v), d(z, w)\}) \tag{2.49}$$

for any $x, y, z \in X$ for which $x \leq u$, $y \geq v$ and $z \leq w$, where $\psi : [0, +\infty) \rightarrow [0, +\infty)$ be such that $\psi(t)$ is monotone, $\psi(t) < t$ and $\lim_{r \rightarrow t^+} \psi(r) < t$ for all $t > 0$. Suppose

- (a) F is continuous, or
 (b) X has the following property:

- (i) if a nondecreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all n ,
 (ii) if a nonincreasing sequence $\{y_n\} \rightarrow y$, then $y_n \geq y$ for all n .

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z, \quad (2.50)$$

that is, F has a tripled fixed point.

Proof. Taking $g(x) = x$ in Theorem 2.1 we obtain Corollary 2.3. \square

Corollary 2.4. Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ are such that F has the mixed monotone property and

$$d(F(x, y, z), F(u, v, w)) \leq k \max\{d(x, u), d(y, v), d(z, w)\} \quad (2.51)$$

for any $x, y, z \in X$ for which $x \leq u, y \geq v$ and $z \leq w$, where $0 < k < 1$. Suppose either

- (a) F is continuous, or
 (b) X has the following property:

- (i) if a nondecreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all n ,
 (ii) if a nonincreasing sequence $\{y_n\} \rightarrow y$, then $y_n \geq y$ for all n .

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z, \quad (2.52)$$

that is, F has a tripled coincidence point.

Proof. Taking $\psi(t) = kt, t > 0$ where $0 < k < 1$, in Corollary 2.3 we obtain Corollary 2.4. \square

The following corollary is the result of Berinde and Borcut in [20].

Corollary 2.5. Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \rightarrow X$ be such that F has the mixed monotone property and

$$d(F(x, y, z), F(u, v, w)) \leq a_1 d(x, u) + a_2 d(y, v) + a_3 d(z, w) \quad (2.53)$$

for any $x, y, z \in X$ for which $x \leq u, y \geq v$ and $z \leq w$, where $a_1 + a_2 + a_3 < 1$. Suppose either

- (a) F is continuous, or
 (b) X has the following property:
 (i) if a nondecreasing sequence $\{x_n\} \rightarrow x$, then $x_n \leq x$ for all n
 (ii) if a nonincreasing sequence $\{y_n\} \rightarrow y$, then $y_n \geq y$ for all n .

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, y_0)$, and $z_0 \leq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y, \quad F(z, y, x) = z, \quad (2.54)$$

that is, F has a tripled fixed point.

Proof. The proof follows from Corollary 2.4, since the inequality in Corollary 2.5 implies that Corollary 2.4. \square

Remark 2.6. The method used in the proof of Corollary 2.5 is different from that used by Berinde and Borcut [20].

Next we discuss an example.

Example 2.7. Let $X = \mathfrak{R}$. Then (X, \leq) is a partially ordered set with the partial ordering defined by $x \leq y$ if and only if $|x| \leq |y|$ and $x \cdot y \geq 0$.

Let $d(x, y) = |x - y|$ for $x, y \in \mathfrak{R}$. Then (X, d) is a complete metric space.

Let $g : X \rightarrow X$ be defined as $g(x) = x^2/10$, for all $x \in X$.

Let $F : X \times X \times X \rightarrow X$ be defined as

$$F(x, y, z) = \frac{x^2 - y^2 + z^2}{9}, \quad \forall x, y, z \in X. \quad (2.55)$$

Then F obeys the mixed g -monotone property.

Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be defined as $\varphi(t) = (1/3)t$ for all $t \in [0, \infty)$.

Let, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences in X such that

$$\begin{aligned} \lim_{n \rightarrow \infty} F(x_n, y_n, z_n) &= \lim_{n \rightarrow \infty} g(x_n) = a, \\ \lim_{n \rightarrow \infty} F(y_n, x_n, y_n) &= \lim_{n \rightarrow \infty} g(y_n) = b, \\ \lim_{n \rightarrow \infty} F(z_n, y_n, x_n) &= \lim_{n \rightarrow \infty} g(z_n) = c. \end{aligned} \quad (2.56)$$

Then explicitly,

$$\lim_{n \rightarrow \infty} \frac{x_n^2 - y_n^2 + z_n^2}{9} = \lim_{n \rightarrow \infty} \frac{x_n^2}{10}, \quad \forall x, y, z \in X, \quad (2.57)$$

or,

$$\frac{10a - 10b + 10c}{9} = a \text{ imply } a - 10b + 10c = 0.$$

Again,

$$\lim_{n \rightarrow \infty} \frac{y_n^2 - x_n^2 + y_n^2}{9} = \lim_{n \rightarrow \infty} \frac{y_n^2}{10}, \quad \forall x, y, z \in X,$$

or,

$$\frac{10b - 10a + 10b}{9} = b \text{ imply } 11b - 10a = 0. \tag{2.58}$$

And

$$\lim_{n \rightarrow \infty} \frac{z_n^2 - y_n^2 + x_n^2}{9} = \lim_{n \rightarrow \infty} \frac{z_n^2}{10}, \quad \forall x, y, z \in X,$$

or,

$$\frac{10c - 10b + 10a}{9} = c \text{ imply } c - 10b + 10a = 0. \tag{2.59}$$

Then from the above relations we have, $a = 0$, $b = 0$, and $c = 0$.

Therefore,

$$\begin{aligned} d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)) &\longrightarrow 0 \text{ as } n \longrightarrow \infty, \\ d(g(F(y_n, x_n, y_n)), F(gy_n, gx_n, gy_n)) &\longrightarrow 0 \text{ as } n \longrightarrow \infty, \\ d(g(F(z_n, y_n, x_n)), F(gz_n, gy_n, gx_n)) &\longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{2.60}$$

Hence, the pair (g, F) is compatible in X .

Also, $x_0 = 0$, $z_0 = c (> 0)$, and $y_0 = 0$ are three points in X such that $g(x_0) = g(0) = 0 < c^2/9 = F(0, 0, c) = F(x_0, y_0, z_0)$, $g(y_0) = g(0) = 0 = F(0, 0, 0) = F(y_0, x_0, y_0)$, and $g(z_0) = g(c) = c^2/10 < c^2/9 = F(c, 0, 0) = F(z_0, y_0, x_0)$.

We next verify inequality (2.1) of Theorem 2.1. We take $x, y, z, u, v, w \in X$, such that $gx \leq gu$, $gz \leq gw$ and $gy \geq gv$, that is, $x^2 \leq u^2$, $z^2 \leq w^2$, and $y^2 \geq v^2$.

Let $A = \max\{d(gx, gu), d(gy, gv), d(gz, gw)\} = \max\{|(x^2 - u^2)|, |(y^2 - v^2)|, |(z^2 - w^2)|\}$.

Then $d(F(x, y, z), F(u, v, w)) = d((x^2 - y^2 + z^2)/9, (u^2 - v^2 + w^2)/9) = (|(x^2 - u^2) - (y^2 - v^2) + (z^2 - w^2)|)/3 \leq (|(x^2 - u^2)| + |(y^2 - v^2)| + |(z^2 - w^2)|)/9 \leq 3A/9 = A/3 = \psi(A) = \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\})$.

Thus it is verified that the functions g , F , and ψ satisfy all the conditions of Theorem 2.1. Here $(0, 0, 0)$ is the tripled coincidence point of g and F in X .

Remark 2.8. It is observed that in Example 2.7 the function F and g do not commute, but they are compatible. Hence Corollary 2.2 cannot be applied to this example. This shows that Theorem 2.1 properly contains Corollary 2.2. Also $g \neq I$, so the results of Berinde and Borcut [20] cannot be applied to this example. This shows that result in [20] is effectively generalised.

References

- [1] M. Abbas, A. R. Khan, and T. Nazir, "Coupled common fixed point results in two generalized metric spaces," *Applied Mathematics and Computation*, vol. 217, no. 13, pp. 6328–6336, 2011.
- [2] H. Aydi, "Some coupled fixed point results on partial metric spaces," *International Journal of Mathematics and Mathematical Sciences*, Article ID 647091, 11 pages, 2011.
- [3] V. Berinde, "Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 18, pp. 7347–7355, 2011.
- [4] B. S. Choudhury and P. Maity, "Coupled fixed point results in generalized metric spaces," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 73–79, 2011.
- [5] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.
- [6] L. Ćirić and V. Lakshmikantham, "Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Stochastic Analysis and Applications*, vol. 27, no. 6, pp. 1246–1259, 2009.
- [7] B. S. Choudhury, N. Metiya, and A. Kundu, "Coupled coincidence point theorems in ordered metric spaces," *Annali dell'Università di Ferrara*, vol. 57, no. 1, pp. 1–16, 2011.
- [8] B. S. Choudhury and A. Kundu, "A coupled coincidence point result in partially ordered metric spaces for compatible mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 73, no. 8, pp. 2524–2531, 2010.
- [9] D. Dorić, Z. Kadelburg, and S. Radenović, "Coupled fixed point results for mappings without mixed monotone property," *Applied Mathematics Letters*. In press.
- [10] J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3403–3410, 2009.
- [11] M. Jleli, V. Ćojbašić-Rajić, B. Samet, and C. Vetro, "Fixed point theorems on ordered metric spaces and applications to Nonlinear beam equations," *Journal of Fixed Point Theory and its Applications*. In press.
- [12] E. Karapinar, "Couple fixed point on cone metric spaces," *Gazi University Journal of Science*, vol. 24, no. 1, pp. 51–58, 2011.
- [13] E. Karapinar, "Couple fixed point theorems for nonlinear contractions in cone metric spaces," *Computers & Mathematics with Applications. An International Journal*, vol. 59, no. 12, pp. 3656–3668, 2010.
- [14] V. Lakshmikantham and L. Ćirić, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 12, pp. 4341–4349, 2009.
- [15] N. V. Luong and N. X. Thuan, "Coupled fixed points in partially ordered metric spaces and application," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 3, pp. 983–992, 2011.
- [16] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.
- [17] S. Radenović, Z. Kadelburg, D. Jandrlić, and A. Jandrlić, "Some results on weakly contractive maps," *Bulletin of Iranian Mathematical Society*. In press.
- [18] B. Samet, "Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 12, pp. 4508–4517, 2010.
- [19] B. Samet and C. Vetro, "Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 74, no. 12, pp. 4260–4268, 2011.
- [20] V. Berinde and M. Borcut, "Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 15, pp. 4889–4897, 2011.



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